P1. (3pts.) Simplify

$$\frac{1}{1 - \cos \theta} + \frac{1}{1 + \cos \theta}$$

Here is the game plan: we get a common denominator, add the fractions together, then FOIL out the new denominator (or use the \((a + b)(a - b) = a^2 - b^2\) identity), and apply the Pythagorean identity, \(\sin^2 \theta + \cos^2 \theta = 1\), to simplify.

\[
\begin{align*}
\frac{1}{1 - \cos \theta} + \frac{1}{1 + \cos \theta} &= \frac{1}{1 - \cos \theta} \left(\frac{1 + \cos \theta}{1 + \cos \theta}\right) + \frac{1}{1 + \cos \theta} \left(\frac{1 - \cos \theta}{1 - \cos \theta}\right) \\
&= \frac{1 + \cos \theta}{(1 - \cos \theta)(1 + \cos \theta)} + \frac{1 - \cos \theta}{(1 + \cos \theta)(1 - \cos \theta)} \\
&= \frac{1 + \cos \theta + 1 - \cos \theta}{2} \\
&= \frac{2}{(1 + \cos \theta)(1 - \cos \theta)} \\
&= \frac{2}{1 - \cos^2 \theta} \\
&= \frac{2}{\sin^2 \theta} \\
&= 2 \csc^2 \theta
\end{align*}
\]

A few remarks:

- It makes no difference to me whether you answered \(\frac{2}{\sin^2 \theta}\) or \(2 \csc^2 \theta\).

- It’s worth mentioning that several people tried to solve the problem by changing \(\frac{1}{1 - \cos \theta}\) into \(\frac{1}{1 - \cos \theta}\). That doesn’t work. You can only break apart a fraction like that with the addition on top, i.e. \(\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}\), but \(\frac{c}{a+b} \neq \frac{c}{a} + \frac{c}{b}\).

- **Reminder:** Lately some of you have been seem to have forgotten my threats about distributing exponents across addition \((a + b)^2 \neq a^2 + b^2\). The ban on this is still in effect: if you did this, you got a 0 on the question. Quit breakin’ the law!
P2. (3pts.) Verify the identity algebraically.

\[
\frac{1 + \sin \psi}{\cos \psi} = \frac{\cos \psi}{1 - \sin \psi}
\]

To verify identities, you pick one side of the equality, then use algebraic manipulation and known identities to change it into the other side of the equality. It’s the same thing as what we do when simplifying (like in P1), except you’re told where you’re supposed to end up.

It doesn’t matter which side you start on, so I’ll show how you would do it for each. In either case, we conjugate, use the Pythagorean identity, then cancel.

*(starting with the left hand side)*

\[
\frac{1 + \sin \psi}{\cos \psi} = \frac{(1 + \sin \psi)(1 - \sin \psi)}{\cos \psi (1 - \sin \psi)}
\]

\[
= \frac{1 - \sin^2 \psi}{\cos \psi (1 - \sin \psi)}
\]

\[
= \frac{\cos^2 \psi}{\cos \psi (1 - \sin \psi)}
\]

\[
= \frac{\cos \psi}{1 - \sin \psi}
\]

*(starting with the right hand side)*

\[
\frac{\cos \psi}{1 - \sin \psi} = \frac{\cos \psi (1 + \sin \psi)}{(1 - \sin \psi)(1 + \sin \psi)}
\]

\[
= \frac{\cos \psi (1 + \sin \psi)}{\cos \psi (1 + \sin \psi)}
\]

\[
= \frac{1 - \sin^2 \psi}{\cos \psi (1 + \sin \psi)}
\]

\[
= \frac{\cos^2 \psi}{\cos \psi}
\]

\[
= \frac{1 + \sin \psi}{\cos \psi}
\]
P3. (3pts.) Verify the identity algebraically.

\[
\frac{\tan^3 \phi - 1}{\tan \phi - 1} = \frac{\sin \phi \cos \phi + 1}{\cos^2 \phi}
\]

This problem depends crucially on whether or not you remember your difference of cubes identity. The rest is just a Pythagorean identity and some algebra.

\[
\frac{\tan^3 \phi - 1}{\tan \phi - 1} = \frac{(\tan \phi - 1)(\tan^2 \phi + \tan \phi + 1)}{\tan \phi - 1} = \tan^2 \phi + \tan \phi + 1 = \tan \phi + (\tan^2 \phi + 1) = \tan \phi + \sec^2 \phi = \sin \phi \cos \phi + \frac{1}{\cos^2 \phi} = \sin \phi \cos \phi + \frac{1}{\cos^2 \phi} = \frac{\sin \phi \cos \phi + 1}{\cos^2 \phi}
\]

P4. (1pt.) Draw a turtle.

I totally drew this with my laptop trackpad in MS Paint just now. I certainly didn’t steal it off the internet.