Expanding Commutator Subgroup

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Abstract

The commutator subgroup of a given group is an important indicator of how close the group is to being abelian. Contrary to the name, however, not all elements of the commutator subgroup are commutators. In this article, we make clear this point by giving an example of a specific group in which we see the commutators inside the commutator subgroup become "proportionally small" as the group expands in size.

1 Commutator Subgroup

We begin first by presenting a definition of a commutator subgroup and it’s immediate consequences.

Definition Let $G$ be a group, and let $x, y \in G$ and let $A, B$ be nonempty subsets of $G$.

1. Define $[x, y] = x^{-1}y^{-1}xy$, called the commutator of $x$ and $y$.

2. Define $[A, B] = \langle [x, y] : a \in A, b \in B \rangle$, the group generated by commutators of elements from $A$ and $B$.

3. Define $G' = \langle [x, y] : x, y \in G \rangle$, the subgroup of $G$ generated by commutators of elements from $G$, called the commutator subgroup of $G$.

Observe that the commutator of $x$ and $y$ is $1$ if and only if $x$ and $y$ commute. Hence, the commutator of two elements "measure the difference" between $xy$ and $yx$. We mentions here some immediate consequence of the definitions above.
Proposition 1.1 Let $G$ be a group, let $x, y \in G$ and let $H \leq G$. Then

1. $xy = yx[x, y]$ (and in particular, $xy = yx \iff [x, y] = 1$)

2. $H \unlhd G$ if and only if $[H, G] \leq H$.

3. $\sigma[x, y] = [\sigma(x), \sigma(y)]$ for any automorphism $\sigma$ of $G$, $G'$ char $G$ and $G/G'$ is abelian.

4. $G/G'$ is the largest abelian quotient of $G$ in the sense that if $H \unlhd G$ and $G/H$ is abelian, then $G' \leq H$, then $H \unlhd G$ and $G/H$ is abelian.

5. If $\varphi : G \longrightarrow A$ is any homomorphism of $G$ onto an abelian group $A$, then $\varphi$ factors through $G'$ i.e., $G' \leq \ker \varphi$ and the following diagram commutes:

We omit the proof of the proposition.

2 Construction

We now construct a group whose commutator subgroup is strictly larger than its set of commutators. Proposition 1.1 gives us a lead as to how we should construct such a group.

We begin by taking $F = \mathbb{Z}/p\mathbb{Z}$ some finite field where $p$ is some odd prime. Take $V$ to be a vector space of dimension $n$ over $F$ with basis

$$\{e_1, e_2, \ldots, e_n\}.$$ 

Let $W$ be a vector space over $F$ of dimension $\frac{n(n-1)}{2} = \binom{n}{2}$ with basis

$$\{b_{i,j} : 1 \leq i < j \leq n\}.$$ 

Now we define a bilinear map

$$\beta : V \times V \longrightarrow W$$

$$\beta(e_i, e_j) = \begin{cases} 
 b_{ij} & \text{if } i < j \\
 -b_{ji} & \text{if } i > j \\
 0 & \text{if } i = j 
\end{cases}$$

$$\beta(\sum_{i=1}^{n} \lambda_i e_i, \sum_{j=1}^{n} \mu_j e_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \mu_j \beta(e_i, e_j).$$
Claim 2.1 $\beta$ is a bilinear map.

Proof

Claim 2.2

1. $\beta(v, w) = -\beta(w, v)$ for all $v, w \in V$.
2. $\beta(v, v) = 0$ for all $v \in V$.

Proof

We can now construct out group $G$. The elements of $G$ are $V \times W$. For $v_1, v_2 \in V$ and $w_1, w_2 \in W$, we define the group operation of $G$ as follows:

$$(v_1, w_1) \cdot (v_2, w_2) = (v_1 + v_2, w_1 + w_2 + \frac{1}{2} \beta(v_1, v_2))$$

Claim 2.3 This binary operation defines a group operation on $V \times W$.

Proof

1. $(0,0)$ is the identity of $\cdot$ in $G$. Take $v \in V$ and $w \in W$, then

$$(0,0) \cdot (v,w) = (v + 0, w + \frac{1}{2} \beta(0,v)) = (v,w).$$

$$(v,w) \cdot (0,0) = (0 + v, w + \frac{1}{2} \beta(v,0)) = (v,w).$$

2. $\cdot$ is associative. Take $v_1, v_2, v_3 \in V$ and $w_1, w_2, w_3 \in W$, then

$$((v_1, w_1) \cdot (v_2, w_2)) \cdot (v_3, w_3)$$

$$= (v_1 + v_2, w_1 + w_2 + \frac{1}{2} \beta(v_1, v_2)) \cdot (v_3, w_3)$$

$$= (v_1 + v_2 + v_3, w_1 + w_2 + w_3 + \frac{1}{2} \beta(v_1, v_2) + \frac{1}{2} \beta(v_1 + v_2, v_3))$$

3. for every element $(v, w) \in V \times W$, there is an inverse, namely $(-v, -w)$.

$$(v, w) \cdot (-v, -w) = (0, \frac{1}{2} \beta(v, -v)) = (0, -\frac{1}{2} \beta(v, v)) = (0, 0)$$

Thus, $(G, \cdot)$ is a group.

Thus, $(G, \cdot)$ is a group. ■
We have now defined a group $G$. Let us now compute the commutators and the commutator subgroup of $G$. We begin by first calculating a general commutator. Let $(v_1, w_1), (v_2, w_2) \in V \times W$. Then,

\[
[(v_1, w_1), (v_2, w_2)] = (-v_1, -w_1) \cdot (-v_2, -w_2) \cdot (v_1, w_1) \cdot (v_2, w_2)
\]

\[
= (-v_1 - v_2, -w_1 - w_2 + \frac{1}{2} \beta(-v_1, -v_2)) \cdot (v_1 + v_2, w_1 + w_2 + \frac{1}{2} \beta(v_1, v_2))
\]

\[
= (0, \beta(v_1 + v_2) + \frac{1}{2} \beta(-v_1 - v_2, v_1 + v_2))
\]

\[
= (0, \beta(v_1, v_2))
\]