1. Consider an economy of identical individuals with preferences given by the utility function

\[ U(x_1, x_2, x_3) = x_1 + \ln(x_2 - a_2) + \ln(x_3 - a_3) \]

Pre-tax prices of all three goods are normalized to one. Individuals supply good 1 (labor) \((x_1 < 0)\) and consume goods 2 and 3. The government can impose \textit{ad valorem} taxes on goods 2 and 3 at rates \(\tau_2 = t_2 / p_2\) and \(\tau_3 = t_3 / p_3\) to raise \(R\) dollars to meet its revenue requirements.

a) Obtain an expression for the indirect utility function as a function of the tax rates. Solve for the consumer’s demands and write down the government budget constraint.

b) Find the first order necessary conditions for the optimal tax rates when \(a_2 = a_3\). Are the tax rates equal?

c) Suppose \(a_3 < 0\) (the consumer has an initial endowment of good 3). Do the relative tax rates depend on the sign of \(a_2\) or on the sign of \(a_2 - a_3\)? Give any intuition for this result you might have.

Pre-tax prices normalized \(\Rightarrow p_1 = p_2 = p_3 = 1\)

Taxes: \(t_i = q_i - p_i \quad (\forall i)\) and \(\tau_j = t_j / p_j = t_j \quad (j = 2, 3) \Rightarrow q_j = \tau_j + 1\)

\(q_1\) untaxed \(\therefore q_1 = 1\)

(a) Consumer problem: \(\max_{x_1,x_2,x_3} U = x_1 + \ln(x_2 - a_2) + \ln(x_3 - a_3) \quad \text{s.t.} \quad q \cdot x = 0\)

Lagrangian: \(\ell = x_1 + \ln(x_2 - a_2) + \ln(x_3 - a_3) - \gamma(q_1x_1 + q_2x_2 + q_3x_3)\)

FOCs: \(U(x_1, x_2, x_3)\) guarantees \(x_2, x_3 > 0\); given \(x_1 < 0\) \(\therefore\) interior solution

\[
\begin{align*}
\frac{\partial \ell}{\partial x_1} &= 1 - \gamma q_1 = 0 \quad \Rightarrow \quad \gamma = \frac{1}{q_1} \quad [1] \\
\frac{\partial \ell}{\partial x_2} &= \frac{1}{x_2 - a_2} - \gamma q_2 = 0 \quad \Rightarrow \quad \gamma = \frac{1}{q_2(x_2 - a_2)} \quad [2] \\
\frac{\partial \ell}{\partial x_3} &= \frac{1}{x_3 - a_3} - \gamma q_3 = 0 \quad \Rightarrow \quad \gamma = \frac{1}{q_3(x_3 - a_3)} \quad [3] \\
-\frac{\partial \ell}{\partial \gamma} &= q_1x_1 + q_2x_2 + q_3x_3 = 0 \quad [4]
\end{align*}
\]

Sub [1] using \(q_1 = 1\) into [2] and [3]; solve for \(x_2\) and \(x_3\):

\[
\begin{align*}
1 &= \frac{1}{q_2(x_2 - a_2)} \quad \Rightarrow \quad q_2(x_2 - a_2) = 1 \quad \Rightarrow \quad x_2 = \frac{1 + a_2q_2}{q_2} = \frac{1}{q_2} + a_2 \\
1 &= \frac{1}{q_3(x_3 - a_3)} \quad \Rightarrow \quad q_3(x_3 - a_3) = 1 \quad \Rightarrow \quad x_3 = \frac{1 + a_3q_3}{q_3} = \frac{1}{q_3} + a_3
\end{align*}
\]

Sub these into [4]; solve for \(x_1\):
\[(1) x_1 + q_2 \left(\frac{1 + a_2 q_2}{q_2}\right) + q_3 \left(\frac{1 + a_3 q_3}{q_3}\right) = 0 \]
\[x_1 + (1 + a_2 q_2) + (1 + a_3 q_3) = 0 \quad \Rightarrow \quad x_1 = -(2 + a_2 q_2 + a_3 q_3)\]

Sub \( q_j = \tau_j + 1 \ (j = 2, 3) \) to get consumer demands in terms of tax rates:

\[
\begin{align*}
  x_1(\tau_2, \tau_3) &= -(2 + a_2 (\tau_2 + 1) + a_3 (\tau_3 + 1)) \\
  x_2(\tau_2, \tau_3) &= \frac{1}{\tau_2 + 1} + a_2 \\
  x_3(\tau_2, \tau_3) &= \frac{1}{\tau_3 + 1} + a_3
\end{align*}
\]

Solve for \( V(\tau_2, \tau_3) = U(x(\tau_2, \tau_3)) \) to get

\[
V(\tau_2, \tau_3) = -(2 + a_2 (\tau_2 + 1) + a_3 (\tau_3 + 1)) + \ln\left(\frac{1}{\tau_2 + 1} + a_2 - a_2\right) + \ln\left(\frac{1}{\tau_3 + 1} + a_3 - a_3\right)
\]

Government Budget Constraint:
\[\tau_2 x_2 + \tau_3 x_3 \geq R\]

Plug in equilibrium values for \( x_2 \) and \( x_3 \):

\[
\tau_2 \left(\frac{1}{\tau_2 + 1} + a_2\right) + \tau_3 \left(\frac{1}{\tau_3 + 1} + a_3\right) \geq R
\]

(b) \[\max_{\tau_2, \tau_3} V(\tau_2, \tau_3) = -(2 + a_2 (\tau_2 + 1) + a_3 (\tau_3 + 1)) + \ln\left(\frac{1}{\tau_2 + 1}\right) + \ln\left(\frac{1}{\tau_3 + 1}\right)\]

s.t. \[\tau_2 \left(\frac{1}{\tau_2 + 1} + a_2\right) + \tau_3 \left(\frac{1}{\tau_3 + 1} + a_3\right) \geq R\]

Lagrangian:
\[
L = -(2 + a_2 (\tau_2 + 1) + a_3 (\tau_3 + 1)) + \ln\left(\frac{1}{\tau_2 + 1}\right) + \ln\left(\frac{1}{\tau_3 + 1}\right) - \\
\lambda \left[\tau_2 \left(\frac{1}{\tau_2 + 1} + a_2\right) + \tau_3 \left(\frac{1}{\tau_3 + 1} + a_3\right) - R\right]
\]

FOCs:
\[
\frac{\partial L}{\partial \tau_2} = -a_2 - \frac{1}{\tau_2 + 1} - \lambda \left[\frac{1}{\tau_2 + 1} + a_2 - \frac{\tau_2}{(\tau_2 + 1)^2}\right] = 0
\]
\[
\frac{\partial L}{\partial \tau_3} = -a_3 - \frac{1}{\tau_3 + 1} - \lambda \left[ \frac{1}{\tau_3 + 1} + a_3 - \frac{\tau_3}{(\tau_3 + 1)^2} \right] = 0
\]
\[
- \frac{\partial L}{\partial \lambda} = \tau_2 \left( \frac{1+a_2(\tau_2 + 1)}{\tau_2 + 1} \right) + \tau_3 \left( \frac{1+a_3(\tau_3 + 1)}{\tau_3 + 1} \right) - R = 0
\]

Take FOC for \( \tau_2 \); pull \( a_2 \) out of brackets and get common denominator
\[
\frac{\partial L}{\partial \tau_2} = -a_2 - \frac{1}{\tau_2 + 1} - \lambda a_2 - \lambda \left[ \frac{\tau_2 + 1}{(\tau_2 + 1)^2} - \frac{\tau_2}{(\tau_2 + 1)^2} \right] = 0
\]

Combine \( a_2 \) terms; get common denominator for second and third terms
\[
-(1+\lambda)a_2 - \frac{\tau_2 + 1}{(\tau_2 + 1)^2} - \frac{\lambda}{(\tau_2 + 1)^2} = 0
\]

Solve for \( a_2 \)
\[
-(1+\lambda)a_2 = \frac{\tau_2 + 1 + \lambda}{(\tau_2 + 1)^2} \quad \Rightarrow \quad a_2 = \frac{\tau_2 + 1 + \lambda}{(1+\lambda)(\tau_2 + 1)^2}
\]

FOC for \( a_3 \) mirrors this derivation: \( a_3 = \frac{\tau_3 + 1 + \lambda}{-(1+\lambda)(\tau_3 + 1)^2} \)

If we have \( a_2 = a_3 \), these terms must be equal:
\[
\frac{\tau_2 + 1 + \lambda}{(\tau_2 + 1)^2} = \frac{\tau_3 + 1 + \lambda}{(\tau_3 + 1)^2}
\]

Cancel the \( -(1+\lambda) \) from both terms
\[
\frac{\tau_2 + 1 + \lambda}{(\tau_2 + 1)^2} = \frac{\tau_3 + 1 + \lambda}{(\tau_3 + 1)^2}
\]

\( f(\tau_2) = f(\tau_3) \)

Rather than make fancy mathematical arguments, I plugged this into Excel using \( \lambda = 0 \) and \( \lambda = 1 \). The graph with \( \lambda = 0 \) shows the only way \( f(\tau_2) = f(\tau_3) \) is for \( \tau_2 = \tau_3 \), however for any \( \lambda > 0 \) it is possible to have \( f(\tau_2) = f(\tau_3) \) with \( \tau_2 \neq \tau_3 \). In these situations, the tax rates will have opposite signs.

\[
\therefore \text{if} \quad \lambda = 0 \quad (\text{the budget constraint is not binding}) \quad \text{or if} \quad \lambda > 0 \quad \text{and the tax rates have the same sign, then the tax rates are equal.} \quad \text{(In the remaining case,} \quad \lambda > 0 \quad \text{and the tax rates have different signs, obviously the tax rates are different.)}
\]
(c) Looking at the consumer’s demands from part (a), if \( a_3 < 0 \) then \( x_3 \) is most likely negative (doesn’t have to be, but it will be less than if \( a_3 > 0 \)). This frees up resources in the consumer’s budget constraint which, based on the consumer’s objective function will be put to better use on good 2, assuming \( a_2 > a_3 \). The point at which \( \text{MRS}_{1,2} = \text{MRS}_{1,3} \) will be determined by the difference between \( a_2 \) and \( a_3 \). This will determine the demands which then will drive the tax rates in order to satisfy the government’s budget constraint. Therefore, the relative tax rates depend on the sign of \( a_2 - a_3 \).

2. Consider Diamond and Mirrlees’s AER papers on optimal commodity taxation. Optimal taxation and productive efficiency are linked together as principles of intervention. Suppose, unlike the traditional model, that both goods 1 and 2 are untaxable—as a prior restriction.

a) To keep it simple, consider a one-consumer economy with 4 goods \( (q_1 = p_1 = 1, \quad q_2 = p_2 \) from the tax restrictions). Start from their original problem (equation (15)) and work through to get an optimization problem analogous to (17). What impact does the constraint that \( t_2 = 0 \) have?

b) Show that productive efficiency is not desirable in this economy.

(a) Original problem:

\[
\begin{align*}
\text{max} & \quad V(q_1, q_2, q_3, q_4) \\
\text{s.t.} & \quad x_i(q) - y_i - z_i = 0, \quad i = 1,2,3,4 \quad (1) \text{market clearing} \\
& \quad y = \arg \max \mathbf{p} \cdot \mathbf{y} \text{ s.t. } y_i = f(y_2, y_3, y_4) \quad (2) \text{private production efficiency} \\
& \quad z_i = g(z_2, z_3, z_n) \quad (3) \text{public production efficiency} \\
& \quad q_2 = p_2 \quad (4) \text{no tax on good 2}
\end{align*}
\]

Use market clearing to solve for private production quantities:

\[
y_i = x_i(q) - z_i = 0, \quad i = 2,3,4
\]

Sub (2) and (3) into market clearing constraint for good 1:

\[
x_i(q) = y_i + z_i = f(y_2, y_3, y_4) + g(z_2, z_3, z_n)
\]

Sub the values for \( y_2, y_3, y_4 \) from the first step:

\[
x_i(q) = f(x_2(q) - z_2, x_3(q) - z_3, x_4(q) - z_4) + g(z_2, z_3, z_n)
\]

Private production efficiency FOCs:

\[
p_i - \lambda = 0 \quad \text{and} \quad p_i - \lambda \frac{\partial f}{\partial y_i} = 0 \quad (i = 2,3,4)
\]

Let \( f_i = \frac{\partial f}{\partial y_i} \) so we have \( p_i = p_1 f_i \) \((i = 2,3,4)\)

\[
\therefore q_2 = p_2 \text{ becomes } q_2 = f_2
\]

New problem:
\[
\begin{align*}
\text{max } V(q) & \quad \text{s.t.} \\
\text{s.t. } x_1(q) &= f(x_2(q) - z_2, x_3(q) - z_3, x_4(q) - z_4) + g(z_2, z_3, z_n) \\
q_2 &= f_2 \\
\end{align*}
\]

The restriction that \( t_2 = 0 \) means there is an additional constraint \( q_2 = f_2 \) which essentially eliminates a decision variable.

(b) Lagrangian:
\[
\ell = V(q) - \lambda \left[ x_1(q) - f(x_2(q) - z_2, x_3(q) - z_3, x_4(q) - z_4) + g(z_2, z_3, z_n) \right] - \mu [q_2 - f_2]
\]

FOCs:
\[
\frac{\partial \ell}{\partial z_k} = -\lambda [f_k - g_k] - \mu \frac{\partial f_2}{\partial z_k} = 0
\]

Assuming an interior solution, we can solve for \( f_k: f_k = g_k - \frac{\mu}{\lambda} \frac{\partial f_2}{\partial z_k} \)

If we assume the private production function \( f \) is twice differentiable and nonlinear in \( y \) (so the \( \frac{\partial f_2}{\partial z_k} \) exists and is \( \neq 0 \)), then \( f_k \neq g_k \) (marginal rate of transformation in private and public sectors are not equal) so we do not have aggregate production efficiency.

3. Go back to Kay’s paper and derive the optimal tax formulas by minimizing excess burden where the utility level in the expenditure function is indirect utility \( v(q, 0) \). Explain why using the compensating variation in this approach will not yield the same results.

Excess burden using equivalent variation:
\[
\Delta L = E(q, u_t) - E(p, u_t) - (q - p) \cdot x(q)
\]

Substitute indirect utility:
\[
\Delta L = E(q, v(q, 0)) - E(p, v(q, 0)) - (q - p) \cdot x(q)
\]

Assume pre-tax prices \( (p) \) are fixed (Hamilton told us to)

Assume government budget constraint is equality (like Kay did)

Problem: \( \min_q \Delta L = E(q, v(q, 0)) - E(p, v(q, 0)) - (q - p) \cdot x(q) \quad \text{s.t.} \quad (q - p) \cdot x(q) = R \)

Lagrangian:
\[
\ell = E(q, v(q, 0)) - E(p, v(q, 0)) - (q - p) \cdot x(q) + \lambda ((q - p) \cdot x(q) - R)
\]

Note: setting it up with +\( \lambda \) rather than the standard −\( \lambda \) in order to get the Diamond and Mirrlees optimal commodity tax rule

FOCs:
\[
\frac{\partial \ell}{\partial q_k} = \frac{\partial E(q, v(q, 0))}{\partial q_k} + \frac{\partial E(q, v(q, 0))}{\partial v} \frac{\partial v(q, 0)}{\partial q_k} + \frac{\partial E(p, v(q, 0))}{\partial v} \frac{\partial v(q, 0)}{\partial q_k} \]

\[
- x_k(q) - \sum_{i=1}^{n} (q_i - p_i) \frac{\partial x_i(q)}{\partial q_k} + \lambda \left\{ x_k(q) + \sum_{i=1}^{n} (q_i - p_i) \frac{\partial x_i(q)}{\partial q_k} \right\} = 0
\]

Note (courtesy of micro notes):
\[
\frac{\partial E(q, v(q, 0))}{\partial q_k} = x_k^*(q, v(q, 0)) = x_k(q)
\]
\[ \sum_{i=1}^{n} (q_i - p_i) \frac{\partial x_i(q)}{\partial q_k} = \lambda \left( x_k(q) + \sum_{i=1}^{n} (q_i - p_i) \frac{\partial x_i(q)}{\partial q_k} \right) \]

Sub in \( t_i = q_i - p_i \) and this is the same as (3'') on p.115 of the Kay article:

\[ \sum_{i=1}^{n} t_i \frac{\partial x_i(q)}{\partial q_k} = \lambda \left( x_k(q) + \sum_{i=1}^{n} t_i \frac{\partial x_i(q)}{\partial q_k} \right) \]

Excess burden using compensating variation and indirect utility:

\[ \tilde{L} = E(q,v(p,0)) - E(p,v(p,0)) - (q - p) \cdot x(q) \]

Problem: \( \min_q \tilde{L} = E(q,v(p,0)) - E(p,v(p,0)) - (q - p) \cdot x(q) \text{ s.t. } (q - p) \cdot x(q) = R \)

Lagrangian: \( \hat{L} = E(q,v(p,0)) - E(p,v(p,0)) - (q - p) \cdot x(q) - \gamma ((q - p) \cdot x(q) - R) \)

FOCs:

\[ \frac{\partial \hat{L}}{\partial q_k} = \frac{\partial E(q,v(p,0))}{\partial q_k} - x_k(q) - \sum_{i=1}^{n} (q_i - p_i) \frac{\partial x_i(q)}{\partial q_k} - \gamma \left( x_k(q) + \sum_{i=1}^{n} (q_i - p_i) \frac{\partial x_i(q)}{\partial q_k} \right) \]

This time \( \frac{\partial E(q,v(p,0))}{\partial q_k} = x_k(q,v(p,0)) \neq x_k(q) \)

Using compensated variation does not yield the same result because the indirect utility is evaluated at pre-tax prices and will not change with respect to post-tax prices (i.e., they're independent of the tax).

**Documentation.**

Prob 1. I set up part (a) on my own and got confirmation from everyone that it was correct. Josh showed me a better way to simplify the demands to make part (b) easier. JC and I tried to get a solution using Mathematica, but it didn't work. The trick in part (b) of solving for \( a_2 \) and \( a_3 \) came from Nick (by way of Christine and Katie). I talked to everyone about part (c), but never really felt comfortable with what I heard. I tried various numerical solutions in Excel, but the problem is very unstable and small changes in parameters caused huge changes so I wasn't able to get a "feel" for what was going on.

Prob 2. Prof. Hamilton confirmed the basics of the problem (going from (15) to (17) was essentially unchanged and we just have a new constraint). Christine pointed out that the aggregate production efficiency result came from the derivative wrt \( z_k \). I checked my results in JC and Josh.

Prob 3. Christine and Prof. Hamilton both pointed me to the optimal tax formulas (3'') in the Kay article. Prof. Hamilton said to use regular demands for all the tax revenue computations. He practically did the whole problem for us because we bugged him with so many questions.