1. Determine the consequences of distributing a fixed total amount of income \( Y \) to maximize the following SWF:

\[
W = \beta_1 y_1^\alpha + \beta_2 y_2^\alpha \quad \text{where} \quad y_1 + y_2 = Y \quad \text{and} \quad \beta_i > 0 \quad (i = 1, 2)
\]

Consider cases:

a) \( \alpha < 0 \)

b) \( \alpha = 0 \)

c) \( 0 < \alpha < 1 \)

d) \( \alpha \geq 1 \)

Setup optimization problem:

\[
\max_{y_1, y_2} W = \beta_1 y_1^\alpha + \beta_2 y_2^\alpha \quad \text{s.t.} \quad y_1 + y_2 = Y
\]

The constraint is linear so second order condition is OK. Concavity of the objective function depends on \( \alpha \).

Lagrangian:

\[
L = \beta_1 y_1^\alpha + \beta_2 y_2^\alpha - \lambda (y_1 + y_2 - Y)
\]

(1) \( \frac{\partial L}{\partial y_1} = \alpha \beta_1 y_1^{\alpha-1} - \lambda \leq 0 \), with equality if \( y_1 > 0 \)

(2) \( \frac{\partial L}{\partial y_2} = \alpha \beta_2 y_2^{\alpha-1} - \lambda \leq 0 \), with equality if \( y_2 > 0 \)

(3) \( -\frac{\partial L}{\partial \lambda} = y_1 + y_2 - Y \leq 0 \), with equality if \( \lambda > 0 \)

a) If \( \alpha < 0 \), then we essentially have welfare equal to each \( \beta_i \) divided by some power of it's respective \( y_i \):

\[
W = \frac{\beta_1}{y_1^{\alpha}} + \frac{\beta_2}{y_2^{\alpha}}
\]

Since neither \( \beta_i \) nor \( y_i \) can be negative, as \( y_i \) increases \( W \) decreases. Therefore, maximizing \( W \) requires setting each \( y_i \) to zero (which is mathematically impossible because of division by zero).

b) \( W = \beta_1 y_1^0 + \beta_2 y_2^0 = \beta_1 + \beta_2 \). Therefore, \( W \) is independent of income so the distribution doesn't matter.

c) Assume an interior solution.

We can solve (1) and (2) for \( \lambda \) and set them equal to each other:
\[ \lambda = \alpha \beta_1 y_1^{\alpha-1} = \alpha \beta_2 y_2^{\alpha-1} \]

Cancel the \( \alpha \) and solve for the income ratio (equivalent to distribution):

\[
\left( \frac{y_1}{y_2} \right)^{\alpha-1} = \frac{\beta_2}{\beta_1} \Rightarrow \frac{y_1}{y_2} = \left( \frac{\beta_2}{\beta_1} \right)^{\frac{1}{\alpha-1}}
\]

Since the constraint is linear, we know it will be binding. We can use the income ratio above to solve for \( y_1 \) as a function of \( y_2 \) in order to determine the amount of income each individual gets in terms of total income:

\[
Y = y_1 + y_2 = y_2 \left( \frac{\beta_2}{\beta_1} \right)^{\frac{1}{\alpha-1}} + y_2 = \left[ 1 + \left( \frac{\beta_2}{\beta_1} \right)^{\frac{1}{\alpha-1}} \right] y_2 \Rightarrow y_2 = \frac{Y}{1 + \left( \frac{\beta_2}{\beta_1} \right)^{\frac{1}{\alpha-1}}}
\]

We could also solve for \( y_1 \) in terms of total income, but that's overkill (and doesn't really add much).

Check that ratio using numbers: \( \beta_1 = 2, \beta_2 = 1, \alpha = 0.5, Y = 10 \)

According to the ratio above, we should have \[ \frac{y_1}{y_2} = \left( \frac{\beta_2}{\beta_1} \right)^{\frac{1}{\alpha-1}} = \left( \frac{1}{2} \right)^{0.5-1} = 4 \]

Excel Solver says... \( y_1 = 8, y_2 = 2 \) :: \[ \frac{y_1}{y_2} = \frac{8}{2} = 4 \]

d) Note in the ratio from part (c), if \( \alpha = 1 \) we get division by zero so we're probably not at an interior solution. In fact, at \( \alpha = 1 \) we have \( W = \beta_1 y_1 + \beta_2 y_2 \). In this case social welfare is maximized by giving all income to a single individual (the one with a larger \( \beta \)).

In the case were \( \beta_1 = \beta_2 \), we have the same result as part (b): income distribution doesn't matter.

These results also hold for \( \alpha > 1 \) because the objective function is not convex and we have corner solutions rather than a tangency condition.
2. Consider the following economy with two people and two goods which are produced at no resource cost. Fortunately, the people become satiated with the goods, so that optima exist. The two public goods are the room temperature in degrees Celsius \((T)\) and the number of games of cards played per day \((C)\). Anne and Bruce are the people in this economy. Preferences are described as follows:

\[
U^a(C,T) = -[(C - 20)^2 + (T - 25)^2]
\]
\[
U^b(C,T) = -[(C - 10)^2 + (T - 15)^2]
\]

a) Sketch the indifference curves on a diagram.
b) Is \(C = 10\), \(T = 15\) Pareto optimal?
c) Find the set of Pareto optima. (Plane geometry may help more than solving the constrained optimization problem.)
d) Find the set of allocations that are Pareto superior to \(C = 9\), \(T = 14\).
e) Find a tangency between the indifference curves which is not Pareto optimal. What is going on here?

\[
\{(C,T) \mid T = C + 5, C \in [10,20]\}
\]

(the blue line in the picture)
e) The point in (d) satisfies that condition: at (9,14) \( u^b = -2 \) and \( u^a = -242 \) are tangent, but this point is not part of the Pareto optimal set defined in (c). Preferences in this example violate the assumption of local nonsatiation so we don't have "well behaved" indifference curves and the usual tangency rule doesn't apply. "Well behaved" assumes better-than-sets don't intersect and in this case they do.

3. Suppose a social planner wishes to maximize a Bergson-Samuelson social welfare function:

\[
W(U^a, U^b)
\]

where

\[
U^a = U^a(x_a, I_a, v^a(x_a)), \quad U^b = U^b(x_b, I_b, v^a(x_a))
\]

\( x_a \) and \( x_b \) are the consumption vectors of \( a \) and \( b \)

and \( I_a \) and \( I_b \) are labor supplies of \( a \) and \( b \)

Assume that

\[
\frac{\partial U^i}{\partial x_{ik}} = \lambda^i_j \frac{\partial U^i}{\partial x_{ik}} \text{ for all } x_{ik}, \ i, j = a, b, \ i \neq j
\]

Note that \( i \) and \( j \) index people, while \( k \) indexes goods.

Another way to think of this condition is:

\[
\frac{\partial v^i}{\partial x_{ik}} = \tilde{\lambda}^i_j \frac{\partial U^i}{\partial x_{ik}} \text{ for all } x_{ik}, \ i, j = a, b, \ i \neq j
\]

a) Show that the allocation which maximizes social welfare has equal marginal rates of substitution across consumers for the commodities (the \( x \)'s)

b) Verify that this optimal allocation is Pareto efficient. Be careful to define Pareto efficiency in this economy.

c) Suppose marginal costs of all consumption goods are constant, labor is the only factor of production, and both individuals sell labor to buy goods at marginal cost. The individuals differ in labor endowments, utility of leisure, or productivity (or all three), so that equal consumption is not the competitive equilibrium. Will the competitive
equilibrium be Pareto efficient? If not, what government policy is needed to sustain a Pareto optimal allocation?

Define the feasible region by letting $F(X, L)$ be the production function, where $x_a + x_b = X$ and $l_a + l_b = L$. This gives the constraint:

$$F(X, L) \leq 0$$

This constraint ensures that what is consumed by the two individuals can actually be produced (or purchased) with the amount of labor they can supply. Another way to do it is to define $Y$ as the total amount of commodities the consumers can jointly purchase given their endowment of labor. Therefore, the constraint becomes

$$x_a + x_b \leq Y$$

$$F(Y, L) \leq 0$$

a) Maximize social welfare:

$$\max_{x_a, x_b, Y} W(U^a, U^b) \quad \text{s.t.} \quad x_a + x_b = Y \quad \text{and} \quad F(Y, L) \leq 0$$

Lagrangian:

$$L = W(U^a, U^b) - \mu(x_a + x_b - Y) - \gamma(F(Y, L))$$

Assume an interior solution and look at first order conditions: (corner solutions may not satisfy the equal marginal rates of substitution property)

First look at person $a$’s consumption of commodity $i$:

$$\frac{\partial L}{\partial x_{ai}} = \frac{\partial W(U^a, U^b)}{\partial x_{ai}} - \mu_i = 0$$

Note:

$$\frac{\partial W(U^a, U^b)}{\partial x_{ai}} = \frac{\partial W}{\partial U^a} \frac{\partial U^a}{\partial x_{ai}} + \frac{\partial W}{\partial U^b} \frac{\partial U^b}{\partial x_{ai}}$$

Substitute

$$\frac{\partial U^b}{\partial x_{ai}} = \lambda_{ab} \frac{\partial U^a}{\partial x_{ai}}$$

Combine terms:

$$\frac{\partial W}{\partial x_{ai}} = \frac{\partial U^a}{\partial x_{ai}} \left( \frac{\partial W}{\partial U^a} + \lambda_{ab} \frac{\partial W}{\partial U^b} \right)$$

So we have:

$$\frac{\partial L}{\partial x_{ai}} = \frac{\partial U^a}{\partial x_{ai}} \left( \frac{\partial W}{\partial U^a} + \lambda_{ab} \frac{\partial W}{\partial U^b} \right) - \mu_i = 0$$

Person $a$’s consumption of commodity $j$:

$$\frac{\partial L}{\partial x_{aj}} = \frac{\partial U^a}{\partial x_{aj}} \left( \frac{\partial W}{\partial U^a} + \lambda_{ab} \frac{\partial W}{\partial U^b} \right) - \mu_j = 0$$

Repeat the math for person $b$’s consumption of commodities $i$ and $j$:

$$\frac{\partial L}{\partial x_{pi}} = \frac{\partial U^b}{\partial x_{pi}} \left( \lambda_{ba} \frac{\partial W}{\partial U^a} + \frac{\partial W}{\partial U^b} \right) - \mu_i = 0$$

$$\frac{\partial L}{\partial x_{pj}} = \frac{\partial U^b}{\partial x_{pj}} \left( \lambda_{ba} \frac{\partial W}{\partial U^a} + \frac{\partial W}{\partial U^b} \right) - \mu_j = 0$$
\[ \frac{\partial L}{\partial x_{ij}} = \frac{\partial U^b}{\partial x_{bj}} \left( \lambda_{ba} \frac{\partial W}{\partial U^a} + \frac{\partial W}{\partial U^b} \right) - \mu_j = 0 \]

Solve for the respective Lagrange multipliers:

\[ \frac{\partial U^a}{\partial x_{ai}} \left( \frac{\partial W}{\partial U^a} + \lambda_{ab} \frac{\partial W}{\partial U^b} \right) = \mu_i \]

\[ \frac{\partial U^b}{\partial x_{bi}} \left( \lambda_{ba} \frac{\partial W}{\partial U^a} + \frac{\partial W}{\partial U^b} \right) = \mu_i \]

\[ \frac{\partial U^a}{\partial x_{aj}} \left( \frac{\partial W}{\partial U^a} + \lambda_{ab} \frac{\partial W}{\partial U^b} \right) = \mu_j \]

\[ \frac{\partial U^b}{\partial x_{bj}} \left( \lambda_{ba} \frac{\partial W}{\partial U^a} + \frac{\partial W}{\partial U^b} \right) = \mu_j \]

Divide the terms in the first column to get \( MRS_{ij}^a \):

\[ \frac{\partial U^a}{\partial x_{ai}} \left( \frac{\partial W}{\partial U^a} + \lambda_{ab} \frac{\partial W}{\partial U^b} \right) = \frac{\partial U^a}{\partial x_{ai}} = \frac{\mu_i}{\mu_j} = MRS_{ij}^a \]

Divide the terms in the second column to get \( MRS_{ij}^b \):

\[ \frac{\partial U^b}{\partial x_{bi}} \left( \lambda_{ba} \frac{\partial W}{\partial U^a} + \frac{\partial W}{\partial U^b} \right) = \frac{\partial U^b}{\partial x_{bj}} = \frac{\mu_i}{\mu_j} = MRS_{ij}^b \]

So \( MRS_{ij}^a = MRS_{ij}^b \)

b) Setup the Pareto problem by having person \( a \) maximize his utility given some arbitrary level of utility for person \( b \):

\[ \max_{x_a, x_b, Y} U^a \quad \text{s.t.} \quad U^b \geq \bar{u}, \ x_a + x_b = Y, \ \text{and} \ F(Y, L) \leq 0 \]

Lagrangian: \[ L = U^a - \phi(U^b - \bar{u}) - \lambda(x_a + x_b - Y) - \gamma(F(Y, L)) \]

Assume an interior solution and look at first order conditions: (corner solutions may not satisfy the equal marginal rates of substitution property)

First look at person \( a \)'s consumption of commodity \( i \):

\[ \frac{\partial L}{\partial x_{ai}} = \frac{\partial U^a}{\partial x_{ai}} - \phi \frac{\partial U^b}{\partial x_{ai}} - \mu_i = 0 \]

Substitute \( \frac{\partial U^b}{\partial x_{ai}} = \lambda_{ab} \frac{\partial U^a}{\partial x_{ai}} \):

\[ \frac{\partial L}{\partial x_{ai}} = \frac{\partial U^a}{\partial x_{ai}} - \phi \lambda_{ab} \frac{\partial U^a}{\partial x_{ai}} - \mu_i = 0 \]

Combine terms:

\[ \frac{\partial L}{\partial x_{ai}} = \frac{\partial U^a}{\partial x_{ai}} (1 - \phi \lambda_{ab}) - \mu_i = 0 \]

Person \( a \)'s consumption of commodity \( j \):
\[
\frac{\partial L}{\partial x_{aj}} = \frac{\partial U^a}{\partial x_{aj}} (1 - \varphi_{ab}) - \mu_j = 0
\]

Repeat the math for person b’s consumption of commodities i and j:
\[
\frac{\partial L}{\partial x_{bi}} = \frac{\partial U^b}{\partial x_{bi}} (\lambda_{ba} - \phi) - \mu_i = 0
\]
\[
\frac{\partial L}{\partial x_{bj}} = \frac{\partial U^b}{\partial x_{bj}} (\lambda_{ba} - \phi) - \mu_j = 0
\]

Solve for the respective Lagrange multipliers:
\[
\frac{\partial U^a}{\partial x_{ai}} (1 - \varphi_{ab}) = \mu_j
\]
\[
\frac{\partial U^b}{\partial x_{bi}} (\lambda_{ba} - \phi) = \mu_i
\]
\[
\frac{\partial U^a}{\partial x_{aj}} (1 - \varphi_{ab}) = \mu_j
\]
\[
\frac{\partial U^b}{\partial x_{bj}} (\lambda_{ba} - \phi) = \mu_j
\]

Divide the terms in the first column to get \( MRS^a_{ij} \):
\[
\frac{\partial U^a}{\partial x_{ai}} (1 - \varphi_{ab}) = \frac{\partial U^a}{\partial x_{aj}} (1 - \varphi_{ab}) \Rightarrow \frac{\mu_j}{\mu_j} = MRS^a_{ij}
\]

Divide the terms in the second column to get \( MRS^b_{ij} \):
\[
\frac{\partial U^b}{\partial x_{bi}} (\lambda_{ba} - \phi) = \frac{\partial U^b}{\partial x_{bj}} (\lambda_{ba} - \phi) \Rightarrow \frac{\mu_j}{\mu_j} = MRS^b_{ij}
\]

So \( MRS^a_{ij} = MRS^b_{ij} \)

c) In competitive equilibrium each person maximizes his own utility given his labor endowment without regard to the other person’s utility (or consumption decision). In this case, there are externalities: one person’s consumption directly affects the other person’s utility. The first fundamental theorem of welfare economics which says any competitive equilibrium is also Pareto efficient does not hold in the presence of externalities. Part of the problem is that the marginal utility of income is not the same for the consumers at equilibrium. The government could put a lump-sum tax on endowments in order to get this relationship to hold in equilibrium. In this case, the consumers do not have an endowment so the lump-sum transfer would be a transfer of purchasing power. If such lump-sum transfers are not feasible, the government tries to get each person to account for the extra costs or benefits caused by their consumption decision through taxes or subsidies.
4. Consider a two-person two-good economy with the following preferences:
   i) both consumers find the goods to be perfect substitutes (although not necessarily at the same rate)
   ii) both consumers find the goods to be perfect complements (although not necessarily in the same proportion)
   iii) one consumer finds the goods to be perfect substitutes and the other finds them to be perfect complements.

a) Find the contract curves for these exchange economies.
b) For an arbitrary initial endowment, find the competitive equilibrium for these exchange economies.
c) For an arbitrary initial endowment, find the core of these exchange economies.
   (This can all be done graphically. Be sure not to choose total levels of the goods which lead to special cases for your results.)

Let $x$ and $y$ represent the quantities of the two goods

**Perfect substitutes:**

$u_1 = ax + by$ and $u_2 = cx + dy$

Slopes of the indifference curves (in $(x,y)$ space) are $-\frac{a}{b}$ and $-\frac{c}{d}$

Consumer with the steeper indifference curve prefers good $y$

**Perfect Complements:**

$u_1 = \min(ax, by)$ and $u_2 = \min(cx, dy)$

Proportion in which consumers like goods depends on parameters as reflected by the corners of their indifference curves which follow lines determined by

$y = \frac{a}{b}x$ and $y = \frac{c}{d}x$

**Edgeworth Box** - shows consumption by both individuals where all available resources are consumed (i.e., no waste). The origin for consumer one is the lower, left corner so his utility improves as he moves to the upper, right corner. The origin for consumer two is the upper, right corner and his utility improves as he moves to the lower, left corner.
a) **Contract Curve** - set of all Pareto optimal allocations

(i) Perfect Substitutes. In the case on the left, consumer 1 has the steeper indifference curves (when viewed with the same origin) so the Pareto optimal allocations favor giving good $y$ to consumer 1 and good $x$ to consumer 2. The case on the right is the opposite. Note, if both consumers thought the goods were perfect substitutes at the same rate, every allocation would be Pareto optimal.

(ii) Perfect Complements. The Pareto optimal allocations are all the points between the lines that determine the proportions in which the consumers find the goods to be perfect complements (i.e., the line through the corners of all the indifference curves).

(iii) One of each. The Pareto optimal allocations are all the points on the line that determines the proportions in which the consumer who views the goods as perfect complements.
b) **Competitive Equilibrium** - allocations where both consumers maximize their utilities as price takers and there is market clearing. The gray areas show the feasible region: those allocations that are Pareto improvements (one or both consumers is better off). Given this is a well behaved problem (convex, individualistic preferences), the competitive equilibrium will be Pareto optimal. Therefore, the CE will be in the PO allocations in the shaded region (also known as the core; see part (c)). The exact equilibrium depends on the price ratio established by the two consumers. This ratio is determined by the relative preferences of the consumers, and since they are price takers, it must either put a consumer on a corner solution (i.e., zero quantity of some commodity) or a kink in the indifference curve or keep the individual indifferent to his original endowment. This will become clear in the answers below.

(i) Perfect Substitutes. In the top graph consumer 1 prefers good \( y \) (relative to consumer 2) so he will trade off all of his endowment of \( x \) in order to get more \( y \). In the lower graph consumer 1 prefers good \( x \). Given the first endowment \( (\omega_1) \), if the price ratio follows consumer 2's indifference curve (the blue lines), consumer 1 will want to trade all of good \( x \) for an infeasible amount of good \( y \). (In the lower graph he trades \( y \) for \( x \).) The infeasible point consumer 1 wants to get to is labeled with a red dot in each graph. Now consider the price ratio being on top of consumer 1's indifference curve (the red ones). In this case, consumer 1 is indifferent between any trades. Consumer 2 trades away all of his endowment of good \( y \) (\( x \) in the lower graph) and ends up on a corner solution (zero good \( y \) in the top graph and zero good \( x \) in the bottom graph). For endowment 2, the argument is similar. If the price ratio follows consumer 1's indifference curve, consumer 2 would want to trade all of his endowment for an infeasible amount of good \( x \) (\( y \) in the lower graph; the blue dot in both graphs). If the price ratio follows consumer 2's indifference curve, consumer 1 ends up on a corner solution (zero good \( x \) on top and zero good \( y \) on bottom) and consumer 2 is indifferent.

The graphs below have purple shaded areas. Any endowments in these areas will result in the equilibria shown in the respective graphs, which are corner solutions for both consumers. The price ratio will be determined by the slope of the line through the endowment and the respective corner of the
Edgeworth box. (It can vary anywhere between the slopes of the indifference curves that border the shaded area.)

(ii) Perfect Complements. In all cases, price taking behavior for both consumers means they want to trade to the intersection of the price ratio (budget line) and the line through the corners of their indifference curves. In the first case, where the corners of their indifference curves touch, consumer 1 wants to be in the upper right corner of the feasible trades; consumer 2 in the lower left corner. At any price ratio (except $p_x = 0$ or $p_y = 0$), the consumers want to be at the same point (where the price ratio intersects their indifference curves). In the zero price cases, one of the consumers will be indifferent and the other will want to be at his optimal corner of the region of feasible trades. In other worse, there exists a competitive equilibrium for every price ratio.

The other three cases are similar (the fourth actually combines all three above it). First consider the price ratio shown (dotted black line). A blue and red dot shows where each consumer would want to be so this price ratio is not a competitive equilibrium. In the second case, only $p_x = 0$ leads to competitive equilibrium (actually multiple equilibria). In the third case, $p_y = 0$ leads to competitive equilibria as shown. The fourth cases an multiple equilibria at $p_x = 0$ and $p_y = 0$ and the price ratio determined by the line through the endowment and the point where both consumers’ indifference curves are tangent (where the dashed blue and red lines intersect).
(iii) One of each. The price ratio in this case must be along consumer 1’s indifference curve (the consumer who views the goods as perfect substitutes). Any other price ratio would make consumer 1 want to be outside the feasible region of trades (i.e., lower consumer 2’s utility). Such a price ratio would also be physically infeasible because consumer 1 will want to trade his entire endowment of $y$ for an amount of $x$ that does not exist. Consumer 2 always wants to trade in such a way that he stays on the line through the corners of all his indifference curves. This is a case where "life is not fair." Consumer 1’s flexibility enables trade to occur, but consumer 2 enjoys all the benefits of trade.

![](diagram1.png)

**c) Core** - collection of allocations that are attainable from a given endowment and aren’t "blocked" (i.e., voted down by consumers who are made worse off). The core is found by looking at those allocations that are Pareto improvements (one or both consumers is better off) which are the allocations shaded in gray. Any points from the contract curves found in part (a) that are in this shaded area form the core (marked in green).

(i) Perfect Substitutes. For the case on the left, consumer 1 prefers good $y$ so he will trade off some of his endowment of $x$ in order to get more $y$. In the other case, consumer 1 trades off good $y$ for $x$. 

![](diagram2.png)

![](diagram3.png)
(ii) Perfect Complements. The lines connecting the corners of the indifference curves define the PO allocations as shown in part (a). The line though the intersections of the indifference curves that go through the endowment defined equilibrium prices (under the assumption of equal negotiating skill). The CE is any point where this second line crosses the set of PO allocations.

(iii) One of each. The price ratio in this case bisects the angle determined by the indifference curves that intersect at the endowment point. As before, the CE is the point where the line determined by the price ratio intersects the set of PO allocations.

Documentation.

I worked out all problems with Josh Kneifel. He specifically helped with problem 1a explaining why $\alpha < 0$ leads to division by zero. For problem 3, Josh got me started with the derivative of $W$ wrt $x_{ik}$. On the rest of the problems we kind of wandered aimlessly through notes and textbooks trying to figure them out together.

Prof Hamilton reviewed the competitive equilibrium graphs for problem 4c in class.