Connectives, truth and truth conditions

Lecture 4

Hana Filip
Propositional Logic

• Propositional logic or statement logic is concerned with rules that allow us to determine the truth values of complex propositions, given the truth values of their parts. The internal structure of individual basic propositions is not taken into account.

We already used the semantics of propositional logic in Toy English when we characterized the truth conditions of complex sentences with the conjunction *and* and the disjunction *or*.

• The rules of propositional logic are formulated using an artificial language, which will be called *PL*. It has its own syntax and semantics. It contains variables for propositions, and ways to combine propositions to complex propositions.
Propositional Logic

• Why is this relevant to you … … even if you don’t want to become a semanticist?

• You will learn a general enough method of analysis by which you can take any bit of English (in books, newspapers, on the TV or radio or internet, etc.) and determine whether:
  – an argument for some claim actually exists, and if one does exist, then
  – show that it is either valid or invalid, and if it is valid, then
  – whether it is also sound.
Why propositional logic?

• Object language vs. Metalanguage

  • *object language*: e.g., English, which is the object of our description
  • *metalanguage*: the language in which we state generalizations and observations about our object language

  “Snow is white” is true if, and only if, snow is white.¹
  “La neige est blanche” is true if, and only if, snow is white.

¹: a sentence made famous by Alfred Tarski (Polish logician, 1901-1983)
Why propositional logic?

If we did not draw the distinction between object language and metalanguage, we would face paradoxes like

- **The liar paradox:**
  
  (1) *This statement is false.*
  
  The oldest known version of the liar paradox is attributed to the Greek philosopher Eubulides of Miletus who lived in the 4th century BC:
  
  (2) *A man says that he is lying. Is what he says true or false?*
  
  If he is lying, then he is telling the truth, and vice versa.

Similarly:

- If (1) is true, then (1) is false.
- But we can also establish the converse, as follows: Assume (1) is false. Because the Liar Sentence is saying precisely that (namely that it is false), the Liar Sentence is true, so (1) is true. We’ve now shown that (1) is true if and only if it is false.
Why propositional logic?

Most logical paradoxes are based on circular definitions or self-referential statements. Such paradoxes are often analyzed by creating metalanguages to separate statements into different levels on which truth and falsity can be assessed independently.

If we did not draw the distinction between object language and metalanguage, we would face paradoxes like

• use vs. mention difficulties

• and other problems, which are problematic for the development of a theory of truth. Recall that the notion of ‘truth’ is the cornerstone of our semantic theory.
Metalanguage

As our metalanguage we have already used certain notions like set and ordered pair, which were developed in mathematics.

We also use
- propositional logic and
- predicate logic.
Metalanguage

• Logical languages are useful for the purpose of analyzing meaning in natural language. They follow well-defined and perspicuous syntactic and semantic principles, and are better understood than natural languages.

• Natural languages involve ambiguity, vagueness, imprecision, etc., and are not as well understood as logical languages.
Today

• the basics of propositional logic, and
• its application to English

will be introduced.

The main focus will be on illustrating
• which parts of the meaning of English sentences can be captured by the tools of propositional logic, and
• which parts of the meaning of English sentences cannot be (easily) rendered by propositional logic.
Propositional Logic

History: Aristotle’s syllogistic logic

• The prime area of logic is the investigation of the laws of inference.
• Aristotle (384 BC-322 BC), one of the founders of logic, was interested in inferences of the following type:

(a) Every man is an animal.
   Every animal is mortal.
   Hence, every man is mortal.

(b) Every sparrow is a bird.
   Every bird has two legs.
   Hence, every sparrow has two legs.

(c) No man has four legs.
   All Greeks are men.
   Hence, no Greek has four legs.

(d) No plant speaks.
   All trees are plants.
   Hence, no tree speaks.
Propositional Logic

History: Aristotle’s syllogistic logic

(a) Every man is an animal.
   Every animal is mortal.
   Hence, every man is mortal.

(c) No man has four legs.
    All Greeks are men.
    Hence, no Greek has four legs.

(b) Every sparrow is a bird.
    Every bird has two legs.
    Hence, every sparrow has two legs.

(d) No plant speaks.
    All trees are plants.
    Hence, no tree speaks.

General Form:

Premise 1
Premise 2
Conclusion

If the premises are true, then the conclusion must be true as well: *transfer of truth.*
Propositional Logic

History: Aristotle’s syllogistic logic

(a) Every man is an animal.
   Every animal is mortal.
   Hence, every man is mortal.

(b) Every sparrow is a bird.
    Every bird has two legs.
    Hence, every sparrow has two legs.

(c) No man has four legs.
    All Greeks are men.
    Hence, no Greek has four legs.

(d) No plant speaks.
    All trees are plants.
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patterns of valid reasoning
or
SYLLOGISMS

All A are B.
All B are C.
Hence, all A are C.

No A is B.
All C are A.
Hence, no C is B.
Propositional Logic

History: Aristotle’s syllogistic logic

• Aristotle provided a survey of valid syllogisms.
• The claim is not that the premises are actually true, and hence
• the claim is not that the consequence is actually true.
• Logic is concerned with the notion of logical consequence.

For example, the following piece of reasoning (inference) is valid:

VALID INFERENCE
Every man is an elephant.
Every elephant has two noses.
Hence, every man has two noses.

If an inference is valid, it is valid by virtue of its logical structure, and independently of what the (possible) world(s) is/are like.
Propositional Logic

History: Aristotle’s syllogistic logic

• Even by chaining valid Aristotelian inference patterns together, we cover only a small part of the variety of valid inference patterns that are implicated in daily reasoning and in the more rigorous, and in complex reasoning that is found in mathematics and other sciences.

• Two problems with Aristotle’s syllogistic logic:
  – the limitations of its concept of logical form;
  – the lack of any sufficiently general method for determining which of the inference patterns expressible in syllogistic notation are valid and which are not.
Propositional Logic

History: Aristotle’s syllogistic logic

• The second deficiency was overcome by the method of **Venn-diagrams** (John Venn, 1834-1923).

All A are B.
All B are C. 
Hence, all A are C.
Propositional Logic

History: Deductive Logic

• The logicians of the late 19th and early 20th century tackled the problems of logic primarily by the deductive method

  – Giuseppe Peano
  – David Hilbert
  – Bertrand Russell
  – Alfred North Whitehead
  – Gerhard Gentzen

• *Deductive or Proof-theoretic method*: A small number of inference patterns is selected as basic and the validity of other patterns is established by chaining two or more applications of the basic patterns together.
Propositional Logic

**Deductive logic vs. inductive logic**

- Logic which is concerned with relations that preserve truth is called *deductive logic*.

- Deductive reasoning uses deductive arguments to move from given statements (premises), which are assumed to be true, to conclusions, which *must be true* if the premises are true. In a valid deductive argument the premises *logically entail* the conclusion. An example of deductive reasoning are Aristotle’s syllogisms.
Propositional Logic

Deductive logic vs. inductive logic

- Inductive logic concerns itself with the preservation of probability, i.e., the premises of an argument are believed to support the conclusion but do not entail it, they do not ensure its truth. In a good inductive argument the premises should provide some degree of support for the conclusion, i.e., the truth of the premises indicates with some degree of strength that the conclusion is true.

Example:

Every raven in a random sample of 3200 ravens is black. This strongly supports the hypothesis that all ravens are black.

- Inductive logic reasoning is based on drawing conclusions from a large number of particular examples to a general rule.
Propositional Logic

Syntactic rules of propositional logic (PL)

• The only syntactic category of PL is a well-formed formula: WFF.
• VOCABULARY of PL
• SYNTAX of PL
Propositional Logic

Syntactic rules of propositional logic (PL)

• VOCABULARY of PL

  1. basic (atomic) propositions: We assume that we have a denumerably infinite (or countably infinite) number of basic propositions, represented by the propositional variables \( p, q, r \), etc.

  2. propositional connectives:
     - \( \neg \) negation
     - \( \land \) ‘and’ conjunction
     - \( \lor \) ‘or’ disjunction
     - \( \rightarrow \) ‘if then’ conditional
     - \( \iff \) ‘if and only if’ biconditional

  3. brackets: (,) or [,]
Propositional Logic

• Syntactic rules of propositional logic (PL)

• SYNTAX of PL
Recursvie definition of the set of all wff’s of PL:
1. Every basic (atomic) proposition is a wff.
2. If p is a wff, then ¬p is a wff (negation, “non p”, also “¬p”).
3. If p and q are wff’s, then
   [p ∧ q], (conjunction)
   [p ∨ q], (disjunction)
   [p → q], (conditional, if p then q)
   [p ↔ q], (biconditional, p if and only if q) are wff’s.
4. Nothing else is a wff.
Propositional Logic

Basic (atomic) wff’s are the logical correspondents of simple declarative sentences in natural languages, i.e., those that do not contain instances of the sentential connectives *and, or, if ...then, if and only if, or not* in English, for ex.

Examples:

- An English sentence like *It is raining* expresses a basic or atomic *proposition* (or a *statement*).

- **Synonymous sentences** express the same proposition:
  - *Paris is the capital of France* \( p \)
  - *France’s capital is Paris* \( p \)

- **Ambiguous sentences** express more than one proposition:
  - *Visiting relatives can be annoying* \( p \) = Relatives who visit you can be annoying.
  - *q* = It can be annoying to visit relatives.
The second and third syntactic rule of PL allow us to form complex wff’s out of atomic or complex propositions.

2. If $p$ is a wff, then $¬p$ is a wff (negation, “non $p$”, also “$¬p$”).
3. If $p$ and $q$ are wff’s, then
   
   - $[p \land q]$, (conjunction)
   - $[p \lor q]$, (disjunction)
   - $[p \rightarrow q]$, (conditional, if $p$ then $q$)
   - $[p \leftrightarrow q]$, (biconditional, $p$ if and only if $q$) are wff’s.

These rules encode the recursion step and make sure that an infinite set of well-formed formulas is generated.
Propositional Logic

- The expressions ¬, ∧, ∨, →, ⊕ are called logical connectives or logical operators.
  They are also logical constants: Their meaning (their semantic value) is CONSTANT in all models (of a given language of symbolic $L$).
- The negation symbol ¬ is a unary operator, because it applies to only one wff to produce a wff.
- Other connectives are binary operators (apply to two wff’s to produce a new wff).
Propositional Logic

A wff or not?

p
  yes

p \land
  no

[p \land \rightarrow q]
  no

[p \land p]
  yes
Propositional Logic

A wff or not?

$\neg[p \land q] \lor r$

yes

$\rightarrow q$

no

[[[p \land q] \rightarrow \neg q] \lor \neg \neg p]]$

no

[[[p \land q] \rightarrow \neg q] \lor \neg \neg p]

yes
Propositional Logic

The last rule

4. Nothing else is a wff.

guarantees that the first three rules are the only way in which wff’s can be construed.
Propositional Logic

We can analyze wff’s in form of syntactic trees

Example: \( \neg [p \land q] \)
Propositional Logic

The parentheses can be seen as auxiliary symbols that help us reconstruct the internal syntactic structure of a complex expression. Without them, expressions might be structurally ambiguous.

For example,

\( \neg p \land q \)

could stand for either

\( \neg [p \land q] \)

or  \( [\neg p \land q] \).

• The language of PL does not allow for structural ambiguity.
Propositional Logic

• We are allowed to drop parentheses in case no ambiguity arises. We can always do this with outermost parentheses, e.g., we can write

\[ [p \land [q \rightarrow r]] \quad \text{or} \quad p \land [q \rightarrow r] \]

• There is a convention saying that ‘\(\land\)’ and ‘\(\lor\)’ bind stronger than ‘\(\rightarrow\)’ and ‘\(\iff\)’. We can write

\[ [p \land q] \rightarrow r \quad \text{or} \quad p \land q \rightarrow r. \]
Propositional Logic

• Syntactic rules of propositional logic (PL)
• Semantic rules for propositional logic (PL)
Propositional Logic

Semantic rules for propositional logic (PL)

• Atomic propositions describe states of affairs, that is, situations which may or may not be the case (true) in a particular context.

• The semantics of propositional logic is defined in terms of truth values.
  – Every atomic proposition can take one of two truth values: true or false.
  – This is an oversimplification, of course; there are systems with more than two values.
  – Let us take “1” for “true” and “0” for “false”.
  – We say that the truth value of a proposition is 0 in case it is false, and 1 in case it is true.
Propositional Logic

Semantic rules for propositional logic (PL)

- Propositional logic allow us to determine the truth values of complex propositions, given the truth values of their parts. Each complex wff receives a truth value, which is determined by

  (1) the truth values of its syntactic component propositions, and
  (2) the syntactic structure of the complex wff: its connectives and their arrangement in the formula.
  (Recall: The Principle of Compositionality)

- The semantic (truth-functional) properties of the logical connectives are usually given by truth tables that give us the truth value of complex expressions with respect to all possible assignments of truth values of their parts.
Propositional Logic

Semantic rules for propositional logic (PL)

<table>
<thead>
<tr>
<th></th>
<th>p</th>
<th>(\neg p)</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
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<tr>
<td>0</td>
<td>1</td>
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</table>

- Negation reverses the truth value of the proposition to which it is attached.

- For example, if *it is raining* is true,
  then *it is not raining* is false.
Propositional Logic

Semantic rules for propositional logic (PL)

• Logical negation is intended to correspond to sentential negation in natural language: We insert *not* into the verb phrase, which has the effect of producing a sentence opposite in truth value to the original.

  The logical connective ‘¬’ can be paraphrased as *it is not the case that* [S].

There are other ways of expressing the negation of a sentence:

- Porcupines are **un**friendly.
- John is **neither** at home **nor** at school.
- **No** one is at home.
- John is **never** at home.
- John is **not** home **yet**.
- John has **never** yet been here.

- Porcupines are **friendly**.
- John is either at home or at school.
- **Someone** is at home.
- John is sometimes at home.
- John is home already.
- John has on occasion been here.
Propositional Logic

Semantic rules for propositional logic (PL)

There are cases in which the insertion of *not* into the verb phrase does not simply or just produce a sentence opposite in truth value to the original:

(1)  

a. *You may smoke in here.*  
    Paraphrase:  
    You are not allowed to smoke in here.  
    It is not the case that [you are allowed [to smoke in here]].  
    \[\text{NEG} \ [\text{MODAL} \ [S]]\]

b. *You may not smoke in here.*

(2)  

a. *They may like the party.*  
    Paraphrase:  
    It is possible that they do not like the party.  
    It is possible that [it is not the case that [they like the party]].  
    \[\text{MODAL} \ [\text{NEG} \ [S]]\]

b. *They may not like the party.*
Propositional Logic

Semantic rules for propositional logic (PL)

What about

*You cannot have too many friends* ?
Propositional Logic

Semantic rules for propositional logic (PL)

**CONJUNCTION**

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p ∧ q</th>
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- A proposition containing two propositions connected by ‘\(\land\)’ is true iff (‘if and only if’) both propositions are true.

*Dean is happy and likes himself.*
Propositional Logic

Semantic rules for propositional logic (PL)

! ‘∧’ does not always correspond to the English sentential conjunction *and*

(1) ‘∧’ corresponds to the English *and* used with the intended speaker’s meaning of ‘and then’, ‘before’, ‘and as a result’ (generalized conversational implicature, Grice 1975¹)

(2) ‘∧’ corresponds to the English *but*, which literally means ‘∧’, and conventionally implicates contrast or unexpectedness (Grice 1975)

(3) ‘∧’ only conjoins propositions, the English *and* can be used to conjoin other categories than sentences

Propositional Logic

Semantic rules for propositional logic (PL)

(1) In PL, ‘p \land q’ always has the same truth value as ‘q \land p’. It is one of the laws of propositional logic, the commutativity law:

\[
\text{Commutativity Law: } [p \land q] \equiv [q \land p]
\]

However, sentences like (i) and (ii) do not express the same situation:

(i)  \textit{John drank a bottle of vodka and fell into a stupor} \neq

(ii) \textit{John fell into a stupor and drank a bottle of vodka}.

Observation:

• Reversing the order of conjoined sentences affects the interpretation of a whole complex sentence.
• In reversing the order of conjunct sentences and asserting (ii), we implicate a very different kind of situation than is expressed by (i).

September 29, 2008  Hana Filip
Propositional Logic

Semantic rules for propositional logic (PL)

• Question: Why should reversing the order of conjoined sentences affect the interpretation of a whole complex sentence?
• Answer: We most naturally understand an utterance of a sentence like

(i)  *John drank a bottle of vodka and fell into a stupor.*

as intended by its speaker to mean

‘John did so in that order’

i.e. we may paraphrase (i) as

*John drank a bottle of vodka and then / before he fell into a stupor.*

*John drank a bottle of vodka and as a result he fell into a stupor.*
Semantics rules for propositional logic (PL)

Analysis:
• The English *and* is often used in utterances that carry the **generalized conversational implicature** ‘and then’, ‘before’ or ‘as a result’ (a part of the intended speaker’s meaning, Grice 1975)
• Generalized conversational implicatures are default inferences (i.e., preferred or normal interpretations) that go through unless they are blocked by specific contextual assumptions (see Grice 1975, Gazdar 1979, Horn 1984, 1989, Levinson 1987).

a. *Mary got married and got pregnant.*

b. *Mary got married and got pregnant, but not in that order.*
(a) conversationally implicates that Mary first got married and **then** she got pregnant, but this implicature is cancelled by adding ‘… not in that order’ in (b).
Propositional Logic

Semantic rules for propositional logic (PL)

(2) ‘∧’ the English *but*

• In translating from English into PL the sentential connective *but* is often rendered as ‘\(\land\)’.

• _The sun is shining, but it is cold outside_  
  
  \[
  p \quad \Box \quad q
  \]

  may be translated as ‘\(p \land q\)’. The English connective *but* means ‘\(\Box\)’ and _conventionally implicates_ contrast or unexpectedness, which ‘\(\Box\)’ does not implicate (Grice 1975).
Propositional Logic

Semantic rules for propositional logic (PL)

(2) ‘∧’ the English but

a. The sun is shining, but it is cold outside.
b. The sun is shining and it is cold outside.
c. p ∧ q

(a) and (b) are synonymous in their truth-conditional meaning and have the same logical form (c). The meaning of contrast is conventionally implied in the meaning of ‘but’.
Propositional Logic

Semantic rules for propositional logic (PL)

(3) In PL, the connective ‘\( \land \)’ only conjoins propositions. And in English can be used to conjoin other categories than sentences, e.g. noun phrases, adjectival phrases or other phrases.

• In many cases, English sentences that contain NP conjunction can be treated as elliptical forms of a sentential conjunction:

(i) a. \textit{John and Mary yawned.} =

b. \textit{John yawned and Mary yawned.}
Propositional Logic

Semantic rules for propositional logic (PL)

• Puzzle: The following English sentences contain a NP conjunction that cannot be treated as an elliptical form of a sentential conjunction:

(ii)  a. John and Mary met last year. ☒
     b. *John met last year and Mary met last year.

(iii) a. Cheech and Chong are fun at parties. ☒
      b. Cheech is fun at parties and Chong is fun at parties.

(They may not be necessarily fun when they are not together.)

The puzzle is to explain why some pairs of examples like those above are semantically equivalent and some are not, although in each case the surface syntactic structure is the same.
Propositional Logic

Semantic rules for propositional logic (PL)

Answers to the puzzle involve

**Lexical meaning of verbs: distributivity**

(i)  
  a. *John and Mary yawned.*  
  b. *John yawned and Mary yawned.*  
  c. *p \land q*

- The verb *yawn* is **distributive**: the property of yawning is attributed to John and separately to Mary.
- Therefore, (1a) can be viewed as elliptical for (1b), and represented as (1c). (1a) and (1b) are synonymous in their truth-conditional meaning and have the same logical form (1c).
Propositional Logic

Semantic rules for propositional logic (PL)

Other examples of sentences with distributive predicates:
Los Angeles and San Diego lie south of San Francisco.
John and Peter are married to Anne, and Betty, respectively.
Propositional Logic

Semantic rules for propositional logic (PL)

Lexical meaning of verbs: collectivity

(ii) a. *John and Mary met last year.
     b. *John met last year and Mary met last year.

• The verb meet is collective: the property of meeting can only be attributed to John and Mary taken as a single collection, i.e., a group of individuals
Propositional Logic

Semantic rules for propositional logic (PL)

Lexical meaning of verbs and pragmatic factors determine the collective interpretation

(iii)  a.  *Cheech and Chong are fun at parties.*  

b.  *Cheech is fun at parties and Chong is fun at parties.*
   (They may not be necessarily fun when they are not together.)
Propositional Logic

Semantic rules for propositional logic (PL)

conjunction of adjectival phrases:

(iv)  
  a. Mary mixed red and blue paint. ✓
  
  b. Mary mixed red paint and Mary mixed blue paint.
Propositional Logic

Semantic rules for propositional logic (PL)

Conclusion (examples (ii) - (iv)):
• English sentences that contain NP conjunction (and other phrasal conjunctions) cannot always be treated as elliptical forms of a sentential conjunction.
Propositional Logic

Semantic rules for propositional logic (PL)

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<tr>
<th>DISJUNCTION</th>
<th>p</th>
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<th>p ∨ q</th>
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- The disjunction of two propositions is true if either one or both disjuncts are true.
- Hence, the logical connective ‘∪’ is **inclusive**: and/or.
Propositional Logic

Semantic rules for propositional logic (PL)

- The logical connective ‘\( \lor \)’ is inclusive, and so are many uses of *or* in natural languages

- **inclusive or**: either one or both disjuncts are true (\( = \) *and/or*)

Example:

\[
\text{At present, we invite all passengers who need some extra help or who are travelling with small children to board the aircraft.}
\]
Propositional Logic

Semantic rules for propositional logic (PL)

• In natural languages, we also find a use of or that corresponds to

• **exclusive or**: one and only one of the disjuncts is true
  \[= \neg p \land q, \text{either } \ldots \text{or}\]

  a. *John is twelve years old or John is fifteen years old.*
  b. *You may have soup or you may have salad, but not both.*
  c. *Mary has a son, or Mary has a daughter.*
Propositional Logic

Semantic rules for propositional logic (PL)

• Questions

1. Is the English disjunction or lexically ambiguous between the inclusive and the exclusive interpretation? I.e., do we want the exclusive interpretation to be also a part of the truth conditions given for or?

2. Is the English disjunction or not lexically ambiguous? I.e., do we want to propose that it is inclusive just like the logical connective ‘∨’ and the exclusive use of or is determined by the context of use and pragmatic principles of interpretation?
Propositional Logic

Semantic rules for propositional logic (PL)

Observation 1

• In some cases, the exclusive meaning of *or* is due to contextual factors. In (i), it is our general world knowledge: We know that John cannot be twelve years old and fifteen years old at the same time.

  a. *John is twelve years old or John is fifteen years old.*
Propositional Logic

Semantic rules for propositional logic (PL)

Observation 2

• In (b), there is a real possibility that both disjuncts can hold, but this possibility is excluded (is cancelled) by adding *but not both*:

  b. *You may have soup or you may have salad, but not both.*
Propositional Logic

Semantic rules for propositional logic (PL)

Observation 3

c.  *Mary has a son or Mary has a daughter* .

• If you knew that Mary has a son *and* a daughter, then you would certainly use *and* as the main connective, because it would result in a more informative sentence.

• By using *or* you *conversationally implicate*
  – that *Mary has a son and a daughter* is false, and
  – that you do not know for sure whether Mary only has a son or whether Mary only has a daughter (see Grice 1975)
Propositional Logic

Semantic rules for propositional logic (PL)

From observations (1) - (3) we can conclude:

• The English disjunction *or* is **not lexically ambiguous**, but rather it only has the inclusive meaning just like the logical connective ‘∨’.

• The exclusive meaning can be predicted using Grice’s (1975) pragmatic principles of interpretation that speakers will commonly use “p or q” to implicate that not both are true.

• The exclusive interpretation of “p or q” as “not p and q”, is a conversational implicature of *or*. So there is no need to postulate a second sense of *or*. (See also Horn 1972, Gazdar 1979, Levison 1983, p.134, Hirschberg 1991, p. 84ff.)
Propositional Logic

The logical disjunction ‘\lor’ vs. the English conjunction *or*:

• The logical disjunction ‘\lor’ may only combine two propositions (simple or complex): \( p \lor q \).

• The English *or* can often be used to conjoin constituents below a level of a simple sentence. For example, it can conjoin two noun phrases:

  (1)  a. *John or Mary yawned.*

  b. *John yawned or Mary yawned.*

  c. \( p \lor q \)

  • (1a) is elliptical for (1b), and therefore represented as (1c).

  • (1a) and (1b) are synonymous in their truth-conditional meaning and have the same logical form (1c).
Propositional Logic

Another example in which the English conjunction *or* differs from the logical disjunction ‘\(\lor\)’

a. *A doctor or a dentist can write prescriptions.*
b. \(p \land q\)

• The intended interpretation is that both doctors and dentists can write prescriptions.
• (2a) would be false if doctors could, but dentists could not, write prescriptions.
• Therefore, the best translation for this sentence into PL would be of the form (2b), not ‘\(p \lor q\)’.
Propositional Logic

Semantic rules for propositional logic (PL)

<table>
<thead>
<tr>
<th>CONDITIONAL</th>
<th>p</th>
<th>q</th>
<th>p → q</th>
</tr>
</thead>
<tbody>
<tr>
<td>material implication</td>
<td>1</td>
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<td>1</td>
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<td>1</td>
</tr>
</tbody>
</table>

In a conditional, we call $p$ antecedent/hypothesis and $q$ consequent/conclusion. The **conditional** is also called the **material implication**.

The truth table is motivated by the following consideration:
- If the antecedent/hypothesis is true, then the consequent must be true.
- If the antecedent/hypothesis is false, nothing follows for the consequent (it can be true or false).
Propositional Logic

- **SIMILARITIES** between the material implication ‘$\rightarrow$’ and the discontinuous connective *if ... then* in English

The logical conditional shares one crucial feature with all the uses of the discontinuous connective *if ... then* in English:

when the *if*-clause (the antecedent) is true, then the *then*-clause (consequent) must be true, as well.

*If it rains, the streets are wet*

- excludes the possibility that it rains and the streets are *not* wet.
- If it does not rain, the streets may be wet or not wet.

To put it slightly differently, *from a falsity we can infer an arbitrary statement $p$*. Or, *from falsehood everything follows* (*ex falso quod libet*).

We also find a rhetorical use of this principle, e.g.,

*If you are smart, then I am Albert Einstein.*
Propositional Logic

• from a falsity we can infer an arbitrary statement $p$. from falsehood everything follows (Latin: *ex falso quodlibet*).
  *the principle of explosion*

We also find a rhetorical use of this principle, e.g.,

*If hell freezes over, I’ll sled there too.*
*I marry you, if/when hell freezes over.*
*If buffalos have wings, I’ll give you my next paycheck.*
Propositional Logic

• **DIFFERENCES** between the material implication ‘→’ and the discontinuous connective *if ... then* in English

• When we use the discontinuous connective *if ... then*, we expect there to be some temporal and/or causal connection between antecedent and consequent.

• This does not hold for the logical connective ‘→’. The following sentences are true, according to the truth table for ‘→’:

  a. *If the earth is flat, then all the square circles are red.*

      0   0   1

  b. *If the earth is flat, then it is raining in Paris.*

      0   0 or 1   1

   All that matters for ‘→’ is the assignment of truth values to atomic propositions.
Propositional Logic

• **DIFFERENCES** between the material implication ‘→’ and the discontinuous connective *if ... then* in English

a.  *If this car is cheaper than $1,000, then I will buy it.*

   1  1 1

b.  *If this car is NOT cheaper than $1,000, then I will NOT buy it.*

   0 0 1

c.  *If this car is NOT cheaper than $1,000, then I will buy it.*

   0 1 1

• Most people will understand (a) in the sense of (b), and not as (c).
• The (c) reading is predicted by the logical conditional, but it is not a reading that we would assign the English conditional sentence (a).
Propositional Logic

**Motivation:** We find a tendency in natural language use to strengthen the conditional. This follows from the rules associated with language use: A speaker typically makes assertions that are as “strong” as he or she is willing to support (Maxim of Quantity, Grice 1975). Given this pragmatic rule, (a) is understood as (b), but not (c), i.e., in a stricter way than the logical conditional.

a.  *If this car is cheaper than $1,000, then I will buy it.*

   1 1

b.  *If this car is NOT cheaper than $1,000, then I will NOT buy it.*

   0 0 1

c.  *If this car is NOT cheaper than $1,000, then I will buy it.*

   0 1
Propositional Logic

*If-then* propositions and necessary & sufficient conditions

*If a number is divisible by 10, then it is divisible by 2.*

\[ p \implies q \]

- The clause introduced by *If* (*a number is divisible by 10*) is called the **hypothesis**. It is what we are given, or what we may assume.
- The clause introduced by *then* (*it is divisible by 2*) is called the **conclusion**. It is the statement that follows from the hypothesis. Or, given the hypothesis, it is the statement that we must prove.
- **When the If-then sentence is true,**
  - the hypothesis is a **sufficient** condition for the conclusion. Thus it is sufficient to know that a number is divisible by 10 in order to conclude that it is divisible by 2.
  - The conclusion is then called a **necessary** condition of that hypothesis. For, if a number is divisible by 10, it necessarily follows that it will be divisible by 2.
Propositional Logic

*If*-then propositions and necessary & sufficient conditions

*If* a number is divisible by 10, *then* it is divisible by 2.

\[ p \implies q \]

- **p** hypothesis
- **q** conclusion
- sufficient condition for the conclusion
- necessary condition of the hypothesis
Propositional Logic

*If-then* propositions and necessary & sufficient conditions

- *If a number ends in 0, then it is a multiple of 5.*

- Is the hypothesis a sufficient condition for that conclusion?
Propositional Logic

*If-then* propositions and necessary & sufficient conditions

- *If a number ends in 0, then it is a multiple of 5.*

- Is the hypothesis a sufficient condition for that conclusion?

- Yes, because the statement is true.
Propositional Logic

*If-then* propositions and necessary & sufficient conditions

- *If a number is a multiple of 5, then it ends in 0.*

- Is the hypothesis a sufficient condition for that conclusion?
Propositional Logic

If-then propositions and necessary & sufficient conditions

• *If a number is a multiple of 5, then it ends in 0.*

• Is the hypothesis a sufficient condition for that conclusion?

• No, because the statement is false. 15 is a multiple of 5, but 15 does not end in 0.
Propositional Logic

*If-then* propositions and necessary & sufficient conditions

- *If a number is prime, then it is odd.*

- Is the conclusion a necessary condition of that hypothesis?

Note:

In mathematics, a prime number (or a prime) is a natural number which has exactly two distinct natural number divisors: 1 and itself.
Propositional Logic

*If*-then propositions and necessary & sufficient conditions

- *If a number is prime, then it is odd.*

- Is the conclusion a necessary condition of that hypothesis?

- No, because the statement is false. 2 is a prime, and it is not odd.
Propositional Logic

*If-then* propositions and necessary & sufficient conditions

- *If two numbers are even, then their product is even.*

- Is the conclusion a necessary condition of that hypothesis?
Propositional Logic

*If-then* propositions and necessary & sufficient conditions

- *If two numbers are even, then their product is even.*

- Is the conclusion a necessary condition of that hypothesis?

- Yes, because the statement is true.
Propositional Logic

Examples:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>getting an A</td>
<td>passing the class</td>
<td>p is sufficient for q</td>
</tr>
<tr>
<td>decapitation</td>
<td>death</td>
<td>p is sufficient for q</td>
</tr>
</tbody>
</table>

In English, the following expressions indicate sufficient conditions:

\( \text{if } p \text{ then } q, p \text{ implies } q, p \text{ is enough for } q \)

**Summary:**

- **IF** \( p \) implies \( q \), then a statement **\( p \text{ is a sufficient condition} \)** of a statement \( q \).
  - \( p \) guarantees \( q \).
  - Whenever you have \( p \), you have \( q \).
  - Anything \( p \) is \( q \).
Propositional Logic

• To show that some statement p (about some property) is NOT a SUFFICIENT CONDITION
  You find cases where p is present but q is not.

Examples:

• Loving someone is not sufficient for being loved. A person who loves someone might not be loved by anyone perhaps because she is a very nasty person.
• Loyalty is not sufficient for honesty because one might have to lie in order to protect the person one is loyal to.
Propositional Logic

- Some state of affairs can have more than one SUFFICIENT CONDITION:

Example: Take the property of being colored.

Being blue is sufficient for being colored, but of course being green, being red etc. are also sufficient for being colored.
If p implies q, then a statement q is a **NECESSARY CONDITION** of a statement p. A necessary condition is sometimes also called ‘an **essential condition**.’

Examples:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>be alive</td>
<td>breathe</td>
<td>q is necessary for p</td>
</tr>
<tr>
<td>car runs</td>
<td>gas in car</td>
<td>q is necessary for p</td>
</tr>
<tr>
<td>being pregnant</td>
<td>being female</td>
<td>q is necessary for p</td>
</tr>
</tbody>
</table>

q is necessary for p if and only if p cannot occur without q, it is impossible to have p without q, the absence of q guarantees the absence of p. Whenever you have p, you have q. Anything p is q.
Propositional Logic

- To show that q is not a NECESSARY CONDITION for p, you find cases where q is present but p is not.

Examples:
- Getting an A in this class is not necessary for passing this class.
- Being rich is not necessary for being happy, since a poor person can be happy too.
Propositional Logic

• *ONLY IF* introduces a necessary condition
  i.e., a condition that must hold in order for something else to hold.

Example:
  Buying a lottery ticket (q) is a necessary condition for winning a lottery (p),
  but it does not guarantee that you will actually win the lottery.

  a. You win only if you buy a lottery ticket.

  b. p only if q

  c. If p then q

  d. p \( \not\rightarrow \) q

  (b) and (c) have the same truth conditions in PL.
  Both are represented as (d).
Propositional Logic

You will win the lottery only if you buy a lottery ticket.

\[ p \quad q \]

- If you buy a lottery ticket, you may or may not win.
- If you do not buy a lottery ticket, you will not win.
- This excludes just one possibility: namely, that you win the lottery, but you did not buy a lottery ticket. This is exactly the case in which if \( p \) then \( q \) is false.
Propositional Logic

Semantic rules for propositional logic (PL)

<table>
<thead>
<tr>
<th>BICONDITIONAL</th>
<th>p</th>
<th>q</th>
<th>p ↔ q</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</table>

• Both $p$ and $q$ are either true at the same time, or false.
  Example: Jane will go to the party if and only if Marc goes.

• $p \iff q$ is logically equivalent to $p \to q \land q \to p$.
  Jane will go to the party if Marc goes and Marc will go to the party if Jane goes.

• Among English expressions translated by the biconditional are
  *if and only if (= iff)*
  *just in case that*, and
  *is a necessary and a sufficient condition for.*
Propositional Logic

IFF or biconditional introduces necessary & sufficient conditions

Examples:
• *Today is the Fourth of July* is a necessary and a sufficient condition for *Today is Independence*
• Having four equal sides is both necessary and sufficient for being a square.
Propositional Logic

• IFF or biconditional introduces necessary & sufficient conditions used in definitions:

a. X is called a Y (or is a Y) iff (= if and only if) X has property P.
b. A geometric figure is a triangle iff it has three sides.

• Using if instead of iff would leave open the possibility that X might also be called a Y (the term being defined) in other circumstances, as well.

Example: X is a student if she attends Princeton University. if she attends Stanford University. if she attends Santa Fe College. etc.
Propositional Logic

• **necessary & sufficient conditions**

**history**

• **Plato (428/427 BC– 348/347 BC)**
  – everything in the world fits into some class, and the classes can be precisely defined
  – it is possible to specify the necessary and sufficient conditions for membership in the class of things denoted by any concept
Propositional Logic

The following classification is very useful when we need clarify how two concepts are related to each other.

Given two conditions X and Y, there are **FOUR WAYS** in which they might be related to each other:

- **X is necessary but not sufficient for Y.**
  Example: Having four sides is necessary but not sufficient for being a square (since a rectangle has four sides but it is not a square).

- **X is sufficient but not necessary for Y.**
  Example: Having a son is sufficient but not necessary for being a parent (a parent can have only one daughter).

- **X is both necessary and sufficient for Y.** (or “jointly necessary and sufficient”)
  Example: Having four equal sides is both necessary and sufficient for being a bachelor.

- **X is neither necessary nor sufficient for Y.**
  Example: Being a tall person is neither necessary nor sufficient for being a successful person.
Propositional Logic

• necessary & sufficient conditions

History (Recall Lecture 1):
• problems related to definitions of concepts (and meaning of words in natural language) in terms of necessary&sufficient conditions
• definitions based on sufficient and necessary conditions rejected by Ludwig Wittgenstein in *Philosophical Investigations* (1953), and instead ‘family resemblance’ proposed as being more suitable as an analogy for capturing the means by which we connect particular uses of the same word
• alternative proposals within mentalistic, cognitive theories of meaning (e.g., the prototype theory of Rosch, starting in the early 1970’s).
Propositional Logic

• How truth values of complex expressions can be computed

As soon as we know the truth values of atomic propositions, we can compute truth values of complex expressions

Example:

a. \[ p \lor q \rightarrow [r \land p] \]

b. Let us assume the following truth values of atomic propositions:
   \[ p = 0, \quad q = 1, \quad r = 1. \]

c. \[
\begin{array}{c|c|c|c|c}
0 & 1 & 1 & 0 \\
\hline
1 & 1 & 0 \\
\hline
\hline
1 & 0 & 0 \\
\hline
0 & 0 & 0
\end{array}
\]
Propositional Logic

- How truth values of complex expressions can be computed
  we can also compute the truth value of the complex expression with respect to
  every possible truth value for the atomic propositions. We then get
  something like a truth table for the complex expression:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>[p ∨ q]</th>
<th>[r ∧ p]</th>
<th>[p ∨ q] → [r ∧ p]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</table>
Propositional Logic

Propositions can be classified according to their truth tables:

- Tautologies
- Contradictions
- Contingencies
Propositional Logic

• **Tautologies**

  A proposition is called a tautology, if the final column in its truth table contains nothing but 1’s, i.e. the proposition is always true, whatever the initial assignment of truth values to its atomic propositions.

  Such propositions are true simply because of the meaning of the connectives.

  __Example of a tautology:__

  \[
  p \lor \neg p
  \]

  Whatever sentence we use for \( p \) and whatever truth-value we give this sentence, the truth-value of the complex proposition will be \( t \).
Propositional Logic

Law of the excluded middle:  \( p \lor \neg p \)

Example: It’s raining or it’s not raining.

- However we change the world (whatever possible world we choose), it will be true that either it is raining or it is not raining.

- The truth-set of \([p \lor \neg p]\) is the union of the set of possible worlds where \( p \) is true and the set of possible worlds where \( \neg p \) is true, which is the same as the set of all the possible worlds (= the universal set). In general, it holds that

  the truth-set of a tautology is the universal set.

- The meaning of any given proposition can be defined as the set of all the possible worlds in which the proposition is true. (Recall David Lewis’s notion of the ‘possible world’.)
Propositional Logic

• **Tautologies**  \( T \)

\[
p \rightarrow p
\]
\[
\neg[p \land \neg p]
\]
\[
p \rightarrow [q \rightarrow p]
\]
Propositional Logic

• Contradictions

A proposition is called a contradiction if the final column in its truth table contains nothing but 0’s, i.e. it is always false, whatever the initial assignment of truth values to its atomic propositions.

\[
\begin{align*}
& p \land \neg p & \text{It is raining and it is not raining.} \\
& p \land \neg p & \\
& \neg[(p \lor q) \lor (q \lor p)]
\end{align*}
\]

No matter how we change the world, \( p \land \neg p \) will always be false. There is no possible world in which a contradiction can be true. Therefore, the truth-set of a contradiction is the empty set (of possible worlds).
Propositional Logic

• Tautologies $T$ and Contradictions

  Complement Law: $\neg \neg p \equiv T$

  • $p$ is a tautology iff $\neg p$ is a contradiction.
  • $p$ is a contradiction iff $\neg p$ is a tautology.

Examples:

  All property is theft $\square$ All circles are round $T$
  $\neg (\text{All property is theft})$ $T$ $\neg (\text{All circles are round}) \square$

  All circles are square $\square$
  $\neg (\text{All circles are square})$ $T$

All property is theft (attributed to Pierre-Joseph Proudhon, 1809-1865, French political phil.)
Propositional Logic

• **Tautologies** and **Contradictions**

  Any proposition (basic or complex) whatever may be substituted for atomic propositions in a tautology or in a contradiction without affecting the truth value of the original expression.

  Example:
  
  \([p \lor \neg p]\) is a tautology.
  
  Replacing \(p\) by \([q \rightarrow r]\), we get
  
  \([ [q \rightarrow r] \lor \neg [q \rightarrow r] ]\), which is also a tautology.

• We can *generate* infinitely many tautologies and contradictions from formulas that are tautologies and contradictions.

• Vice versa, we can check whether a given formula is a tautology or contradiction by *reducing* it to a known tautology or contradiction.
Propositional Logic

• Verifying Tautologies T and Contradictions

How can we prove whether a proposition of PL is a tautology or a contradiction? By applying one of the two main types of proofs in mathematics:

1. DIRECT PROOF: we compute all the possible assignments of truth values to the atomic propositions, and check whether every such assignment yields 0 (contradiction) or whether every such assignment yields 1 (tautology). Aka
   PROOF BY CASES
   PROOF BY EXHAUSTION
   PERFECT INDUCTION or the BRUTE FORCE METHOD.

2. INDIRECT PROOF: QUICK FALSIFICATION test which makes use of the general reasoning strategy of
   INDIRECT PROOF BY CONTRADICTION, also closely related to the INDIRECT PROOF BY REDUCTIO AD ABSURDUM.
Propositional Logic

• **Example: Verifying Tautologies T**
  
  • DIRECT PROOF, PROOF BY CASES
  
  • Consider the formula \(((A \land B) \land C) \lor (A \land (B \lor C))\)
  
  • There are 8 possible valuations for the atomic propositions A, B, C, represented by the first 3 columns of the table. The remaining columns show the truth of subformulas of the main formula, the final column shows the truth value of the whole formula under each valuation.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>A \land B</th>
<th>(A \land B) \rightarrow C</th>
<th>B \rightarrow C</th>
<th>A \rightarrow (B \rightarrow C)</th>
<th>((A \land B) \rightarrow C) \iff (A \rightarrow (B \rightarrow C))</th>
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<tbody>
<tr>
<td>T</td>
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Propositional Logic

• Verifying Tautologies T and Contradictions

We can convince ourselves that this method works for an arbitrary formula of PL. That is, it will give us a result in a finite number of steps, because every PL formula is of finite length. Hence, it has only a finite number of different atomic propositions $n$ and we have to consider a finite number of truth value assignments to atomic propositions, $2^n$ (2 truth values, $n$ number of atomic propositions). We can check in a finite number of steps for each truth value assignment whether the formula is true.
Propositional Logic

- **Verifying Tautologies T and Contradictions**

For example, for three atomic propositions we have $2 \times 2 \times 2 = 8$ possible assignments.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
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<tbody>
<tr>
<td>1</td>
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</table>
Propositional Logic

• Verifying Tautologies $T$ and Contradictions 🎯

2. **INDIRECT PROOF: QUICK FALSIFICATION** test which makes use of the general reasoning strategy of
   **INDIRECT PROOF BY CONTRADICTION**, also closely related to the **INDIRECT PROOF BY REDUCTIO AD ABSURDUM**.

**PROOF BY CONTRADICTION** uses the

Law of Indirect Reasoning:
If valid reasoning from a statement $p$ leads to a false conclusion, then $p$ is false.

**PROOF BY REDUCTIO AD ABSURDUM** is a process of refutation on grounds that absurd - and patently untenable consequences would follow from accepting the item at issue.

*If Fido becomes the next President, then pigs can fly.*
Propositional Logic

• Law of Indirect Reasoning:
  If valid reasoning from a statement $p$ leads to a false conclusion, then $p$ is false.

IMPLEMENTATION may be done by means of truth tables:
• The test systematically searches for a line on a truth table whose final value is 0.
• If the search is completed and no such line is found, then we know for sure that the proposition is a tautology.
Propositional Logic

(p \rightarrow (q \rightarrow p)) \text{ tautology?}

1. assumption to the contrary: contradiction

2. follows from truth table for \rightarrow

3. uniform assignment of truth value to atomic propositions

4. no assignment for q that makes q\rightarrow p false

If p is true, then q\rightarrow p is true, according to the truth table for \rightarrow.

Therefore, (p \rightarrow (q \rightarrow p)) is a tautology.
Propositional Logic

Is \((p \rightarrow (q \rightarrow p))\) a tautology?

… previous slide, in words:

• We enter the truth value directly under the principle or “highest” connective, i.e., the last one added in the syntactic construction of the formula.

2. Then reasoning from that assumption: the antecedent \(p\) of this conditional must be true and the consequent \(q\) must be false, since this is the only 0-case for conditionals.

3. We fill out the 1-assignment for the atomic proposition \(p\) in the consequent of the whole formula (uniform assignment of truth values to the atomic proposition, the same throughout the entire formula).

4. Conflicting assignment: the consequent should be false, this cannot be the case, since \(p\) is true and the consequent can only be false if \(p\) is false. Hence, we conclude that the assumption that there is a line on the truth table for this proposition which ends in false is itself false. Thus all lines must be 1.
Is a given formula a tautology?

General strategy:

1. We assume to the contrary that it is not: we assume that it is a contradiction. This means that we assume that there is at least one line whose value is 0 for the whole formula.

2. For there to be a line whose final value is 0 there must be at least one assignment of truth values to the atomic formulas that yields the truth value 0 for the whole formula. We try to construct such an assignment by reasoning “backwards” to see whether we can find such an assignment of truth values to the atomic propositions without running into contradictory or conflicting assignments.

3. If we fail, we have to retract our assumption that the formula is a contradiction, and have shown that our formula is a tautology.
Propositional Logic

• Contingency
  All propositions with both 1 and 0 under the main connective in their truth tables are called contingent propositions or contingencies.
Propositional Logic

- Logical equivalence
- Logical consequence
Propositional Logic

• Logical equivalence

• $p \equiv q$ iff $p \equiv q$ is a tautology

  In words: The two propositions $p$ and $q$ are logically (semantically) equivalent, i.e., either they are both true or both false at the same time, written as $p \equiv q$, if and only if $p \equiv q$ (a biconditional proposition) is a tautology.

<table>
<thead>
<tr>
<th>CONDITIONAL</th>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>material implication</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

• $p \equiv q$ means that $p$ implies $q$ and $q$ implies $p$: $p \rightarrow q \land q \rightarrow p$.
• Also: $p \equiv q$
Propositional Logic

• **Logical equivalence**

Saying that two propositions are equivalent, either true or false at the same time, does not mean that they have the same meaning (sense).

Example:  

\[ p = \text{The triangle ABC has two equal sides.} \]
\[ q = \text{The triangle ABC has two equal angles.} \]

\[ p \rightarrow q \]
If the triangle ABC has two equal sides, then the triangle ABC has two equal angles.

\[ q \rightarrow p \]
If the triangle ABC has two equal angles, then the triangle ABC has two equal sides.
Propositional Logic

• Logical equivalence
  
  The relation of logical equivalence allows for a replacement of any subformula by a logically equivalent expression, whereby the truth value of the original formula is preserved, i.e. both its truth and falsity is preserved.

Example:

In the proposition $p \lor q$
replacement of $p$ by the logically equivalent complex proposition $(p \land p)$
yields a statement $((p \land p) \lor q)$

whose truth value is exactly the same as the original proposition: $(p \lor q) \equiv (p \land p) \lor q)$
Propositional Logic

• **Logical consequence**

If a conditional proposition is a tautology, we say that
• the consequent is a *logical consequence* of the antecedent, or
• the antecedent *logically implies (entails)* the consequent.

We write this as:

\[ p \implies q \]
\[ p \implies q \iff p \implies q \text{ is a tautology} \]

Example: \( p \land q \implies p \)

*Archibald wears a bowtie and a silly grin* logically implies/entails that
*Archibald wears a bowtie.*
Propositional Logic

• Laws of Propositional Logic

Logical equivalence allows us to define the relations between the connectives of propositional logic: such relations are expressed by the laws of propositional logic.
Propositional Logic

1. Idempotency

(a) \[ p \land p \] \[\equiv\] p
(b) \[ p \lor p \] \[\equiv\] p

*Pigs can fly and pigs can fly* is logically equivalent to *pigs can fly.*

*Pigs can fly or pigs can fly* is logically equivalent to *pigs can fly.*
Propositional Logic

2. Associativity

(a) \([[[p \land q] \land r] \implies [p \land [q \land r]]]\)

(b) \([[[p \lor q] \lor r] \implies [p \lor [q \lor r]]]\)
Propositional Logic

3. Commutativity

a. \([p \land q] \Leftrightarrow [q \land p]\)

b. \([p \lor q] \Leftrightarrow [q \lor p]\)

It’s raining and the streets are wet. ☂
The streets are wet and it’s raining.

Tom is short or Tom is vertically challenged ☂
Tom is vertically challenged or Tom is short.
Propositional Logic

4. Distributivity

a. \([p \land [q \lor r]] \implies [[p \land q] \lor [p \land r]]\)

b. \([p \lor [q \land r]] \implies [[p \lor q] \land [p \lor r]]\)

Example for (a):
I’ll have salad and then (I’ll order a pizza or a I’ll order a cheese cake).

(I’ll have salad and then I’ll order a pizza) or
(I’ll have salad and then I’ll order a cheese cake).
Propositional Logic

5. Identity

a. \([p \land \text{F}] \not\implies \text{F}\)

b. \([p \land T] \not\implies p\)

c. \([p \lor \text{F}] \not\implies p\)

d. \([p \lor T] \not\implies T\)

• Take \(T\) to be a formula that is always true (a tautology), and \(\text{F}\) to be a formula that is always false (a contradiction).
Propositional Logic

6. Complement

a. \[ p \lor \neg p \] T (excluded middle)
b. \neg \neg p \iff p (double negation)
c. \[ p \land \neg p \] \# (contradiction)
d. \neg \neg \neg p \iff T (see above)

Examples:

a. Either Bart knows the answer or he does not know the answer.
b. We Cannot Not Change the World!
c. Bart knows the answer and Bart does not know the answer.
Propositional Logic

7. DeMorgan’s Laws

a. \( \neg [p \land q] \equiv [\neg p \lor \neg q] \)
b. \( \neg [p \lor q] \equiv [\neg p \land \neg q] \)

Examples:

a. It’s not true that you can eat lots of food and stay thin. Either you do not eat lots of food or you do not stay thin.

b. It’s not true that milk is in the fridge or on the table. Milk is not in the fridge and not on the table.
Propositional Logic

7. De Morgan’s Laws

• Probably the most important logical equivalence
• To negate \( p \lor q \) (or \( p \land q \)), you “flip” the sign, and negate BOTH \( p \) and \( q \)
  – Thus, \( \neg(p \lor q) \equiv \neg p \land \neg q \)
  – Thus, \( \neg(p \land q) \equiv \neg p \lor \neg q \)

\[
\begin{array}{cccccccc}
p & q & \lor & \land & \lor(p \land q) & \lor(p \lor q) & p \land q & \lor(p \land q) \\
\hline
T \ T & F & F & T & F & F & T & F \\
T \ F & F & T & F & T & T & T & F \\
F \ T & T & F & T & T & T & T & F \\
F \ F & T & T & F & T & F & T & T \\
\end{array}
\]
Propositional Logic

8. Conditional Laws

a. \[p \rightarrow q \] [\neg p \lor q] 

b. \[p \rightarrow q \] [\neg q \rightarrow \neg p] \quad \text{(contraposition)}

c. \[p \rightarrow q \] [\neg p \land \neg q] 

d. \[p \rightarrow q \] [p \lor q] \uparrow q 

e. \[p \rightarrow q \] [p \land q] \uparrow p 

Examples:

a. If John loves Jen, then they are together in NY City. [✓]
   John does not love Jen or they are together in NYC.

c. If John loves Jen, then they are together in NY City. [x]
   It cannot be the case that John loves Jen and they are not together in NY City.
Propositional Logic

9. Biconditional Laws

a. \[ p \iff q \iff [p \rightarrow q] \land [q \rightarrow p] \]

b. \[ p \iff q \iff [\neg p \land \neg q] \lor [p \land q] \]
Propositional Logic

- **Laws of Propositional Logic**

- We may verify that these logical equivalences hold by the truth table method. Logically equivalent propositions have the same truth value for any assignment of truth values to the atomic propositions.

- The laws of propositional logic (i.e., relations between the connectives of propositional logic, their interdefinability) are useful when we try to simplify complex formulas by rewriting them in a more transparent way and reducing them to simpler formulas.
  
  I.e., we show that a complex formula is logically equivalent to a simpler formula and we do so by means of the laws of propositional logic.

- Propositional logic can be defined in terms of negation plus one other connective (see also de Swart, p.85).
Propositional Logic

- Laws of Propositional Logic

Show that $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$.

\[
\begin{align*}
\neg(p \lor (\neg p \land q)) & \equiv \neg p \lor \neg (\neg p \land q) \\
& \equiv \neg p \lor (\neg \neg p) \lor \neg q \\
& \equiv \neg p \lor (p \lor \neg q) \\
& \equiv (\neg p \land p) \lor (\neg p \land \neg q) \\
& \equiv \neg p \land \neg q \\
\end{align*}
\]

(De Morgan’s law)

(De Morgan’s law)

(Double negation law)

(Distributive law)

(Contradiction law)

(Commutative law)

(Identity law)
Propositional Logic

- **Laws of Propositional Logic**

Show that \( \neg(p \lor \neg q) \lor (\neg p \land \neg q) \equiv \neg p \).

\[
\begin{align*}
\neg(p \lor \neg q) & \lor (\neg p \land \neg q) \\
\equiv (\neg p \land q) & \lor (\neg p \land \neg q) \quad \text{(De Morgan’s law)} \\
\equiv \neg p & \land (q \lor \neg q) \quad \text{(Distributive law)} \\
\equiv \neg p & \land T \quad \text{(Excluded middle law)} \\
\equiv \neg p & \quad \text{(Identity law)}
\end{align*}
\]
Propositional Logic

• Propositional Logic and Set Theory

• Propositional logic and set theory show the same underlying mathematical structure, which is called a Boolean algebra (logic).
• George Boole (1854) *An Investigation of the Laws of Thought*.
• Boolean algebra is the algebra of two-valued logic (whose members are most commonly denoted 0 and 1, or false and true) with only sentential connectives, or equivalently the algebra of sets under union and complementation.
Propositional Logic

Set Theory

SETS

∪ (union)

∩ (intersection)

⊆ (subset)

= 

‘ ~ C (complement)

U

∅

Propositional Logic

PROPOSITIONS (WFF’S)

∨

∧

→

fl

¬

T

Ø

U: is a set of all sets (the universal set, the truth-set of a tautology)

∪: is the union of the sets in S.
Propositional Logic

- Laws of Propositional Logic and Set Theory: Venn Diagrams

∪ (union)
- raining ☐ the sun is shining.

∩ (intersection)
- raining ☐ the sun is shining.
Propositional Logic

- material implication ‘→’ and subset ‘⊆’

*If a car runs, then it has gas.*

A car runs.  ⊆ A car has gas.

run ⊆ have gas

The set of situations in which a car runs is included within the set of situations in which that car has gas. Having gas is necessary for a car to run, but it does not guarantee that (is not sufficient for) a car (to) run, because a car may have gas and not run.
Propositional Logic

<table>
<thead>
<tr>
<th>set notation</th>
<th>pronunciation</th>
<th>meaning</th>
<th>Venn diagram</th>
<th>answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \cup B$</td>
<td>&quot;A union B&quot;</td>
<td>everything that is in either of the sets</td>
<td><img src="image1.png" alt="Venn diagram 1" /></td>
<td>{1, 2, 3}</td>
</tr>
<tr>
<td>$A \cap B$</td>
<td>&quot;A intersect B&quot;</td>
<td>only the things that are in both of the sets</td>
<td><img src="image2.png" alt="Venn diagram 2" /></td>
<td>{2}</td>
</tr>
<tr>
<td>$\sim A$</td>
<td>&quot;A complement&quot;, or &quot;not A&quot;</td>
<td>everything in the universe outside of $A$</td>
<td><img src="image3.png" alt="Venn diagram 3" /></td>
<td>{3, 4}</td>
</tr>
</tbody>
</table>
## Propositional Logic

<table>
<thead>
<tr>
<th>set notation</th>
<th>pronunciation</th>
<th>meaning</th>
<th>Venn diagram</th>
<th>answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A - B$</td>
<td>&quot;A minus B&quot;, or &quot;A complement B&quot;</td>
<td>everything in $A$ except for anything in its overlap with $B$</td>
<td><img src="image" alt="Venn diagram" /></td>
<td>{1}</td>
</tr>
<tr>
<td>$\sim(A \cup B)$</td>
<td>&quot;not (A union B)&quot;</td>
<td>everything outside $A$ and $B$</td>
<td><img src="image" alt="Venn diagram" /></td>
<td>{4}</td>
</tr>
<tr>
<td>$\sim(A \wedge B)$ or $\sim(A \cap B)$</td>
<td>&quot;not (A intersect B)&quot;</td>
<td>everything outside of the overlap of $A$ and $B$</td>
<td><img src="image" alt="Venn diagram" /></td>
<td>{1, 3, 4}</td>
</tr>
</tbody>
</table>
Propositional Logic

• **Boolean algebra and programming**

• In 1938, Claude Shannon proved that a two-valued Boolean algebra (whose members are most commonly denoted 0 and 1, or false and true) can describe the operation of two-valued electrical switching circuits.

• Boolean algebra and Boolean functions are indispensable in the design of computer chips, integrated circuits.

• Boolean logic has many applications in electronics, computer hardware and software, and is the base of digital electronics.
Propositional Logic

• Boolean algebra and programming

• In 1938, Claude Shannon (1916-2001) proved that a two-valued Boolean algebra (whose members are most commonly denoted 0 and 1, or false and true) can describe the operation of two-valued electrical switching circuits, the basic building blocks of digital electronic circuits.
Propositional Logic

- **Boolean algebra and programming**

- **Basic combinational circuits**

  In electronic switching circuits, so-called gates are used to perform logical functions equivalent to networks of switch contacts. In this sense, an electronic gate is an elementary combinational circuit.

  Gates do not function by physically inserting or removing metallic conduction paths between contacts of manually operated switches or remotely controlled relays. Instead, they function by control of voltage or current levels at their output.

  The most commonly encountered gates are the AND and the OR gates. The AND gate produces an output only if all its inputs are concurrently present.
Propositional Logic

• Bitwise operation: A bit is a binary digit, taking a value of either 0 or 1.
• In computer programming, a bitwise operation operates on binary numerals (numbers based on 0 and 1) at the level of their individual bits. An ordered collection of bits is a byte, with each bit denoting a single binary value of 1 or 0. The size of a byte can vary and is generally determined by the underlying computer operating system or hardware, although the 8-bit byte is the standard in modern systems.
• A byte is the basic unit of measurement of information storage in computer science.
Propositional Logic

• Bitwise AND
  The bitwise AND operator simply takes the logical AND of the bits in each position of a number in binary form.
  
  \[
  \begin{array}{c}
  01001000 \\
  \text{ampersand} \\
  10111000 = \\
  00001000
  \end{array}
  \]

• Bitwise OR
  
  \[
  \begin{array}{c}
  01001000 \\
  \text{pipe line line} \\
  10111000 = \\
  11111000
  \end{array}
  \]