MAC2234: Quiz 5

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Score: E E E

P1. Derive the general solution to the logistic equation,
\[ \frac{dy}{dt} = ky \left( 1 - \frac{y}{N} \right). \]

Write the solution in terms of \( y_0 \), the initial value of \( y \) at \( t = 0 \).

\[ \frac{dy}{y(N-y)} = k \frac{dt}{dt} \]

\[ \int \left( \frac{1}{y} + \frac{1}{N-y} \right) dy = \int k \ dt \]

\[ \ln y - \ln(N-y) = kt + C \]

\[ e^{\ln y} - e^{\ln(N-y)} = e^{kt} + C \]

We need a lot of room to fully explain every step of P1, so for this week, the solutions will be 2 pages.

Now, for convenience, we will write \( A \) for \( e^C \).

\[ y = A e^{kt} \]

\[ \frac{y}{N-y} = A e^{kt} \]

\[ y = A e^{kt}(N-y) = A e^{kt}N - A e^{kt}y \]

\[ y + A e^{kt}y = A N e^{kt} \]

\[ y(1 + A e^{kt}) = A N e^{kt} \]

\[ y = \frac{A N e^{kt}}{1 + A e^{kt}} \]

Now, for convenience, we write \( B \) for \( \frac{1}{A} \).

\[ \begin{aligned}
2y + e^{2t} - y e^{2t} &= 0 \\
y \left( \frac{1}{3} \right) &= \sqrt{3}
\end{aligned} \]

Describe the long term behavior of the solution.

To get the solution in terms of \( y_0 \), we must solve \( y(0) = y_0 \) for \( B \).

\[ \frac{y_0}{y_0(1+B)} = N \]

\[ B = \frac{N - y_0}{y_0} \]

So, the final answer is

\[ y(t) = \frac{N}{1 + \left( \frac{N-y_0}{y_0} \right) e^{-kt}} \]
Name: Rev. Dr. D. Answers

**P1.** Derive the general solution to the logistic equation,
\[
\frac{dy}{dt} = ky \left(1 - \frac{y}{N}\right).
\]

Write the solution in terms of \( y_0 \), the initial value of \( y \) at \( t = 0 \).

\[
y(t) = \frac{C}{1 + C e^{-kt}}
\]

\[
u(t) = \int \frac{1}{1 + C e^{-kt}} dt = 2 \ln t = \ln t^2
\]

\[
u(t) = e^{\mu(t)} = e^{\ln t^2} = t^2
\]

\[
t^2 y = \int t^2 e^{\mu(t)} dt = \int t e^{2 \ln t} dt
\]

\[
v = t, \quad dv = e^{2 \ln t} dt = e^{2 \ln t} dt = e^{2 \ln t}
\]

\[
u = \frac{1}{2} e^{2 \ln t}
\]

\[
t^2 y = \frac{e^{2 \ln t} - \frac{1}{2} \int e^{2 \ln t} dt}{2}
\]

\[
= \frac{e^{2 \ln t}}{2} \left(t - \frac{1}{2}\right) + c
\]

\[
y = \frac{e^{2 \ln t}}{2} \left(1 - \frac{1}{2t}\right) + \frac{c}{2t}
\]

(for complete solution, we solve for \( c \))

\[
y(t) = \frac{e^{2 \ln t}}{2} \left(1 - \frac{1}{2t}\right) + \frac{c}{(1/4)}
\]

\[
y(\frac{1}{2}) = \frac{e^{\ln (\frac{1}{2})}}{2} \left(1 - \frac{1}{2(\frac{1}{2})}\right) + \frac{c}{(1/4)}
\]

\[
y(\frac{1}{2}) = \frac{4}{4} \Rightarrow c = \frac{\sqrt{3}}{4}
\]

So our final answer is:

\[
y(t) = \frac{e^{2 \ln t}}{2} \left(1 - \frac{1}{2t}\right) + \frac{\sqrt{3}}{4t^2}
\]

**P2.** Solve the initial value problem.

\[
\begin{align*}
2y + t \frac{dy}{dt} - e^{2t} &= 0 \\
y \left(\frac{1}{2}\right) &= \sqrt{3}
\end{align*}
\]

Describe the long term behavior of the solution.

\[
\lim_{t \to \infty} y(t) = \left(\lim_{t \to \infty} \frac{e^{2t}}{2t}ight)(1) + 0 \quad \text{(by linearity)}
\]

\[
= \lim_{t \to \infty} \frac{e^{2t}}{2t} \quad \text{(by L'Hopital's)}
\]

\[
= \infty
\]

So the solution increases without bound.