We generalize the logistic equation to a variable population size: \( \frac{dy}{dt} = ky \left( 1 - \frac{y}{N(t)} \right) \)

To solve this equation we make the substitution \( u = \frac{y}{N(t)} \). This gives us \( \frac{du}{dt} = -\frac{1}{y^2} \), and thus

\[
\frac{du}{dt} = \frac{dy}{dt} \frac{dy}{dt} = -u^2 \cdot k \cdot y \left( 1 - \frac{y}{N(t)} \right) = -u^2 \cdot k \cdot \left( 1 - \frac{1}{vN(t)} \right)
\]

\[
\Rightarrow \frac{du}{dt} = -ku + \frac{k}{N(t)}
\]

This is a first order linear differential equation in \( u \) that we can solve using the method of the integrating factors.

\[
\frac{du}{dt} + ku = \frac{k}{N(t)}
\]

\( y(t) = \int k \, dt = ke^t \Rightarrow I(t) = e^{kt} \)

\[
e^{kt}u(t) = \int \frac{ke^{kt}}{N(t)} \, dt
\]

\[
u(t) = e^{-kt} \int \frac{ke^{kt}}{N(t)} \, dt \quad \text{so} \quad y(t) = \frac{e^{kt}}{\int \frac{ke^{kt} \, dt}{N(t)}}
\]

Ex. Logistically growing subset of an exponentially growing population:

In this case, \( N(t) = No \cdot e^{rt} \). So \( y(t) = \frac{e^{kt}}{\int e^{kt} \, dt} = \frac{e^{kt}}{\frac{e^{kt}}{e^{kt}}} = \frac{e^{kt}}{(k-r)e^{kt} + C} = \frac{k}{ke^{-rt} + C e^{-kt}} \)

Exercise: Derive an equation modeling a logistically growing subset of a logistically growing population.