P1 (2pts.) Evaluate the following expression. Express the resulting fraction in lowest terms.

$$\frac{1}{2} \left( -\frac{1}{2} + \frac{4}{3} \right) + \frac{1}{4}$$

P2 (4pts.) Evaluate the following expression.

$$\left( \sqrt{2} - \sqrt{3} \right)^3 \left( \sqrt{2} + \sqrt{3} \right)^3$$

P3 (4pts.) Fully simplify the radical expression. Assume only that $a$ and $b$ are real numbers.

$$\sqrt[3]{\frac{36a^2b}{3b^5}}$$

P4 If you were a monster truck driver, what would you name your monster truck?
P1 (2pts.) Evaluate the following expression. Express the result fraction in lowest terms.

This is just a basic fraction arithmetic problem, which I put on here to be sure everybody really does know how to do this. If you don’t, please get caught up ASAP, as fractions are prerequisite knowledge for this course. Come see me during office hours if you have trouble.

\[
\frac{1}{2} \left( \frac{-1}{2} + \frac{4}{3} \right) + \frac{1}{4} = \frac{1}{2} \left( \frac{-1}{2} \times \frac{3}{3} + \frac{4}{3} \times \frac{2}{2} \right) + \frac{1}{4} \\
= \frac{1}{2} \left( \frac{-3}{6} + \frac{8}{6} \right) + \frac{1}{4} \\
= \frac{1}{2} \left( \frac{5}{6} \right) + \frac{1}{4} \\
= \frac{5}{12} + \frac{3}{12} \\
= \frac{8}{12} \\
= \frac{2}{3}
\]

P2 (4pts.) Evaluate the following expression.

Here we distribute the exponent across multiplication, then use the identity \((a + b)(a - b) = a^2 - b^2\).

\[
\left( \sqrt{2} - \sqrt{3} \right)^3 \left( \sqrt{2} + \sqrt{3} \right)^3 = \left( \left( \sqrt{2} - \sqrt{3} \right) \left( \sqrt{2} + \sqrt{3} \right) \right)^3 \\
= \left( \left( \sqrt{2} \right)^2 - \left( \sqrt{3} \right)^2 \right)^3 \\
= \left( 2 - 3 \right)^3 \\
= \left( -1 \right)^3 \\
= -1
\]
P3 (4pts.) Fully simplify the radical expression. Assume only that $a$ and $b$ are real numbers.

Problem 3 is subtraction and division of powers. The tricky part is remembering that $\sqrt{a^2} = |a|$. 

$$\frac{\sqrt{36a^2b}}{3b^5} = \frac{\sqrt{12a^2}}{b^4}$$
$$= \frac{2\sqrt{3}|a|}{b^2}$$

I would also have accepted the following answer.

$$\frac{\sqrt{36a^2b}}{3b^5} = \frac{6|a|}{\sqrt{3b^2}}$$

Strictly speaking, the first answer is better, because we’ve rationalized the denominator. But for now this is okay.

P4 If you were a monster truck driver, what would you name your monster truck?

I’d name it “Bill Murray.”
P1 (2pts.) Evaluate the following expression. Express the resulting fraction in lowest terms.

\[ \frac{1}{2} \left( -\frac{1}{2} + \frac{2}{3} \right) + \frac{1}{4} \]

P2 (4pts.) Evaluate the following expression.

\[ \left( \sqrt{5} - \sqrt{3} \right)^3 \left( \sqrt{5} + \sqrt{3} \right)^3 \]

P3 (4pts.) Fully simplify the radical expression. Assume only that \( a \) and \( b \) are real numbers.

\[ \sqrt{\frac{36a^2b^5}{2b}} \]

P4 If you were a monster truck driver, what would you name your monster truck?
P1. (3pts.) Simplify the following expression.

\[
\frac{(x^3 + 1)(x^2 - 2x + 1)}{(x^2 - 1)}
\]

P2. (3pts.) Solve for \( x \). \textit{Hint: Use the method of substitution.}

\[
(3x + 2)^2 + 2(3x + 2) - 15 = 0
\]

P3. (3pts.) For which values of \( c \) does the equation \( 3x^2 + 4x + c = 0 \) have no solutions?

P4. (1pt.) If you could turn any activity into an Olympic sport, what would it be?
P1. (3pts.) Simplify the following expression.

\[
\frac{x^2 - 1}{(x^3 - 1)(x^2 + 2x + 1)}
\]

This is a simple application of the identities for difference of squares, sum/difference of cubes, and square of binomial. Know how to spot these so you can easily cancel factors in problems involving rational functions.

\[
\frac{x^2 - 1}{(x^3 - 1)(x^2 + 2x + 1)} = \frac{(x + 1)(x - 1)}{(x^3 - 1)(x^2 + 2x + 1)}
\]

\[
= \frac{(x + 1)(x - 1)}{(x - 1)(x^2 + x + 1)(x^2 + 2x + 1)}
\]

\[
= \frac{(x + 1)(x - 1)}{(x - 1)(x^2 + 2x + 1)(x + 1)^2}
\]

\[
= \frac{1}{(x^2 + x + 1)(x + 1)}
\]

P2. (3pts.) Solve for \(x\). \textit{Hint: Use the method of substitution.}

\[
(3x + 1)^2 - 8(3x + 1) + 15 = 0
\]

We make the substitution \(u = 3x + 1\).

\[
(3x + 1)^2 - 8(3x + 1) + 15 = 0
\]

\[
u^2 - 8u + 15 = 0
\]

\[
(u - 5)(u - 3) = 0
\]

\[
u = 5, 3
\]
We now solve for $x$ using each value of $u$.

\[
\begin{align*}
  u &= 5 \\
  3x + 1 &= 5 \\
  3x &= 4 \\
  x &= \frac{4}{3} \\

  u &= 3 \\
  3x + 1 &= 3 \\
  3x &= 2 \\
  x &= \frac{2}{3}
\end{align*}
\]

So $x = \frac{4}{3}$ or $\frac{2}{3}$.

**P3. (3pts.)** For which values of $c$ does the equation $5x^2 + 4x + c = 0$ have no solutions?

We wish for the equation to have no solution, so we set

\[
b^2 - 4ac < 0
\]

and solve this inequality for $c$.

\[
\begin{align*}
  4^2 - 4 \times 5 \times c &< 0 \\
  16 &< 20c \\
  \frac{16}{20} &< c \\
  \frac{4}{5} &< c
\end{align*}
\]

So, in interval notation, the given equation has no solution precisely when $c$ belongs to $\left(\frac{4}{5}, \infty\right)$.

**P4. (1pt.)** If you could turn any activity into an Olympic sport, what would it be?

Break dancing.
P1. (3pts.) Simplify the following expression.

\[
\frac{x^2 - 1}{(x^3 - 1)(x^2 + 2x + 1)}
\]

P2. (3pts.) Solve for \(x\). Hint: Use the method of substitution.

\[
(3x + 1)^2 - 8(3x + 1) + 15 = 0
\]

P3. (3pts.) For which values of \(c\) does the equation \(5x^2 + 4x + c = 0\) have no solutions?

P4. (1pt.) If you could turn any activity into an Olympic sport, what would it be?
P1. (3pts.) Find the domain of $f(x) = \frac{1}{\sqrt{x}}$. Then compute $f\left(4\right) - f\left(\frac{1}{16}\right)$.

P2. (3pts.) Take $P = (-1, 2)$ and $Q = (7, 6)$. Write the equation of a circle passing through the origin whose center is the midpoint of the line segment connecting $P$ and $Q$.

P3. (3pts.) Find the distance between the point $(-2, 7)$ and the center of the circle $x^2 + 4x + y^2 - 6x - 3 = 0$. Then write the equation for the line passing through these two points.

P4. (1pt.) What’s the best name for a pet tiger?
P1. (3pts.) Find the domain of $f(x) = \frac{1}{\sqrt{x}}$. Then compute $f(4) - f\left(\frac{1}{16}\right)$.

First, we find the domain. In the expression $1/\sqrt{x}$, we see two things: a denominator and an expression under a square root. We need the denominator to be non-zero, so we write

$$\sqrt{x} \neq 0$$

Squaring both sides, this gives us

$$x \neq 0$$

Now, any expression under a square root must be non-negative (i.e. greater than or equal to 0), and $x$ appears under a square root, so we have

$$x \geq 0$$

Combining $x \neq 0$ and $x \geq 0$ gives us $x > 0$, so we conclude that the domain of $f$ is

$$(0, \infty)$$

Next, we carry out the requested computation.

$$f(4) - f\left(\frac{1}{16}\right) = \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{\frac{1}{16}}}$$

$$= \frac{1}{2} - \frac{1}{\frac{1}{4}}$$

$$= \frac{1}{2} - 4$$

$$= \frac{1}{2} - \frac{8}{2}$$

$$= \frac{-7}{2}$$

For those of you having trouble evaluating $\frac{1}{\sqrt{\frac{1}{16}}}$, it may help you to rewrite $f(x) = x^{-1/2}$ (which we can do, because $\sqrt{x} = x^{1/2}$, so $1/\sqrt{x} = x^{-1/2}$). Then we can simply use exponent laws:

$$f(16^{-1}) = (16^{-1})^{-1/2} = 16^{1/2} = 4$$
P2. (3pts.) Take $P = (-1, 2)$ and $Q = (7, 6)$. Write the equation of a circle passing through the origin whose center is the midpoint of the line segment connecting $P$ and $Q$.

First let’s find the midpoint $m$ of $PQ$.

$$m = \left( \frac{P_x + Q_x}{2}, \frac{P_y + Q_y}{2} \right)$$
$$= \left( \frac{-1 + 7}{2}, \frac{2 + 6}{2} \right)$$
$$= \left( \frac{6}{2}, \frac{8}{2} \right)$$
$$= (3, 4)$$

Plugging the coordinates of $m$ into the standard form of the circle equation gives us

$$(x - 3)^2 + (y - 4)^2 = r^2$$

All that remains is finding $r$. Every point $(x, y)$ on the circle must satisfy the equation above, and we’re given in the problem that $(0, 0)$ is a point on the circle, so we can substitute the point in to solve for $r$.

$$(0 - 3)^2 + (0 - 4)^2 = r^2$$
$$9 + 16 = r^2$$
$$25 = r^2$$
$$5 = r$$

So our final answer is

$$(x - 3)^2 + (y - 4)^2 = 5^2$$

Many of you who got this problem wrong made the mistake of substituting the coordinates for $P$ or $Q$ instead of $(0, 0)$ to solve for $r$. Note that the question didn’t say the circle should pass through either $P$ or $Q$, only that it should pass through the origin. Be sure to read questions like this carefully.

P3. (3pts.) Find the distance between the point $(-2, 7)$ and the center of the circle $x^2 + 4x + y^2 - 6y - 3 = 0$. Then write the equation for the line passing through these two points.

We begin by converting its equation to standard form.

$$(x^2 + 4x) + (y^2 - 6y) - 3 = 0$$
$$(x^2 + 4x + 4) - 4 + (y^2 - 6y + 9) - 9 - 3 = 0$$
$$(x + 2)^2 - 4 + (y - 3)^2 - 9 - 3 = 0$$
$$(x + 2)^2 + (y - 3)^2 - 16 = 0$$
$$(x + 2)^2 + (y - 3)^2 = 4^2$$

The center of the circle is, therefore, the point $(-2, 3)$. We need the distance between this and $(-2, 7)$. 


We pause here briefly to make an important observation: both points have the same the $x$-coordinate. Therefore, the distance is simply the difference between their $y$-coordinates. Alternatively, we could also notice that $(-2, 7)$ is a point on the circle (at the "top"), so the distance between it and the center is precisely the radius of the circle, which we have already calculated above. These are easy to see if you draw the circle after finding its equation, which I suggest you always do.

After making either of the above observations, it would be overkill to actually compute the distance using the distance formula, but here is how to do it in case you didn’t notice them.

$$
\text{distance} = \sqrt{(-2 - (-2))^2 + (7 - 3)^2} \\
= \sqrt{(0)^2 + (4)^2} \\
= \sqrt{4^2} \\
= 4
$$

Finally, the question asks for the equation of the line passing through these two points. Since $(-2, 7)$ and $(-2, 3)$ both have the same $x$-coordinate, the line passing through them is vertical. Vertical lines have the form $x = \text{some constant}$, so the line is

$$
x = -2
$$

\textbf{P4. (1pt.)} What’s the best name for a pet tiger?

Azealia Banks.
P1. (3pts.) Find the domain of \( f(x) = \frac{1}{\sqrt{x}} \). Then compute \( f\left(\frac{1}{4}\right) - f(16) \).

P2. (3pts.) Take \( P = (5, 8) \) and \( Q = (3, -2) \). Write the equation of a circle passing through the origin whose center is the midpoint of the line segment connecting \( P \) and \( Q \).

P3. (3pts.) Find the distance between the point \((1, -9)\) and the center of the circle \(x^2 - 2x + y^2 + 8x - 5 = 0\). Then write the equation for the line passing through these two points.

P4. (1pt.) What’s the best name for a pet tiger?
P1. (4pts.) Let $f(x) = \frac{1}{x^2-x+6}$ and $g(x) = \sqrt{2x-1}$. Find the domain of $(f \circ g)(x)$ and write $f(g(x))$ in its simplest form.

P2. (3pts.) Let $f(x) = \frac{1}{2-x}$ and $g(x) = \frac{1}{2x-1}$. Write $f(g(x))$ in its most simplified form and find the domain of $(f \circ g)(x)$.

P3. (2pts.) Let $f(x) = \frac{x}{3x-2} + 1$. Find the inverse function $f^{-1}$.

P4. (1pt.) If you had a machine that could freeze time for everyone but you, what would you do?
P1. Let $f(x) = \frac{1}{x^3 - x^2 - 6}$ and $g(x) = \sqrt{2x - 1}$. Find the domain of $(f \circ g)(x)$ and write $f(g(x))$ in its simplest form.

First we’ll compute the domain of $g$, since we’ll need that later for the domain of $f \circ g$. Since $2x - 1$ is under a square root, it must be non-negative, so we write

$$2x - 1 \geq 0$$

This tells us that $x \geq 1/2$. Let’s set that aside for now and compute $f(g(x))$ and its domain.

We begin to compute $f(g(x))$.

$$f(g(x)) = \frac{1}{(g(x))^4 - (g(x))^2 - 6}$$

$$= \frac{1}{(\sqrt{2x - 1})^4 - (\sqrt{2x - 1})^2 - 6}$$

$$= \frac{1}{(2x - 1)^2 - (2x - 1) - 6}$$

Here, one may notice that we can do a $u$-substitution to find when the denominator is 0. Using $u = 2x - 1$, the denominator becomes $u^2 - u - 6 = (u - 3)(u + 2)$, so $u = 3, -2$. In the first case, $2x - 1 = 3$, we see $x = 2$. In the second case, $2x - 1 = -2$, so $x = -1/2$. Thus we find that the domain of $f(g(x))$, the composed function, is $x \neq 2, -1/2$. Combining this with the domain of $g$, we find that the domain of $f \circ g$ is $x \geq 1/2, x \neq 2, -1/2$, or, in other words,

$$\left(-\infty, \frac{1}{2}\right) \cup \left[\frac{1}{2}, 2\right) \cup (2, \infty)$$

Furthermore, knowing the roots of the denominator immediately gives us its factorization: $2(2x+1)(x-2)$. We could also have arrived at this by direct computation, i.e.

$$\frac{1}{(2x - 1)^2 - (2x - 1) - 6} = \frac{1}{(4x^2 - 4x + 1) - (2x - 1) - 6}$$

$$= \frac{1}{4x^2 - 6x - 4}$$

$$= \frac{1}{2(2x+1)(x-2)}$$

which is the most simplified form of $f(g(x))$.

To solve this problem, you must know how to compose functions, how to multiply exponents, how to find the domain of a function, and either how to use $u$-substitution or how to factor $4x^2 - 6x - 4$. It’s not an
easy problem, but it should be straightforward to those who are keeping up and have not forgotten the older material.

**P2.** Let \( f(x) = \frac{1}{2-x} \) and \( g(x) = \frac{1}{2x-1} \). Write \( f(g(x)) \) in its most simplified form and find the domain of \((f \circ g)(x)\).

Once again, let’s start by computing the domain of \( g \). The denominator is \( 2x - 1 \), so we know that \( 2x - 1 \neq 0 \), that is, \( x \neq 1/2 \).

Now we compute the composed function \( f(g(x)) \).

\[
f(g(x)) = \frac{1}{2 - g(x)} = \frac{1}{2 - \frac{1}{2x-1}}
\]

To find the most simplified form of \( f(g(x)) \), recall how to simplify a complex fraction.

\[
\frac{1}{2 - \frac{1}{2x-1}} = \frac{1}{2 - \frac{2x - 1}{2x - 1}} \times \frac{2x - 1}{2x - 1} = \frac{2x - 1}{2(2x - 1) - 1} = \frac{2x - 1}{4x - 3}
\]

We have a denominator of \( 4x - 3 \), so \( 4x - 3 \neq 0 \), meaning \( x \neq 3/4 \). Intersecting this with the domain of \( g \), we find that the domain of \( f \circ g \) is \( x \neq 1/2, 3/4 \), or

\[
(-\infty, \frac{1}{2}) \cup \left( \frac{1}{2}, \frac{3}{4} \right) \cup \left( \frac{3}{4}, \infty \right)
\]

This is an easier domain of composite function problem than **P1.** Most people who got this wrong forgot how to simplify complex fractions. I can just about guarantee that something resembling either **P1** or **P2** will appear on an exam.

**P3.** Let \( f(x) = \frac{x}{3x-2} + 1 \). Find the inverse function \( f^{-1} \). We replace \( f(x) \) with \( y \),

\[
y = \frac{x}{3x-2} + 1
\]

switch the \( x \)'s and the \( y \)'s,

\[
x = \frac{y}{3y-2} + 1
\]
then solve for $y$ in terms of $x$.

\[
\begin{align*}
    x - 1 &= \frac{y}{3y - 2} \\
    (3y - 2)(x - 1) &= y \\
    3yx - 3y - 2x + 2 &= y \\
    3yx - 3y - y &= 2x - 2 \\
    3yx - 4y &= 2x - 2 \\
    y(3x - 4) &= 2x - 2 \\
    y &= \frac{2x - 2}{3x - 4}
\end{align*}
\]

So, we conclude that

\[f^{-1}(x) = \frac{2x - 2}{3x - 4}\]

Collecting terms is key here. When solving for $y$, you should bring terms containing $y$ to one side of the equation so that you can factor one out. This example is a pretty canonical application of this technique, so make sure you can mimic this process when called for.

**P4.** If you had a machine that could freeze time for everyone but you, what would you do?

Remove all cell phones from the world; watch world population develop mindfulness.
P1. (4pts.) Let \( f(x) = \frac{1}{x^2+x-6} \) and \( g(x) = \sqrt{3x-1} \). Find the domain of \((f \circ g)(x)\) and write \( f(g(x))\) in its simplest form.

P2. (3pts.) Let \( f(x) = \frac{1}{3-x} \) and \( g(x) = \frac{1}{2x-3} \). Write \( f(g(x))\) in its most simplified form and find the domain of \((f \circ g)(x)\).

P3. (2pts.) Let \( f(x) = \frac{x}{2x-3} + 1 \). Find the inverse function \( f^{-1} \).

P4. (1pt.) If you had a machine that could freeze time for everyone but you, what would you do?
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**P1. (2pts.)** Rationalize and write in standard form.

\[
\frac{1 + 3i}{5 - 2i}
\]

**P2. (3\(\frac{1}{2}\)pts.)** Find a polynomial with real coefficients of minimal degree that has zeroes at both 1 and \(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\). Write this polynomial in standard form.

**P3. (3\(\frac{1}{2}\)pts.)** Find all complex zeroes of \(f(x) = 3x^4 + 8x^3 + 24x^2 + 72x - 27\), given the information that \(f(-3i) = 0\).

**P4. (1pt.)** Tell me one thing that is legal, but shouldn’t be.
**P1.** Rationalize and write in standard form.

\[
\frac{1 + 3i}{5 - 2i}
\]

We multiply by the conjugate of the denominator, then separate the fraction into the form \(a + ib\).

\[
\frac{1 + 3i}{5 - 2i} \cdot \frac{5 + 2i}{5 + 2i} = \frac{5 + 2i + 15i + 6i^2}{25 + 4} = \frac{5 + 2i + 15i - 6}{25 + 4} = \frac{-1 + 17i}{29} = \frac{-1}{29} + \frac{17}{29}i
\]

Something to keep in mind here is that multiplying a complex number by its conjugate has an easy formula: \((a + ib)(a - ib) = a^2 + b^2\). You can foil it out to see that this is true, or alternatively you can apply the identity \((x + y)(x - y) = x^2 - y^2\) with \(x = a\) and \(y = ib\).

Also, keep in mind that while \(-\frac{1 + 17i}{29}\) is not necessarily wrong, it is not in "standard form." Standard form refers to writing the real part and imaginary part of a complex number separately. I didn’t count off if you didn’t do this because it’s not a big deal, but know it for future reference.

**P2.** Find a polynomial with real coefficients of minimal degree that has zeroes at both \(1\) and \(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\). Write this polynomial in standard form.

By the conjugate pairs theorem, we know that if \(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\) is the root of a polynomial with real coefficients, then its conjugate, \(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\), must also be a root. So the polynomial we’re looking for must be of the form

\[
(x - 1) \left(x - \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\right) \left(x - \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\right)
\]

First let’s multiply the second and third polynomial together. There are lots of ways to do this, but the
following is probably the shortest.

\[
(x - \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)) \left(x - \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\right) = \left(x + \frac{1}{2}ight) - \frac{i\sqrt{3}}{2} \left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)
\]

\[
= \left(x + \frac{1}{2}\right)^2 - \left(\frac{i\sqrt{3}}{2}\right)^2
\]

\[
= x^2 + x + \frac{1}{4} - \frac{3}{4}
\]

\[
= x^2 + x + 1
\]

So we have

\[
(x - 1) \left(x - \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\right) \left(x - \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\right) = (x - 1)(x^2 + x + 1) = x^3 - 1
\]

There were two common mistakes in this problem. The first was conjugating incorrectly by changing negative before the \(\frac{1}{2}\) as well as the negative before the \(i\) term. The second was being unable to multiply the second and third polynomial together. There's no doubt that is a long and somewhat tedious calculation, but if you go through it step by step, you should be able to get through it. Remember to break up expressions to make your math less complicated, like what I did above when I calculated the product of the second and third polynomials off to the side before doing the whole thing.
**P3.** Find all complex zeroes of \( f(x) = 3x^4 + 8x^3 + 24x^2 + 72x - 27 \), given the information that \( f(-3i) = 0 \). Note: whenever you’re given that a polynomial has a certain zero, use that information first!

We know that \(-3i\) is a zero of \( f \), so by the conjugate pairs theorem, \( 3i \) has to be a zero too. That means that \((x - 3i)(x + 3i) = x^2 + 9\) is a factor of \( f(x) \).

\[
\begin{array}{c|cccc}
3x^4 & + & 8x^3 & + & 24x^2 & + & 72x & - & 27 \\
3x^4 & + & 27x^2 \\
8x^3 & - & 3x^2 & + & 72x & - & 27 \\
8x^3 & + & 72x \\
-3x^2 & - & 27 \\
-3x^2 & - & 27 \\
0
\end{array}
\]

which means we have

\[
f(x) = 3x^4 + 8x^3 + 24x^2 + 72x - 27 = (x^2 + 9)(3x^2 + 8x + 3) = (x - 3i)(x + 3i)(3x^2 + 8x - 3) = (x - 3i)(x + 3i)(x + 3)(3x - 1)
\]

Therefore, the zeroes of \( f \) are \( 3i, -3i, -3, \) and \( \frac{1}{3} \).

A lot of people tried to go in directly and find the zeroes with the Rational Root Theorem. In general, this is a bad idea; the Rational Root Theorem takes tons of computation, so it should be used only as a last resort. Remember: whenever you’re given a degree four polynomial and told that a certain complex number is a zero of that polynomial, use the conjugate pairs theorem and polynomial long division first. This will leave you with a quadratic, which will be much easier to solve. (You can always fall back on the quadratic formula.)

**P4.** Tell me one thing that is legal, but shouldn’t be.

Webassign / distance learning in general.
P1. (2pts.) Rationalize and write in standard form.

\[
\frac{3 - 2i}{4 - 3i}
\]

P2. (3½ pts.) Find a polynomial with real coefficients of minimal degree that has zeroes at both \(-1\) and \(\frac{1}{2} - \frac{i\sqrt{3}}{2}\). Write this polynomial in standard form.

P3. (3½ pts.) Find all complex zeroes of \(f(x) = 4x^4 + 7x^3 + 14x^2 + 28x - 8\), given the information that \(f(2i) = 0\).

P4. (1pt.) Tell me one thing that is legal, but shouldn’t be.
P1a. (3pts.) Solve the polynomial inequality \(2x^2 + 5x \leq \frac{6}{1+2x}\).

P1b. (1pts.) What is the domain of \(f(x) = \sqrt{\frac{4x^3 + 12x^2 + 5x - 6}{2x + 1}}\) ?

P2. (2pts.) Solve \(e^{2x^2 + 2x - 3} = 1\).

P3. (3pts.) Solve each linear system.

\[
\begin{align*}
-9x + 15y &= 3 \\
-6x + 10y &= 3
\end{align*}
\]

\[
\begin{align*}
-9x + 15y &= 3 \\
-6x + 10y &= 2
\end{align*}
\]

\[
\begin{align*}
-9x + 15y &= 3 \\
-3x + 10y &= 2
\end{align*}
\]

P4. (1pt.) What should I ask for P4 on Quiz 7?
P1a. (3pts.) Solve the polynomial inequality $2x^2 + 5x \leq \frac{6}{1+2x}$.

We begin by bringing everything to one side and turning this into a rational function.

$$2x^2 + 5x < \frac{6}{1+2x}$$

$$2x^2 + 5x - \frac{6}{1+2x} < 0$$

$$\frac{(2x^2 + 5x)(1+2x) - 6}{1+2x} \leq 0$$

$$\frac{4x^3 + 12x^2 + 5x - 6}{1+2x} < 0$$

This is a rational inequality, so our game plan is to find out the critical values, then determine the sign of the rational function between those values.

The critical values of the function are the zeroes of the numerator and the zeroes of the denominator. The bottom is already factored, so we now factor the top. This is degree 3, so I know that all I have to find is one root, and I’ll be left with an easy to factor quadratic. To find the first root, I use the rational root theorem. The possible rational zeroes are, after eliminating duplicates,

$$\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}.$$ 

Starting from the left, I check each value using synthetic division. The first zero I find is $-2$.

$$-2 \mid \begin{array}{cccc} 4 & 12 & 5 & -6 \\ \hline & -8 & -8 & 6 \\ \end{array}$$

So $4x^3 + 12x^2 + 5x - 6 = (x + 2)(4x^2 + 4x - 3) = (x + 2)(2x + 3)(2x - 1)$ has zeroes at $-2, -\frac{3}{2}, \frac{1}{2}$. Therefore, the critical values for the rational inequality are $-2, -\frac{3}{2}, \frac{1}{2}$, and $-\frac{1}{2}$, so our table is

<table>
<thead>
<tr>
<th>Score</th>
<th>3</th>
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<th>2</th>
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<tbody>
<tr>
<td>P1a</td>
<td>P1b</td>
<td>P2</td>
<td>P3</td>
<td>P4</td>
<td>Q6</td>
<td></td>
</tr>
</tbody>
</table>

We can determine the left hand behavior by simply plugging in $-3$, yielding $\frac{(-1)(-7)(-3)}{-6}$ which is positive.

$$\begin{array}{cccccc} -2 & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ + & & & & & \\ \end{array}$$
Now we note that \(-2\) and \(-3/2\) each have odd multiplicity, so function changes sign at those points.

\[
\begin{array}{cccccc}
-2 & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\
+ & - & + & \square & \square \\
\end{array}
\]

Since \(-\frac{1}{2}\) is an asymptote, we can’t use multiplicity arguments. Instead, we simply pick any number in \((-1/2, 1/2)\) and plug it in. The easiest is 0, which gives us \(-6\).

\[
\begin{array}{cccccc}
-2 & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\
+ & - & + & - & \square \\
\end{array}
\]

Finally, \(1/2\) has odd multiplicity, so we change signs again.

\[
\begin{array}{cccccc}
-2 & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\
+ & - & + & - & + \\
\end{array}
\]

Thus the answer is

\[
\left(-2, -\frac{3}{2}\right) \cup \left(-\frac{1}{2}, \frac{1}{2}\right). 
\]

**P1b.** (1pts.) What is the domain of \(f(x) = \sqrt{\frac{4x^3 + 12x^2 + 5x - 6}{2x + 1}}\)?

First we see that the denominator can’t be 0, so \(x \neq -1/2\). Next, we know that the inside of the radical must be non-negative, i.e.

\[
\frac{4x^3 + 12x^2 + 5x - 6}{1 + 2x} \geq 0
\]

If you did part a correctly, you already have the sign table drawn for this rational function, so we already have our answer:

\[
(-\infty, -2] \cup \left[-\frac{3}{2}, -\frac{1}{2}\right] \cup \left(\frac{1}{2}, \infty\right)
\]

If you answered part b with the opposite of what you got as your answer for part a, I gave you full credit regardless of whether or not your answer to part a was correct. For those of you who did do part a correctly, but missed this one, you need to review - you should know how to find domains by now.

**P2.** (2pts.) Solve \(e^{2x^2 + 2x - 3} = 1\).

The one-to-one property of exponential functions tells us that \(e^a = e^b\) if and only if \(a = b\). We can rewrite the given equation as

\[
e^{2x^2 + 2x - 3} = e^0.
\]

Thus it suffices to solve \(2x^2 + 2x - 3 = 0\). Using the quadratic formula, the solutions are \(x = \frac{-1 \pm \sqrt{7}}{2}\).

This problem was easy if you knew about the one-to-one property and impossible if not. If you didn’t see the trick, don’t worry too much about it, but remember it for later because this frequently shows up on exams.
P3. (3pts.) Solve each linear system.

\[
\begin{align*}
-9x + 15y &= 3 \\
-6x + 10y &= 3
\end{align*}
\]

This problem was designed to test whether or not you understand dependent systems of linear equations and inconsistent systems of linear equations. The arithmetic is essentially identical for all three cases.

Multiplying \(-6x + 10y = 3\) by \(3/2\) we get \(-9x + 15y = 9/2\), which contradicts the top equation. Therefore, the system of equations to the far left is indeterminate.

Multiplying \(-6x + 10y = 2\) by \(3/2\) we get \(-9x + 15y = 3\), which is identical to the top equation. Therefore, the system of equations in the middle is dependent and has infinitely many solutions. In particular, the solutions are given by \((x, y) = (t, \frac{1+3t}{5})\), where \(t\) is any real number.

Multiplying \(-3x + 10y = 2\) by \(3/2\) we get \(-3x + 15y = 3\). Subtracting this from the top equation we get \(-6x = 0\), so \(x = 0\). Plugging \(x\) back into either equation then yields \(y = 1/5\). Therefore, the system of equations to the far right is independent with solution \((0, \frac{1}{5})\).

P4. (1pt.) What should I ask for P4 on Quiz 7?

I think I’m going to ask you to draw a turtle.
P1a. (3pts.) Solve the polynomial inequality $3x^2 + 11x + 3 \leq \frac{21}{3x + 2}$.

P1b. (1pts.) What is the domain of $f(x) = \sqrt{\frac{9x^3 + 39x^2 + 31x - 15}{3x + 2}}$?

P2. (2pts.) Solve $e^{x^2 - x - 1} = 1$.

P3. (3pts.) Solve each linear system.

\begin{align*}
6x - 4y &= 2 \\
9x - 6y &= 3
\end{align*}

\begin{align*}
6x - 4y &= 2 \\
9x - 6y &= 3
\end{align*}

\begin{align*}
6x - 4y &= 2 \\
9x - 3y &= 2
\end{align*}

P4. (1pt.) What should I ask for P4 on Quiz 7?
P1. (3pts.) Evaluate \[
\frac{(\log 2)^2 + (\log 16)^2}{\log 2 \log 16}
\]

P2. (3pts.) Solve for \(x\).
\[
\left(\frac{\ln x}{\ln 2}\right)^2 - \log_2 \left(\frac{2}{x^3}\right) + 1 = 0
\]

P3. (3pts.) Solve for \(x\).
\[
1 = 3^{x+1}7^{1-x}
\]

P4. (1pt.) If you were a type of function, which one would you be and why?
P1. Evaluate

\[ \frac{(\log 2)^2 + (\log 16)^2}{\log 2 \log 16} \]

We distribute the denominator to create two fractions, cancel terms, and apply the change of base formula.

\[
\frac{(\log 2)^2 + (\log 16)^2}{\log 2 \log 16} = \frac{(\log 2)^2}{\log 2 \log 16} + \frac{(\log 16)^2}{\log 2 \log 16}
\]

\[
= \frac{\log 2}{\log 16} + \frac{\log 16}{\log 2}
\]

\[
= \log_{16} 2 + \log_2 16
\]

To find the value of \( \log_{16} 2 \), you can either notice that 2 is the fourth root of 16 (so \( 16^{1/4} = 2 \), which means that \( \log_{16} 2 = 1/4 \)), or you can use the change of base formula as follows.

\[
\log_{16} 2 + \log_2 16 = \frac{\log_2 2}{\log_2 16} + \log_2 16
\]

\[
= \frac{1}{4} + 4
\]

\[
= \frac{17}{6}
\]

The point of this problem was to present a situation in which you might be tempted to use fake log properties, such as \((\log a)^b = \log(a^b)\) or \((\log a) (\log b) = \log(a + b)\). These are false: they look a bit like the log properties we learned, but they are not actually identities. Be careful to avoid these types of mistakes. Only use the log properties we learned in lecture 22 (along with \( a^{\log_a x} = x = \log_a a^x \)), and watch your parenthesis while doing so.
P2. Solve for $x$.

\[
\left(\frac{\ln x}{\ln 2}\right)^2 - \log_2 \left(\frac{2}{x^3}\right) + 1 = 0
\]

We begin by applying log properties one by one, which I will now demonstrate slowly, step by step. If you don’t see an immediate way to solve an expression, be persistent - keep simplifying in small steps until a clearer pattern appears.

\[
\left(\frac{\ln x}{\ln 2}\right)^2 - \log_2 \left(\frac{2}{x^3}\right) + 1 = 0 \\
(\log_2 x)^2 - \log_2 \left(\frac{2}{x^3}\right) + 1 = 0 \\
(\log_2 x)^2 - (\log_2 (2) - \log_2 (x^3)) + 1 = 0 \\
(\log_2 x)^2 - (\log_2 (2) - 3\log_2 (x)) + 1 = 0 \\
(\log_2 x)^2 - (1 - 3\log_2 (x)) + 1 = 0 \\
(\log_2 x)^2 - 1 + 3\log_2 (x) + 1 = 0 \\
(\log_2 x)^2 + 3\log_2 (x) = 0
\]

This is as far as the equation can be simplified. To solve, we substitute $u = \log_2 x$. The equation becomes $u^2 + 3u = 0$, the solutions of which are $u = 0$ and $u = -3$. We solve each of these equations for $x$.

\[
\begin{align*}
u &= 0 & \quad & u = -3 \\
\log_2 x &= 0 & \quad & \log_2 x = -3 \\
2^{\log_2 x} &= 2^0 & \quad & 2^{\log_2 x} = 2^{-3} \\
x &= 1 & \quad & x = \frac{1}{8}
\end{align*}
\]

This is one of those "quadratic equations in disguise" problems. I tried to make it clear by putting the terms in order, but many people didn’t see it. If you used the change of base formula on the squared term and expanded the second term correctly, I gave you two and a half points out of three. In the future, be wary of quadratic $u$-substitution problems, especially with logs and exponentials; people love to put that type of thing on exams.
P3. (3pts.) Solve for $x$.

\[ 1 = 3^{x+1}7^{1-x} \]

Once again, we manipulate the expression using logs until we can isolate $x$.

\[
\begin{align*}
1 &= 3^{x+1}7^{1-x} \\
\log_3 1 &= \log_3 (3^{x+1}7^{1-x}) \\
0 &= \log_3 (3^{x+1}) + \log_3 (7^{1-x}) \\
0 &= x + 1 + \log_3 (7^{1-x}) \\
0 &= x + 1 + (1 - x) \log_3 7 \\
0 &= x + 1 + \log_3 7 - x \log_3 7 \\
-x + x \log_3 7 &= 1 + \log_3 7 \\
x (-1 + \log_3 7) &= 1 + \log_3 7 \\
x &= \frac{1 + \log_3 7}{-1 + \log_3 7}
\end{align*}
\]

At this point, you’re done - this is a perfectly acceptable answer to this question. If you want to be fancy, you can continue to simplify it using log properties. (Even though it’s not strictly necessary for this particular problem, I recommend you work through the following steps to practice simplifying logarithmic expressions.)

\[
\begin{align*}
x &= \frac{1 + \frac{\log 7}{\log 3}}{-1 + \frac{\log 7}{\log 3}} \\
x &= \left( \frac{1 + \frac{\log 7}{\log 3}}{-1 + \frac{\log 7}{\log 3}} \right) \left( \frac{\log 3}{\log 3} \right) \\
x &= \frac{\log 3 + \log 7}{-\log 3 + \log 7} \\
x &= \frac{\log 21}{\log (7/3)} \\
x &= \log_{\frac{7}{3}} 21
\end{align*}
\]

Solving the equation is more important than simplifying it, so it’s fine if you didn’t write it as a single fraction. Don’t be afraid to play around with expressions like these - it’s a good way to learn how to manipulate log properties.

P4. If you were a type of function, which one would you be and why?

I’d be an elliptic function, because I gained a lot of weight over Spring Break.
P1. (3pts.) Evaluate \[
\frac{(\ln 27)^2 - (\ln 3)^2}{\ln 3 \ln 27}
\]

P2. (3pts.) Solve for \(x\).
\[
\left(\frac{\log x}{\log 3}\right)^2 + \log_3 \left(\frac{3}{x^2}\right) - 1 = 0
\]

P3. (3pts.) Solve for \(x\).
\[
1 = 11^{x-1}5^{1+x}
\]

P4. (1pt.) If you were a type of function, which one would you be and why?
P1. (5pts.) Fill in all unlabeled angles and side lengths for each triangle. **Show your work.**

\[ \begin{array}{c}
\triangle 2 \quad \pi/6 \\
\end{array} \quad \begin{array}{c}
\triangle \pi/4 \quad 1 \\
\end{array} \]

P2. (2pts.) Simplify \((\cos \theta - \sin \theta)^2 - 1\). Then find its value when \(\theta = \frac{31}{4}\pi\).

P3. (2pts.) Find all values of \(\theta\) for which \(\sin^2 \theta = \frac{1}{2}\) and \(0 \leq \theta < 2\pi\).

P4. (1pt.) If you had to write your autobiography today, what would you call it?
P1. (5pts.) Fill in all unlabeled angles and side lengths for each triangle. Show your work.

For the first triangle,

\[ \sin \frac{\pi}{6} = \frac{2}{h} = \frac{1}{2} \]

so \( h = 4 \)

\[ \tan \frac{\pi}{6} = \frac{2}{x} = \frac{1}{\sqrt{3}} \]

so \( x = 2\sqrt{3} \)

For the second triangle,

\[ \sin \frac{\pi}{4} = \frac{y}{1} = \frac{\sqrt{2}}{2} \]

so \( y = \frac{\sqrt{2}}{2} \)

This problem was designed to test the fundamentals of trigonometry. Before we get into any fancy stuff, I want to be sure you know how to use basic trig functions, as well as how to evaluate them. It’s important to learn how to do this before we get into crazy trig identity stuff or graph drawing. This is the “meaning” of trig - if you take physics college, they’ll expect you to know this before walking in the door.

P2. (2pts.) Simplify \((\cos \theta - \sin \theta)^2 - 1\). Then find its value when \(\theta = \frac{31}{4} \pi\).

\[
(\cos \theta - \sin \theta)^2 - 1 = \cos^2 \theta + \sin^2 \theta - 2 \cos \theta \sin \theta - 1
= \cos^2 \theta + \sin^2 \theta - 2 \cos \theta \sin \theta - 1
= 1 - 2 \cos \theta \sin \theta - 1
= -2 \cos \theta \sin \theta
\]
Now it’s fully simplified, so let’s plug in \(31/4\) and see what we get. First I’ll compute the sin and cos separately.

\[
\cos \left( \frac{31}{4} \pi \right) = \cos \left( \frac{32}{4} \pi - \frac{\pi}{4} \right) = \cos \left( 8 \pi - \frac{\pi}{4} \right) = \cos \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2}
\]

\[
\sin \left( \frac{31}{4} \pi \right) = \sin \left( \frac{32}{4} \pi - \frac{\pi}{4} \right) = \sin \left( 8 \pi - \frac{\pi}{4} \right) = \sin \left( \frac{-\pi}{4} \right) = -\sin \left( \frac{\pi}{4} \right) = -\frac{\sqrt{2}}{2}
\]

Altogether, we have

\[
-2 \cos \left( \frac{31}{4} \pi \right) \sin \left( \frac{31}{4} \pi \right) = -2 \left( \frac{\sqrt{2}}{2} \right) \left( -\frac{\sqrt{2}}{2} \right) = 1
\]

I have no idea why, but many people began this problem by writing \((\cos \theta - \sin \theta)^2 - 1 = 0\). There is no “equals” in the problem - you were asked to simplify an expression, not solve an equation. This mistake leaves me clueless.

**P3. (2pts.)** Find all values of \(\theta\) for which \(\sin^2 \theta = \frac{1}{2}\) and \(0 \leq \theta < 2\pi\).

\[
\sin^2 \theta = \frac{1}{2} \rightarrow \sin \theta = \pm \frac{1}{\sqrt{2}}
\]

The angles on the unit circle which have a sine of either positive or negative \(1/\sqrt{2}\) are \(\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\). There are several ways of seeing this, but the easiest is by drawing the \(\pi/4\) triangle, then reflecting it across the \(x\) and \(y\) axes into a "bowtie" shape. I’ll leave it up to you to verify this at home.

**P4. (1pt.)** If you had to write your autobiography today, what would you call it?

Finite Gruber Theory.
**P1. (5pts.)** Fill in all unlabeled angles and side lengths for each triangle. Show your work.

```
/\  π/3
\  /
\ / 2
```

```
/\    π/4
\  /
\ / 1
```

**P2. (2pts.)** Simplify \((\sin \theta - \cos \theta)^2 - 1\). Then find its value when \(\theta = \frac{19}{4}\pi\).

**P3. (2pts.)** Find all values of \(\theta\) for which \(\cos^2 \theta = \frac{1}{2}\) and \(0 \leq \theta < 2\pi\).

**P4. (1pt.)** If you had to write your autobiography today, what would you call it?
P1. (3pts.) Graph both $\sin(x)$ and $\cos(x + \pi/2)$ using the axes below.

\[\text{Graph both } \sin(x) \text{ and } \cos(x + \pi/2) \text{ using the axes below.}\]

P2. (3pts.) Rewrite as an algebraic expression. Then find the value when $x = 2$.

\[\csc(\tan^{-1}(x))\]

P3. (3pts.) One of Kanye West’s roadies is using a ten foot long board as a ramp to load equipment into the back of his fabulous pickup truck. The bed of the truck is five feet above the ground. What angle does the board make with the ground?

P4. (1pt.) Tell me a band / artist / musician I may not have heard about.
**P1. (3pts.)** Graph both $\sin(x)$ and $\cos(x + \pi/2)$ using the axes below. Phase shifts correspond to horizontal translations of the $\sin$ / $\cos$ graphs, so we get the following. The solutions for each quiz form were a bit different, so I included both graphs below.
P2. (3pts.) Rewrite as an algebraic expression. Then find the value when \(x = 2\).

\[ \csc \left( \tan^{-1} (x) \right) \]

Write \( \theta = \tan^{-1} (x) \), so that \( \tan \theta = x \). We know that for any triangle containing angle \( \theta \), the ratio of the opposite side to \( \theta \) and the adjacent side to \( \theta \) is \( x \). One such triangle is the following.

![Diagram of a right triangle with sides labeled \(x\), \(\sqrt{x^2 + 1}\), and \(\theta\).]

The hypotenuse \( \sqrt{x^2 + 1} \) is deduced using the Pythagorean theorem. So,

\[ \csc \theta = \frac{1}{\sin \theta} = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{x^2 + 1}}{x} \]

Evaluating this at \(x = 2\), we obtain \( \csc \left( \tan^{-1} (2) \right) = \frac{\sqrt{2^2 + 1}}{2} = \frac{\sqrt{5}}{2} \).

P3. (3pts.) One of Kanye West’s roadies is using a ten foot long board as a ramp to load equipment into the back of his fabulous pickup truck. The bed of the truck is five feet above the ground. What angle does the board make with the ground?

![Diagram of an isosceles triangle with sides labeled 10, 5, and \(\theta\).]

From the picture, we can see that

\[ \sin \theta = \frac{5}{10} = \frac{1}{2} \]

and thus

\[ \theta = \sin^{-1} \left( \frac{1}{2} \right) = \frac{\pi}{6} \]

P4. (1pt.) Tell me a band / artist / musician I may not have heard about.

Die Antwoord. Zef side represent.
P1. (3pts.) Graph both $\cos(x)$ and $\sin(x + \pi/2)$ using the axes below.

\[ \begin{array}{cc}
0 & 2\pi \\
\end{array} \]

P2. (3pts.) Rewrite as an algebraic expression. Then find the value when $x = 2$.

$$\sec(\tan^{-1}(x))$$

P3. (3pts.) One of Ronnie James Dio’s roadies is using a twelve foot long board as a ramp to load equipment into the back of his fabulous pickup truck. The bed of the truck is six feet above the ground. What angle does the board make with the ground?

P4. (1pt.) Tell me a band / artist / musician I may not have heard about.
P1. (3pts.) Simplify

\[
\frac{1}{1 - \cos \theta} + \frac{1}{1 + \cos \theta}
\]

P2. (3pts.) Verify the identity algebraically.

\[
\frac{1 + \sin \psi}{\cos \psi} = \frac{\cos \psi}{1 - \sin \psi}
\]

P3. (3pts.) Verify the identity algebraically.

\[
\frac{\tan^3 \phi - 1}{\tan \phi - 1} = \frac{\sin \phi \cos \phi + 1}{\cos^2 \phi}
\]

P4. (1pt.) Draw a turtle.
P1. (3pts.) Simplify

\[ \frac{1}{1 - \cos \theta} + \frac{1}{1 + \cos \theta} \]

Here is the game plan: we get a common denominator, add the fractions together, then FOIL out the new denominator (or use the \((a + b)(a - b) = a^2 - b^2\) identity), and apply the Pythagorean identity, \(\sin^2 \theta + \cos^2 \theta = 1\), to simplify.

\[
\frac{1}{1 - \cos \theta} + \frac{1}{1 + \cos \theta} = \frac{1}{1 - \cos \theta} \left( \frac{1 + \cos \theta}{1 + \cos \theta} \right) + \frac{1}{1 + \cos \theta} \left( \frac{1 - \cos \theta}{1 - \cos \theta} \right)
= \frac{1 + \cos \theta}{(1 - \cos \theta)(1 + \cos \theta)} + \frac{1 - \cos \theta}{(1 + \cos \theta)(1 - \cos \theta)}
= \frac{(1 + \cos \theta)(1 + \cos \theta) + (1 - \cos \theta)(1 - \cos \theta)}{2}
= \frac{2}{1 + \cos \theta} \cdot \frac{1 - \cos \theta}{\sin^2 \theta}
= \frac{2}{\sin^2 \theta}
= 2 \csc^2 \theta
\]

A few remarks:

- It makes no difference to me whether you answered \(\frac{2}{\sin^2 \theta}\) or \(2 \csc^2 \theta\).

- It’s worth mentioning that several people tried to solve the problem by changing \(\frac{1}{1 - \cos \theta}\) into \(\frac{1}{1 - \cos \theta}\). That doesn’t work. You can only break apart a fraction like that with the addition on top, i.e. \(a + b + c = \frac{a}{c} + b + \frac{c}{c}\), but \(\frac{c}{a + b} \neq \frac{c}{a} + \frac{c}{b}\).

- **Reminder:** Lately some of you have been seem to have forgotten my threats about distributing exponents across addition \((a + b)^2 \neq a^2 + b^2\). The ban on this is still in effect: if you did this, you got a 0 on the question. Quit breakin’ the law!
P2. (3pts.) Verify the identity algebraically.

\[
\frac{1 + \sin \psi}{\cos \psi} = \frac{\cos \psi}{1 - \sin \psi}
\]

To verify identities, you pick one side of the equality, then use algebraic manipulation and known identities to change it into the other side of the equality. It’s the same thing as what we do when simplifying (like in P1), except you’re told where you’re supposed to end up.

It doesn’t matter which side you start on, so I’ll show how you would do it for each. In either case, we conjugate, use the Pythagorean identity, then cancel.

(starting with the left hand side)

\[
\frac{1 + \sin \psi}{\cos \psi} = \frac{(1 + \sin \psi)(1 - \sin \psi)}{\cos \psi (1 - \sin \psi)}
= \frac{1 - \sin^2 \psi}{\cos \psi (1 - \sin \psi)}
= \frac{\cos^2 \psi}{\cos \psi (1 - \sin \psi)}
= \frac{\cos \psi}{1 - \sin \psi}
\]

(starting with the right hand side)

\[
\frac{\cos \psi}{1 - \sin \psi} = \frac{\cos \psi (1 + \sin \psi)}{(1 - \sin \psi)(1 + \sin \psi)}
= \frac{\cos \psi (1 + \sin \psi)}{1 - \sin^2 \psi}
= \frac{\cos \psi (1 + \sin \psi)}{\cos^2 \psi}
= \frac{1 + \sin \psi}{\cos \psi}
\]
P3. (3pts.) Verify the identity algebraically.

\[
\frac{\tan^3 \phi - 1}{\tan \phi - 1} = \frac{\sin \phi \cos \phi + 1}{\cos^2 \phi}
\]

This problem depends crucially on whether or not you remember your difference of cubes identity. The rest is just a Pythagorean identity and some algebra.

\[
\frac{\tan^3 \phi - 1}{\tan \phi - 1} = \frac{(\tan \phi - 1)(\tan^2 \phi + \tan \phi + 1)}{\tan \phi - 1}
\]

\[
= \tan^2 \phi + \tan \phi + 1
\]

\[
= \tan \phi + (\tan^2 \phi + 1)
\]

\[
= \tan \phi + \sec^2 \phi
\]

\[
= \frac{\sin \phi}{\cos \phi} + \frac{1}{\cos^2 \phi}
\]

\[
= \frac{\sin \phi \cos \phi}{\cos^2 \phi} + \frac{1}{\cos^2 \phi}
\]

\[
= \frac{\sin \phi \cos \phi + 1}{\cos^2 \phi}
\]

P4. (1pt.) Draw a turtle.

I totally drew this with my laptop trackpad in MS Paint just now. I certainly didn’t steal it off the internet.
P1. (3pts.) Simplify
\[ \frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} \]

P2. (3pts.) Verify the identity algebraically.
\[ \frac{1 + \cos \psi}{\sin \psi} = \frac{\sin \psi}{1 - \cos \psi} \]

P3. (3pts.) Verify the identity algebraically.
\[ \frac{\cot^2 \phi - 1}{\cot \phi - 1} = \frac{\sin \phi \cos \phi + 1}{\sin^2 \phi} \]

P4. (1pt.) Draw an octopus.