A Binary Adverse Selection/Limited Liability Problem

A risk-neutral employer designs a contract to hire a worker and motivate him to perform a task. The contract specifies payments \( S \) to the worker depending on his observed non-negative output level \( x \). The worker’s reservation utility level is known by all to be zero, whatever the worker’s ability. The personal costs incurred by the worker in producing observable output \( x \) are \( x^2/\theta \), where \( \theta \) is the worker’s ability level. Thus, the utility function for the risk-neutral worker is \( U(S, x; \theta) = S - x^2/\theta \). \( \theta \) takes on one of two possible values \{2, 4\}. The worker learns his ability level before agreeing to the terms of the contract proposed by the employer. The employer is never able to observe the worker’s ability directly. The employer believes the smaller realization of \( \theta \) occurs with probability \( p \in (0, 1) \). The employer values output at one dollar per unit.

(a) Characterize completely the incentive contract that maximizes the employer’s expected net payoff in this environment.

(b) Explain intuitively the rationale for any deviations from the "first-best" (full-information) contract.

(c) How does the output of the worker when \( \theta = 2 \) vary with \( p \)? Prove your answer formally, and provide an intuitive explanation.

a. Employer’s Problem:

\[
\begin{align*}
\max_{x_1, x_2, S_1, S_2} & \quad p(x_1 - S_1) + (1 - p)(x_2 - S_2) \\
\text{s.t.} & \quad S_1 - x_1^2 / 2 \geq 0 \\
& \quad S_2 - x_2^2 / 4 \geq 0 \\
& \quad S_1 - x_1^2 / 2 \geq S_2 - x_2^2 / 2 \\
& \quad S_2 - x_2^2 / 4 \geq S_1 - x_1^2 / 4 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

\( \ell = p(x_1 - S_1) + (1 - p)(x_2 - S_2) + \lambda_1(S_1 - x_1^2 / 2) + \lambda_2(S_2 - x_2^2 / 4) + \mu_1(S_1 - x_1^2 / 2 - S_2 + x_2^2 / 2) + \mu_2(S_2 - x_2^2 / 4 - S_1 + x_1^2 / 4) \)

(1) \( \ell_{x_1} = p + \lambda_1(-x_1) + \mu_1(-x_1) + \mu_2(x_1 / 2) \leq 0 \) and \( x_1 \ell_{x_1} = 0 \)

(2) \( \ell_{x_2} = (1 - p) + \lambda_2(-x_2 / 2) + \mu_1(x_2) + \mu_2(-x_2 / 2) \leq 0 \) and \( x_2 \ell_{x_2} = 0 \)

(3) \( \ell_{S_1} = -p + \lambda_1 + \mu_1 - \mu_2 = 0 \)

(4) \( \ell_{S_2} = -(1 - p) + \lambda_2 - \mu_1 + \mu_2 = 0 \)

(5) \( \ell_{\lambda_1} = S_1 - x_1^2 / 2 \geq 0 \) and \( \lambda_1 \ell_{\lambda_1} = 0 \)

(6) \( \ell_{\lambda_2} = S_2 - x_2^2 / 4 \geq 0 \) and \( \lambda_2 \ell_{\lambda_2} = 0 \)

(7) \( \ell_{\mu_1} = S_1 - x_1^2 / 2 - S_2 + x_2^2 / 2 \geq 0 \) and \( \mu_1 \ell_{\mu_1} = 0 \)
(8) $\ell_{\mu_2} = S_2 - x_2^2 / 4 - S_1 + x_1^2 / 4 \geq 0$ and $\mu_2 \ell_{\mu_2} = 0$

Assume $\text{IR}_{\theta=4}$ and are $\text{IC}_{\theta=2}$ not binding. (This will be proved later.)
That means $\lambda_2 = \mu_1 = 0$
From (4): $-(1-p) + \mu_2 = 0 \Rightarrow \mu_2 = (1-p) > 0 \Rightarrow \ell_{\mu_2} = 0$ (i.e., $\text{IC}_{\theta=4}$ is binding)
From (3): $-p + \lambda_1 - (1-p) = 0 \Rightarrow \lambda_1 = 1 > 0 \Rightarrow \ell_{\lambda_1} = 0$ (i.e., $\text{IR}_{\theta=2}$ is binding)
Assume $x_1 = 0$
That means $\ell_{x_1} = p < 0$ which is a contradiction $\therefore x_1 > 0$ and $\ell_{x_1} = 0$
Sub $\lambda_1 = 1$, $\mu_1 = 0$, and $\mu_2 = (1-p)$ into (1): $p + (1)(-x_1) + (1-p)(x_1 / 2) = 0$
Combine terms: $(1+p) \frac{x_1}{2} = p$
Solve for $x_1$: $x_1 = \frac{2p}{1+p}$
Assume $x_2 = 0$
That means $\ell_{x_2} = (1-p) < 0$ which is a contradiction $\therefore x_2 > 0$ and $\ell_{x_2} = 0$
Sub $\lambda_2 = \mu_1 = 0$ and $\mu_2 = (1-p)$ into (2): $(1-p) + (1-p)(-x_2 / 2) = 0$
Solve for $x_2$: $x_2 = 2 \frac{1-p}{1-p} \Rightarrow x_2 = 2$
Sub $x_1$ into (5): $S_1 = \frac{1}{2} \left( \frac{2p}{1+p} \right)^2 \Rightarrow S_1 = 2 \left( \frac{p}{1+p} \right)^2$
Sub $x_1$, $x_2$, and $S_1$ into (8): $S_2 - \frac{(2)^2}{4} - 2 \left( \frac{p}{1+p} \right)^2 + \frac{1}{4} \left( \frac{2p}{1+p} \right)^2 = 0$
Simplify: $S_2 = 1 + \left( \frac{p}{1+p} \right)^2$
Now verify $\text{IR}_{\theta=4}$ does not bind:
Sub $x_2$ and $S_2$ into (6): $1 + \left( \frac{p}{1+p} \right)^2 - \frac{(2)^2}{4} = \left( \frac{p}{1+p} \right)^2 > 0$
Now verify $\text{IC}_{\theta=2}$ does not bind:
Sub $x_1$, $x_2$, $S_1$ and $S_2$ into (7):
$2 \left( \frac{p}{1+p} \right)^2 - 1 \left( \frac{2p}{1+p} \right)^2 - 1 - \left( \frac{p}{1+p} \right)^2 + \frac{(2)^2}{2} = 1 - \left( \frac{p}{1+p} \right)^2 > 0$
Optimal Contract: $\{(x_1, S_1), (x_2, S_2)\} = \left\{ \left( \frac{2p}{1+p}, 2 \left( \frac{p}{1+p} \right)^2 \right), \left( 2, 1 + \left( \frac{p}{1+p} \right)^2 \right) \right\}$
b. First-best solutions:  
If $\theta = 2$, principal solves:

$$\max_{x_1, S_1} x_1 - S_1$$  
net payoff

s.t.  
$$S_1 - x_1^2 / 2 \geq 0$$  
IR$\theta=2$ (i.e., $\theta = 2$ participates)

$$x_1 \geq 0$$  
non-negative output

Lagrangian:  
$$L = x_1 - S_1 + \gamma_1 (S_1 - x_1^2 / 2)$$

K-T Conditions:

(i)  
$$L_{x_1} = 1 - \gamma_1 x_1 \leq 0 \quad \text{and} \quad x_1 L_{x_1} = 0$$

(ii)  
$$L_{S_1} = -1 + \gamma_1 = 0$$

(iii)  
$$L_{\gamma_1} = S_1 - x_1^2 / 2 \geq 0 \quad \text{and} \quad \gamma_1 L_{\gamma_1} = 0$$

From (ii):  
$$\gamma_1 = 1 > 0 \quad \therefore \quad L_{\gamma_1} = 0 \quad \text{(i.e., IR$\theta=2$ is binding)}$$

Assume $x_1 = 0$

That means $L_{x_1} = 1 < 0$ which is a contradiction, \therefore $x_1 > 0$ and $L_{x_1} = 0$

Sub $\gamma_1 = 1$ into (i):  
$$1 - (1)x_1 = 0 \quad \Rightarrow \quad x_1 = 1$$

Sub $x_1$ into (iii):  
$$S_1 - \frac{(1)^2}{2} = 0 \quad \Rightarrow \quad S_1 = \frac{1}{2}$$

Note: This is the same as plugging $p = 1$ into $(x_1, S_1)$ from optimal private information contract from part (a)

If $\theta = 4$, principal solves:

$$\max_{x_2, S_2} x_2 - S_2$$  
net payoff

s.t.  
$$S_2 - x_2^2 / 4 \geq 0$$  
IR$\theta=4$ (i.e., $\theta = 4$ participates)

$$x_2 \geq 0$$

non-negative output

Lagrangian:  
$$l = x_2 - S_2 + \gamma_2 (S_2 - x_2^2 / 4)$$

K-T Conditions:

(iv)  
$$L_{x_2} = 1 - \frac{1}{2} \gamma_2 x_2 \leq 0 \quad \text{and} \quad x_2 L_{x_2} = 0$$

(v)  
$$L_{S_2} = -1 + \gamma_2 = 0$$

(vii)  
$$L_{\gamma_2} = S_2 - x_2^2 / 4 \geq 0 \quad \text{and} \quad \gamma_2 L_{\gamma_2} = 0$$

From (v):  
$$\gamma_2 = 1 > 0 \quad \therefore \quad L_{\gamma_2} = 0 \quad \text{(i.e., IR$\theta=4$ is binding)}$$

Assume $x_2 = 0$

That means $L_{x_2} = 1 < 0$ which is a contradiction, \therefore $x_2 > 0$ and $L_{x_2} = 1 < 0$

Sub $\gamma_2 = 1$ into (iv):  
$$1 - \frac{1}{2} (1)x_2 = 0 \quad \Rightarrow \quad x_2 = 2$$

Sub $x_2$ into (vi):  
$$S_2 - \frac{(2)^2}{4} = 0 \quad \Rightarrow \quad S_2 = 1$$

Note: This is the same as plugging $p = 0$ into $(x_2, S_2)$ from optimal private information contract from part (a)
Summary:

<table>
<thead>
<tr>
<th>$\theta = 2$</th>
<th>$\theta = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$S_1$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

First-Best

<table>
<thead>
<tr>
<th>Private Info</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{2p}{1+p}$</td>
</tr>
</tbody>
</table>

In both the first-best and private information cases, the principal wants the $\theta = 4$ agent to produce $x_2 = 2$. For the $\theta = 2$ agent, however, the principal wants the agent to be inefficient (compared to the first-best output). This is because the $\theta = 4$ agent has an incentive to take the bundle intended for the $\theta = 2$ agent. In order to keep the $\theta = 4$ agent indifferent between the bundles, the principal has to pay rent to the agent (this is in exchange for the agent revealing his private information and taking the $(x_2, S_2)$ option). The amount of efficiency-rent tradeoff depends on the likelihood of encountering a $\theta = 2$ agent ($p$).

c. Output for a $\theta = 4$ agent does not vary between the first-best and private information cases, nor does it vary with $p$.

Output for the $\theta = 2$ agent is the same in the first-best and private information cases at the limit where $p = 1$ (i.e., there is certainty that the agent is a $\theta = 2$ type). As $p$ decreases, the output from a $\theta = 2$ agent decreases to compensate for the greater likelihood of having to pay rent to a $\theta = 4$. At the other limit where $p = 1$, the $\theta = 2$ agent produces no output and the $\theta = 4$ agent receives no rent.

Documentation

I only used my class notes.