Optimal Commodity Taxation

Previous section discussed changing commodity taxes to reduce excess burden (i.e., making consumers better off), but assumed there were lump-sum taxes to keep revenue constant. This section will assume we don't have lump-sum taxes; we'll solve a general equilibrium model that gives us "bonus" answers:

1. How should government evaluate operations of its activities when lump-sum taxation is infeasible?
2. How should taxation that's used affect governments operating decisions?
3. How to levy taxes for 2nd best outcome (i.e., lump-sum taxes not available)?

**Goal** - maximize social welfare subject to revenue and production constraints; decision variables include taxes and public production.

**Public Production** - government can buy finished goods for public consumption or can produce goods itself; that means government buys output and sells output; since it's producing it is competing with private sector for factors of production.

**Areas** - we're combining taxation, public production and welfare economics (and GE).

**Consumer Budget Constraint** - \( q \cdot x + \hat{x} = q \cdot \omega \) (value of consumption = value of endowment);
problem is we don't see consumption, just net trades (\( x + \omega = \hat{x} \)):
Sub this into the budget constraint: \( q \cdot (x + \omega) = q \cdot \omega \Rightarrow q \cdot x = 0 \)
(This allows us to suppress income when we go to multiple consumers)

Want to end up with economy of many consumers with public and private production; want to compare different tax systems; want to determine types/amount of public consumption.

**Training Wheels** - we'll start with single consumer, no public consumption (but no private production; all public production), and only use commodity taxes; 1 consumer and 2 goods (1 = labor; 2 = consumption good).

**Weird Graphs** -
Define origin as "no net trades" (i.e., consumer keeps his endowment);
Assume free disposal (any point below the PPF is feasible);
Consumer sells labor which is used for production.

**Problems** - special cases don't have convex feasible regions, but are realistic so we won't assume them away.

**Fixed Cost** - PPF doesn't cross origin, but origin is feasible.

**Increasing Returns** - reason for natural monopolies.
Other Assumptions -
- **No Lump-Sum Tax** - government can't tax endowments
- **Uniform Prices** - government can only trade with consumer in market
- **Price Consumption Curve** - all trades government makes with consumer lie on consumer's offer curve: tangencies of consumer budget line and indifference curves; all budget lines go through origin (endowment); moving along offer curve away from origin means consumer is better off

Two Constraints - government chooses point on offer curve lying in (on) PPF

More Than Two Goods
Consumer Demands - \( x = (x_1, x_2, \ldots, x_n) \)

Where \( x(q,0) \) solves \( \max_x u(x) \) s.t. \( q \cdot x = 0 \) (\( q \) is consumer prices; 0 is money income)

PPF - defined by \( G(z) \leq 0 \), where \( z \) is public production

Government Objective - \( V(q) \) (technically it's \( \tilde{V}(x(q)) \)); government chooses taxes so it can effectively change consumer prices

\[
\max_q V(q) \text{ s.t. } G(x(q)) \leq 0
\]

Note: this ensures \( x(q) = z \) (i.e., demand = supply)

**Individualistic** - if government's concern is social welfare and SW is individualistic (based on consumer utility), then \( V(q) \) increases as \( u(x) \) increases; government's objective essentially becomes maximizing consumer's utility; solution is last intersection between offer curve and boundary of PPF \( \therefore \) we get production efficiency (trivial result for this economy)

Add Private Production
Private Sector - constant returns to scale
Public Sector - government runs any industry with increasing returns or fixed costs
Decreasing Returns - assume we have none

**Individualistic**

\[
\frac{\partial V}{\partial q_i} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial q_i} = -\alpha x_i, \text{ where } \alpha \text{ is marginal utility of income to society}
\]

\[
\therefore \frac{\partial V}{\partial q_i} = -\alpha x_i, \quad \frac{\partial V}{\partial q_j} = -\alpha x_j
\]

\( p = (p_1, p_2, \ldots, p_n) \) = producer prices
\( y = (y_1, y_2, \ldots, y_n) \) = quantities of private production (> 0 is output; < 0 is input)
\( z = (z_1, z_2, \ldots, z_n) \) = quantities of public production

Single Firm - if prices are given, constant returns allows us to consider the private sector as being a single firm

**Private Production Efficient** - we'll assume \( y_i = f(y_2, \ldots, y_n) \) (production is efficient)

Also assume \( f \) is differentiable and \( y_i \neq 0 \ \forall \ i = 1, \ldots, n \) (i.e., private firm is either a net producer or consumer of each good)
Private Objective - \( \max_y \mathbf{p} \cdot \mathbf{y} \) s.t. \( y_1 = f(y_2, \ldots, y_n) (\lambda) \)

Note: since \( y_i > 0 \) is output and \( y_i < 0 \) is input, \( \mathbf{p} \cdot \mathbf{y} \) is profit

1st Order Conditions - \( p_i - \lambda = 0 \) and \( p_i - \lambda \frac{\partial f}{\partial y_i} = 0 \)

Notation: \( f_i = \frac{\partial f}{\partial y_i} \) ... we can combine 1st order conditions: \( p_i = p_1 f_i \ \forall \ i = 2, \ldots, n \)

Note: \( \mathbf{p} \cdot \mathbf{y} = 0 \) otherwise problem would be unbounded \( \ldots \) if we know \( y_2, \ldots, y_n \) we know \( y_1 \) and \( \mathbf{p} / p_1 \) (ratio of prices relative to good 1)

Public Production Efficient - used to have \( G(z) \leq 0 \) for public (government) constraint, but now we’ll use \( z_i = g(z_2, \ldots, z_n) \)

Production Efficiency - we assumed private production is efficient and public production is efficient, but we didn’t say anything about them being jointly efficient (i.e., \( \text{MRT}^P = \text{MRT}^G \))

Consumer - used to have \( \mathbf{x}(\mathbf{q}) = \mathbf{z} \), but now it becomes \( \mathbf{x}(\mathbf{q}) = \mathbf{y} + \mathbf{z} \) (market clearing; demand = supply)

Walras’ Law - since we have market clearing, if we know \( n - 1 \) markets clear, we know all \( n \) markets clear because all agents must satisfy budget constraints

Reverse - we showed all \( n \) markets clear \( \ldots \) if we know all but one agent satisfy their budget constraints, then they all do (so we can delete one budget constraint); in this case, we’ll delete the government’s budget constraint: \( (\mathbf{q} - \mathbf{p}) \cdot \mathbf{x} + \mathbf{p} \cdot \mathbf{z} = 0 \)

Normalizing - we get to do two of them because we have \( \mathbf{x}(\mathbf{q}) \) homogeneous of degree zero in \( \mathbf{q} \) and \( \mathbf{y}(\mathbf{p}) \) homogeneous of degree zero in \( \mathbf{p} \)

(1) Producer Price - from the private production optimization we looked at earlier, we know only the price ratios \( (\mathbf{p} / p_1) \) matter so we’ll set \( p_1 = 1 \)

(2) Consumer Price - conventional choice is \( q_1 = 1 \)

Tax Problem? - since taxes are \( \mathbf{t} = \mathbf{q} - \mathbf{p} \), this looks like we just said good 1 is not taxed, but it’s just a result of the normalization, not a model restriction... don’t blink or you might miss the explanation...

\( x_i > 0 \Rightarrow \) good purchased by the consumer

\( x_i < 0 \Rightarrow \) good provided by the consumer; in this case \( t_i > 0 \) is actually a subsidy because \( (q_i - p_i) \cdot x_i < 0 \) (i.e., it costs the government money)

Proportional Tax - desirable because it keeps all the \( \text{MRS}_{ij} = \text{MRT} \) (doesn’t distort consumption), but if tax is proportional to producer prices for all goods, then it raises zero revenue

Proof: proportional tax means \( \tau = \frac{\mathbf{q} - \mathbf{p}}{\mathbf{p}} \) or \( \tau \mathbf{p} = \mathbf{q} - \mathbf{p} \) or \( (1 + \tau)\mathbf{p} = \mathbf{q} \)

From consumer budget constraint: \( \mathbf{q} \cdot \mathbf{x} = 0 \)

Proportional tax: \( \mathbf{q} = (1 + \tau)\mathbf{p} \)

Substitute into budget constraint: \( (1 + \tau)\mathbf{p} \cdot \mathbf{x} = 0 \)

We can’t have \( \tau = -1 \) so that means \( \mathbf{p} \cdot \mathbf{x} = 0 \)

\( \therefore (\mathbf{q} - \mathbf{p}) \cdot \mathbf{x} = 0 \) (the tax raises zero revenue)
No Subsidies - in order to ensure positive tax revenue the government could set $q_i > p_i$ (i.e., $t_i > 0$) if $x_i > 0$ and $q_i < p_i$ (i.e., $t_i < 0$) if $x_i < 0$

Good - raises positive revenue

Bad - this tax is distortionary

Normalizing - if we normalize $p_1 = q_1 = 1$ we get $t_1 = 0$; if we do $\hat{p}_1 = 5$ and $\hat{q}_1 = 1$ we get $\hat{t}_1 = 4$ ... this shifts all taxes up by a factor of 4 (a proportional tax) so it doesn't change revenue).

Welfare Maximization Problem - we're doing optimal taxation, but we don't look at government setting taxes ($t = q - p$) because prices aren't tied down so we can get a multiplicity problem; instead we focus on setting prices (except $p_1 = q_1 = 0$)

\[
\max V(q) \quad \text{s.t.} \quad (i) \ x_i(q) - y_i - z_i = 0, \ i = 1, \ldots, n \quad \text{(market clearing)}
\]

\[
\text{(ii) } y = \arg \max_p p \cdot y \quad \text{s.t.} \quad y_1 = f(y_2, \ldots, y_n)
\]

\[
\text{(iii) } z_1 = g(z_2, \ldots, z_n)
\]

Simplification - $x_i(q)$ is consumer demand (solution to the consumer maximization problem on top of page 2); we can't do the same for $y$ because of constant returns to scale ($y(p)$ would be unbounded), but we can play with the constraints to simplify the problem

Price Vector - $p$ only enters in constraint (ii) so if we can drop this constraint, we don't have to worry about $p$ (for now)

- First step is to use the market clearing condition (constraint (i)):

\[
y_j = x_j(q) - z_j, \ j = 2, \ldots, n
\]

(note we didn't use $j = 1$ because we already normalized $p_1 = 0$)

- Next sub the production constraints ((ii) and (iii)) into the market clearing condition for good 1:

\[
x_i(q) = y_i + z_i = f(y_2, \ldots, y_n) + g(z_2, \ldots, z_n)
\]

- Now sub the $y_2, \ldots, y_n$ we found in the first step:

\[
x_i(q) = f(x_2(q) - z_2, \ldots, x_n(q) - z_n) + g(z_2, \ldots, z_n)
\]

New problem -

\[
\max V(q) \quad \text{s.t.} \quad x_i(q) = f(x_2(q) - z_2, \ldots, x_n(q) - z_n) + g(z_2, \ldots, z_n)
\]

Note 1: $z_1$ is no longer a decision since $z_1 = g(z_2, \ldots, z_n)$

Note 2: we're using = constraint (not $\leq$) so we need to make sure second order conditions hold

Lagrangian - $L = V(q) - \lambda \left[ x_i(q) - f(x_2(q) - z_2, \ldots, x_n(q) - z_n) - g(z_2, \ldots, z_n) \right]$

FOC - simplified notation:

\[
\frac{\partial V}{\partial q_k} = V_k, \quad \frac{\partial f}{\partial y_k} = f_k
\]

\[
\frac{\partial L}{\partial q_k} = V_k - \lambda \left[ \frac{\partial x_i}{\partial q_k} - \sum_{i=2}^n f_i \frac{\partial x_i}{\partial q_k} \right] = 0, \ k = 2, \ldots, n
\]

\[
\frac{\partial L}{\partial z_k} = -\lambda [f_k - g_k] = 0, \ k = 2, \ldots, n
\]
Aggregate Production Efficiency - if it's an interior solution (i.e., \( \lambda \neq 0 \)), the \( \frac{\partial L}{\partial z_k} \)

FOCs imply \( f_k = g_k \) ... that means \( \text{MRTS}^p = \text{MRTS}^G \) (marginal rates of technical substitution in private and public [government] sectors are equal; could also think of it as \( \text{MRT}^p = \text{MRT}^G \) (marginal rates of transformation)

Result - with aggregate production efficiency, we can't reallocate \( y \) or \( z \) to produce more or produce same amount more efficiently

Graphs - we assumed production efficiency for private sector and public sector; that means each sector is operating on the frontier of it's PPF; for aggregate production efficiency, they're at points on their PPF that have the same slope

**Interpretations** - 3 ways to look at aggregate production efficiency

1. **No Intermediate Goods Tax** - if we disaggregate private production sector we need a price vector for each sector so we couldn't use \( y(p) \) like we did, but aggregate production efficiency means all price vectors should be equal; that means business to business transactions are untaxed

2. **Untaxable Sectors** - subsistence agriculture (people who grow own food and eat it) or household production (home schooling, laundry, cooking, etc.) all gets lumped into consumer sector, but this model is focused on transactions (net trades), not final consumption so these activities are not taxed

3. **Const-Benefit Analysis** - don't have to confine interpretation to static model; Arrow & Debreu view it as dynamic (just relabel commodity name for time periods); have to worry about discount rate

**Tax on Interest Income** - consumer sees \( r(1-t) \) and producer sees \( r \);

Aggregate production efficiency says government project should use producer discount rate

**Interior Solution** - what's required to guarantee \( \lambda \neq 0 \) (i.e., we have aggregate production efficiency)? Look at in terms of many consumer economy so rather than using \( V(q) \) (derived from \( x(q) \)) a modification of a Samuelson social welfare function
SWF - \( W(x) = W(u^1(x^1), u^2(x^2), \ldots, u^H(x^H)) \)

Modified - sub consumer demands: \( V(q) = W(u^1(x^1(q)), u^2(x^2(q)), \ldots, u^H(x^H(q))) \)

Note: this is assuming individualistic SWF; otherwise we'd have
\( V(q) = B(x^1(q), x^2(q), \ldots, x^H(q)) \) which is not based on utilities

Individual Consumer - showed production will be at last intersection of offer curve and PPF so we know \( \lambda \neq 0 \) (see top of page 2)

Multiple Consumers - change in prices can bring gains to some consumers and losses to others (based on consumption possibilities) \( \therefore \) we're not guaranteed to have an optimum that has aggregate production efficiency (i.e., could have \( \lambda = 0 \))

Aggregate Offer Curve - \( X(q) = \sum_{i=1}^H x^i(q) \)

Guarantee Improvement - want to find conditions under which \( \Delta q \) makes everyone better off so we'll have \( \lambda \neq 0 \) (aggregate production efficiency)

Assume... individualistic SWF: \( V(q) = W(u^1(x^1(q)), u^2(x^2(q)), \ldots, u^H(x^H(q))) \)

Assume... \( \exists \) good \( j \) such that \( x^h_j \leq 0 \ \forall \ h \) and \( x^h_j < 0 \) for some \( h \) (i.e., some consumers are net sellers of good \( j \) and others don't trade good \( j \)); could also use \( j \) such that \( x^h_j \geq 0 \ \forall \ h \) and \( x^h_j > 0 \) for some \( h \) (i.e., some consumers are net buyers of good \( j \) and others don't trade good \( j \)); important thing is to have all consumers on same side of market for one good

Sub individual's indirect utility functions into SWF:
\( V(q) = W(v^1(q), v^2(q), \ldots, v^H(q)) \)

How does price change affect social welfare? Take derivative:
\[
\frac{\partial V}{\partial q_j} = \sum_{h=1}^H \frac{\partial V}{\partial u^h} \frac{\partial u^h}{\partial x^h} \frac{\partial x^h}{\partial q_j} = \sum_{h=1}^H V_h(-a^h x^h_j)
\]

Note: "individualistic" means \( \frac{\partial V}{\partial q_i} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial q_i} = -\alpha x_i \) (see bottom of page 2)

We know \( V_h = \frac{\partial V}{\partial u^h} > 0 \) (make 1 consumer better off and [all else equal]

social welfare improves)

Also know \( \alpha^h > 0 \) (marginal utility of income)

Sign of \( x^h_j \) depends \( \therefore \) if all \( x^h_j \geq 0 \Rightarrow \frac{\partial V}{\partial q^h_j} < 0 \); or if all \( x^h_j \leq 0 \Rightarrow \frac{\partial V}{\partial q^h_j} > 0 \)

i.e., if all consumers are on same side of market for some good, we can make all consumers better off (hence raise social welfare) by changing price \( \therefore \) everyone is made better off by moving to the last intersection of aggregate offer curve and PPF (i.e. aggregate production efficiency)

Labor - we assume this condition holds for labor (all consumers are suppliers); but this doesn't work if we categorize labor (e.g., high and low skill)
Tax Rules - observations of the optimal commodity tax (by torturing the FOCs)

Go back to FOC for \( q_k \) (from bottom of p.4): \[
\frac{\partial L}{\partial q_k} = V_k - \lambda \left[ \frac{\partial x_i}{\partial q_k} + \sum_{i=2}^{n} f_i \frac{\partial x_i}{\partial q_k} \right] = 0, \quad k = 2, \ldots, n
\]

We know \( p_i = -p_i f_i \forall i = 2, \ldots, n \) (comes from private producer max profit; top of p.3)

Since good 1 is numeraire \( p_1 = 1 \vdash p_i = f_i \forall i = 2, \ldots, n \)

Sub that into the FOC for \( q_k \): \[
V_k - \lambda \sum_{i=2}^{n} p_i \frac{\partial x_i}{\partial q_k} = 0, \quad k = 2, \ldots, n
\]

We know \( q_i = p_i + t_i \); gov’t sets both \( p \) and \( q \) so \( \frac{\partial x_i}{\partial q_k} = \frac{\partial x_i}{\partial t_k} \)

Sub that into the FOC for \( q_k \): \[
V_k - \lambda \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial t_k} = 0, \quad k = 2, \ldots, n
\]

Derivative is linear operator so: \[
V_k = \lambda \frac{\partial}{\partial t_k} \left[ \sum_{i=1}^{n} p_i x_i \right], \quad k = 2, \ldots, n
\]

Go back to tax: \( t = q - p \Rightarrow t \cdot x = q \cdot x - p \cdot x \), but \( q \cdot x = 0 \) (middle of p.1) \( \therefore t \cdot x = -p \cdot x \)

Sub that in: \[
V_k = -\lambda \frac{\partial}{\partial t_k} \left[ \sum_{i=1}^{n} t_i x_i \right], \quad k = 2, \ldots, n
\]

(1) \( \Delta Tax\ Revenue > 0 \Rightarrow \Delta Welfare < 0 \):

\( V_k = \) marginal utility to society of price change of good \( k \); for 1 consumer case, \( V_k = -\alpha x_k \)

(bottom of p.2); that means, if consumer is net buyer \( (x_k > 0) \), raising the price lowers social welfare \( (V_k < 0) \)

\[
\frac{\partial}{\partial t_k} \left[ \sum_{i=1}^{n} t_i x_i \right] = \text{marginal tax revenue with respect to } t_k \text{ (i.e., how total tax revenue changes based on a change in the tax on good } k \text{)}; \text{it's not as simple to compute as just } \Delta t_k \Delta x_k
\]

because changing the price of good \( k \), potentially changes the amount consumed of other goods

For the case where \( V_k < 0 \) (social welfare declines from raising price on good for which consumer is net buyer), we must have marginal tax revenue > 0 (i.e., positive tax revenue must be raised in order to make up for hurting consumers)

(2) Marginal Tax Revenue vs. Consumption:

For one consumer case \( V_k = -\alpha x_k = -\lambda \frac{\partial}{\partial t_k} t \cdot x \Rightarrow x_k = \frac{\lambda}{\alpha} \frac{\partial}{\partial t_k} t \cdot x, \quad k = 2, \ldots, n \)

That says marginal tax revenue wrt \( t_k \) (tax on good \( k \)) is proportional to consumer’s consumption of good \( k \)

(3) These Also Hold for Numeraire (Good 1)

The previous rules were for \( k = 2, \ldots, n \), but they also hold for \( k = 1 \)

Go back to using \( p_i \) instead of \( t_i \) in the FOC: \[
V_k = \lambda \frac{\partial}{\partial t_k} \left[ \sum_{i=1}^{n} p_i x_i \right], \quad k = 2, \ldots, n
\]
$V$ is homogeneous of degree 0 in $q$: $\sum_{k=1}^{n} q_k V_k = 0$ (changing all prices by the same amount doesn't change social welfare)

Demand is homogeneous of degree 0 in $q$: $\sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k} q_k = 0$

Get tricky: multiply the demand condition by $\lambda \sum_{i=1}^{n} p_i$ (a constant times zero still equals zero):

$$\lambda \sum_{i=1}^{n} p_i \left( \sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k} q_k \right) = 0$$

Add this to the first homogeneity condition: $\sum_{k=1}^{n} q_k V_k - \lambda \sum_{i=1}^{n} p_i \sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k} q_k = 0$

Do some fancy rearranging: $\sum_{k=1}^{n} \left[ V_k - \lambda \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial q_k} \right] q_k = 0$

Return to FOC of $q_k$: $V_k - \lambda \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial q_k} = 0, \ k = 2, \ldots, n$ (version from the top of p.7)

We can multiple both sides by $q_k$: $\left[ V_k - \lambda \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial q_k} \right] q_k = 0, \ k = 2, \ldots, n$

Add these up: $\sum_{k=2}^{n} \left[ V_k - \lambda \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial q_k} \right] q_k = 0$

Combine this with the sum from 1 to $n$ (4 lines up) and we must have $V_i - \lambda \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial q_i} = 0$

(4) **Ramsey Rule** - compensated demands for all goods change in equal proportion to the consumption of the goods; a tax that raises revenue causes fewer net trades

Consider 1 consumer economy with individualistic SWF
Gov't objective is $\max_{q} V(q)$ (consumer's indirect utility function)

$$V_k = -\alpha x_k = -\lambda \frac{\partial}{\partial t_k} \sum_{i=1}^{n} t_i x_i$$  (middle of p.7, near "\(\Delta\) Tax Revenue > 0 \(\Rightarrow\) \(\Delta\) Welfare < 0")

$$\frac{\partial t_i x_i}{\partial t_k} = \begin{cases} x_i + t_i \frac{\partial x_i}{\partial t_k} & i = k \\ t_i \frac{\partial x_i}{\partial t_k} & i \neq k \end{cases}$$

$$\therefore -\alpha x_k = -\lambda \left( x_k + \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial t_k} \right)$$

**Slutsky Equation** - $\frac{\partial x_i}{\partial q_k} = S_{ik} - x_k \frac{\partial x_i}{\partial t}$, where $S_{ik} = \frac{\partial x_i^c}{\partial q_k}$
Recall $\frac{\partial x_i}{\partial q_k} = \frac{\partial x_i}{\partial t_k}$ (top of p.7), so we can sub the Slutsky equation

$$-\alpha x_k = -\lambda x_k + \sum_{i=1}^{n} t_i \left( S_{ik} - x_k \frac{\partial x_i}{\partial t} \right)$$

$$-\alpha x_k = -\lambda x_k - \lambda \sum_{i=1}^{n} t_i S_{ik} + \lambda x_k \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial t}$$

$$\lambda \sum_{i=1}^{n} t_i S_{ik} = \left( \alpha - \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial t} \right) x_k$$

$$\frac{\sum_{i=1}^{n} t_i S_{ik}}{x_k} = \frac{\alpha}{\lambda} - 1 + \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial t} \quad \text{... note the right hand side is independent of } k$$

Define $-\theta = \frac{\sum_{i=1}^{n} t_i S_{ik}}{x_k} = \frac{\sum_{i=1}^{n} t_i S_{ki}}{x_k}$ (although defined this way $\theta$ is independent of $k$)

$(S_{ik} = S_{ki}$ because they are elements of the matrix of derivatives of compensated demand; $S$ is symmetric and negative semidefinite)

Multiply both sides by $\sum_{k=1}^{n} t_k x_k$ : $\sum_{k=1}^{n} t_k x_k \theta = -\sum_{k=1}^{n} t_k x_k \frac{\sum_{i=1}^{n} t_i S_{ki}}{x_k}$

$$\theta \sum_{k=1}^{n} t_k x_k = -\sum_{k=1}^{n} \sum_{i=1}^{n} t_k t_i S_{ki} = -t' St$$

This is quadratic form; $S$ is negative semidefinite ($t' St \leq 0$) so $-t' St \geq 0$

$$\theta \sum_{k=1}^{n} t_k x_k \geq 0 \quad \Rightarrow \quad \theta \text{ has same sign as net tax revenue}$$

Back to $-\theta = \frac{\sum_{i=1}^{n} t_i S_{ki}}{x_k}$; Sub $S_{ki} = \frac{\partial x_k^c}{\partial q_i}$

$$\theta = \frac{\sum_{i=1}^{n} t_i \frac{\partial x_k^c}{\partial q_i}}{x_k}$$

**Gradient Approximation** - $t_i$ is the price (tax) change; $\frac{\partial x_k^c}{\partial q_i}$ is the change in compensated demand for good $k$ from the price change on good $i$; if we assume $p$ is constant, the numerator approximates $\Delta x_k^c$ (change in compensated demand for good $k$) due to all taxes (Note: the results below still hold if $p$ varies)

**Ramsey Rule** - if there is positive net tax revenue for a tax change ($\theta > 0$), the change in compensated demand for good $k$ due to the tax is proportional to $-x_k$
Less Trades - $x_k > 0$ if buying; $x_k < 0$ if selling; change in proportion to $-x_k$ means $|x_k|$ gets smaller; that means there are fewer trades (less buying and selling)

(5) Regular Demands and Income - regular demands for all goods change in proportion (not equal) to the consumption of the goods; as income elasticity rises, the distortion (change in demand) is greater

\[ \sum_{i=1}^{n} t_i S_{ki} \]

Go back to $-\theta = \frac{\sum_{i=1}^{n} t_i S_{ki}}{x_k}$

Reverse the Slutsky equation: $\frac{\partial x_k}{\partial q_i} = S_{ki} - x_k \frac{\partial x_k}{\partial t} \Rightarrow S_{ki} = \frac{\partial x_k}{\partial q_i} + x_k \frac{\partial x_k}{\partial t}$

$-\theta = \frac{\sum_{i=1}^{n} t_i \frac{\partial x_k}{\partial q_i}}{x_k} + \frac{\sum_{i=1}^{n} t_i x_i}{x_k}$

Gradient Approximation - same as bottom of p.9; $t_i$ is the price (tax) change; $\frac{\partial x_k}{\partial q_i}$ is the change in (regular) demand for good $k$ from the price change on good $i$; the numerator of the left hand side approximates $\Delta x_k$ (change in demand for good $k$) due to all taxes

Interpretations - similar to Ramsey Rule in that the change in regular demand is proportional to the amount demanded, but there are a variety of interpretations based on the specific demand structure (e.g., taxes, after tax prices, etc.) and is "usually messy"; quantities (regular demands) are distorted by the tax anyway so it's easier to use the Ramsey Rule

Income Elasticity - from the right side, we can see that $\Delta x_k$ (left side) is larger if

$(\frac{\partial x_k}{\partial t} / x_k)$ is larger; if we multiply that term by $I$ we have income elasticity $\Rightarrow$ larger income elasticity implies larger distortion from tax (i.e., more impact on $\Delta x_k$)

(6a) Deviate from Proportional Tax Locally - Corlett & Hague considered a proportional tax on all goods except the numeraire and looked to see if there are any local improvements (similar to Dixit, but without a lump-sum tax); got the same result as Dixit (i.e., want to deviate away from a proportional tax)

(6b) Deviate from Proportional Tax in Optimal 2nd Best - Diamond & Mirrlees; no assumption about proportional tax

Consider 3 goods (1 being the numeraire so $t_1 = 0$)

Apply this to $-\theta = \frac{\sum_{i=1}^{n} t_i S_{ki}}{x_k}$:

$t_2 S_{22} + t_3 S_{23} = -\theta x_2$ and $t_2 S_{32} + t_3 S_{33} = -\theta x_3$
After lots of magical algebra the end result is:

$$\frac{t_2}{q_2} > \varepsilon = \frac{t_3}{q_3} \text{ as } \sigma_{21} > \varepsilon = \sigma_{31}$$

**English** - optimal tax rates (based on consumer prices) follow the same order as compensated demand elasticities (between the good and the numeraire)

Assume $$x_1 < 0, x_2 > 0, x_3 > 0$$ (i.e., good 1 is labor which consumer sells to buy goods 2 and 3); the optimal tax rate is higher on the good that is more complementary to leisure ($x_1$)... i.e., tax rates are not proportional to amount consumed

### (7) Special Cases

#### (7a) Cobb-Douglas has Proportional Tax

Cobb-Douglas utility function with endowment of only one good will have proportional taxes on all but the numeraire

$$U(x) = b_1 \ln(x_1 + \omega_1) + \sum_{i=2}^{n} b_i \ln x_i$$

This is a Cobb-Douglas utility function

To make life even easier, assume $$\sum_{i=1}^{n} b_i = 1$$ (constant utility of income)

(Results below still hold without the constant utility of income)

**Endowment** - $$\omega_1$$ is amount of good 1 available (doesn't matter which good is the endowment as long as it's only one good)

**Buying & Selling** - note structure of utility function implies the consumer will sell the endowment to buy other goods:

- (a) $$x_1 + \omega_1 \geq 0 \Rightarrow$$ sell good 1 so $$x_1 < 0$$, but can’t sell more than you have, $$\omega_1$$
- (b) $$x_i \geq 0 \Rightarrow$$ goods 2,...,n are what the consumer buys

**Budget Constraint** - $$q \cdot x = q \cdot \omega = \omega_i$$

**Cobb-Douglas** - know structure of optimal consumption:

$$x_j = \frac{b_j \omega_1}{q_j} \quad (j = 2,...,n) \quad \text{and} \quad x_i = \frac{(b_i - 1) \omega_i}{q_i} = (b_i - 1) \omega_i$$

**Uncompensated Elasticities** -

$$\frac{\partial x_i}{\partial q_k} = 0 \text{ for } i \neq k \Rightarrow \varepsilon_{ik} = 0$$

$$\frac{\partial x_k}{\partial q_k} = \frac{\partial}{\partial q_k} \left( \frac{b_k \omega_1}{q_k} \right) = -\frac{b_k \omega_1}{q_k^2} = -\frac{b_k \omega_1}{q_k} \Rightarrow \frac{q_k \frac{\partial x_k}{\partial q_k}}{x_k} = \varepsilon_{kk} = -1 \quad (k = 2,...,n)$$

**FOC wrt $$q_k$$:** $$-\lambda x_k = -\lambda \left( x_k + \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial t_k} \right)$$ (version used in Ramsey Rule, bottom of p.8)

Recall $$\frac{\partial x_i}{\partial q_k} = \frac{\partial x_i}{\partial t_k}$$ (top of p.7): $$-\lambda x_k = -\lambda x_k - \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial q_k}$$

Apply results of uncompensated elasticities: $$-\lambda x_k = -\lambda x_k + \lambda t_k \frac{x_k}{q_k} \quad (k = 2,...,n)$$
\[ \frac{\lambda - \alpha}{\lambda} = \frac{t_k}{q_k} \quad (k = 2, \ldots, n) \]

So all goods (except numeraire) are taxed at the same rate; the tax is proportional

(7b) **Inverse Elasticity Rule** - if \( \varepsilon_{ik} = 0 \; \forall \; i = 1, \ldots, n, \; k \neq i \) (i.e. goods are not related), then the tax rate on good \( k \) is inversely proportional to \( \varepsilon_{kk} \)

FOC wrt \( q_k \):

- \[ -\alpha x_k = -\lambda \frac{\partial x_k}{\partial t_k} = -\lambda \frac{\partial x_k}{\partial t_k} \sum_{i=1}^{n} t_i x_i \]

Pull out the \( k \)th term:

- \[ -\alpha x_k = -\lambda x_k - \lambda \frac{\partial x_k}{\partial t_k} \sum_{i=1}^{n} t_i x_i \]

Move derivative into the summation; recall \( \frac{\partial x_i}{\partial q_k} = \frac{\partial x_k}{\partial t_k} \):

- \[ -\alpha x_k = -\lambda x_k - \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial q_k} \]

From \( \varepsilon_{ik} = 0 \; \forall \; i \neq k \) \Rightarrow \frac{\partial x_i}{\partial q_k} = 0 \; \forall \; i \neq k:\n
- \[ -\alpha x_k = -\lambda x_k - \lambda t_k \frac{\partial x_k}{\partial q_k} \]

Multiply both sides by \( \frac{q_k}{x_k} : \frac{q_k}{x_k} (\lambda - \alpha) x_k = -\lambda t_k \frac{\partial x_k}{\partial q_k} \]

Recognize \( \varepsilon_{ik} = \frac{\partial x_k}{\partial q_k} \frac{q_k}{x_k} : q_k (\lambda - \alpha) = -\lambda t_k \varepsilon_{kk} \)

\[ \frac{\lambda - \alpha}{\lambda} = \frac{t_k}{q_k} \varepsilon_{kk} \ldots \text{tax rate } \ast \text{ uncompensated demand elasticity constant across goods} \]

\[ \frac{t_k}{q_k} = \frac{1}{\lambda} \frac{(\lambda - \alpha)}{\varepsilon_{kk}} \ldots \text{tax rate is inversely proportional to elasticity} \]

**General Rule** - result if approximation if own effects are much larger than cross effects

**Cobb Douglas** - original assumption is just \( \varepsilon_{ik} = 0 \; \forall \; i = 1, \ldots, n, \; k \neq i \), but if we look at budget constraint \( (q \cdot x = 0) \) we get interesting result (it only happens with Cobb-Douglas preferences):

Differentiate wrt \( q_k \):

- \[ x_k + \sum_{i=1}^{n} q_i \frac{\partial x_i}{\partial q_k} = 0 \]

\[ \varepsilon_{ik} = 0 \; \forall \; i \neq k \Rightarrow \frac{\partial x_i}{\partial q_k} = 0, \; i \neq k : x_k + q_k \frac{\partial x_k}{\partial q_k} = 0 \]

Multiply both sides by \( \frac{1}{x_k} : 1 + \frac{q_k}{x_k} \frac{\partial x_k}{\partial q_k} = 0 \Rightarrow \frac{q_k}{x_k} \frac{\partial x_k}{\partial q_k} = \varepsilon_{kk} = -1 \)

Constant elasticity of demand results from Cobb-Douglas preferences

**Demand Restriction** - doesn’t make sense that \( \varepsilon_{kk} = 0 \) if good 1 has the endowment (i.e., consumer sells good 1 to buy all other goods); changes FOC above:

- \[ x_k + q_i \frac{\partial x_i}{\partial q_k} + q_k \frac{\partial x_k}{\partial q_k} = 0 \Rightarrow q_k \frac{\partial x_k}{x_k q_k} = \varepsilon_{kk} = -1 - q_i \frac{\partial x_i}{x_k q_k} \]

**Complex** - don’t know what preferences result in these demand elasticities

**Dangerous** - asking numeraire (good 1) to do a lot
Don't Add Much - this rule applies to Cobb Douglas preferences ($\epsilon_i = -1 \ \forall \ i = 1, \ldots, n$) which agrees with rule 7a that tax rate is constant across all goods

Private Production - so fare we only considered having constant returns in private production sector; anything with increasing returns was put in government sector (to avoid regulation issue); we ignored decreasing returns; adding that to the private sector could result in profits

Profit Tax - if there's a 100% profits tax (on pure profit, not counting return to capital), the same optimal tax results go through

< 100% - papers by Munk; profits get returned to consumers in closed GE model so it changes the budget constraint: \( q \cdot x = I = (1 - \tau)p \cdot y \) (where \( \tau \) is the profits tax rate)

Problem - in work above, we claimed \( q \) and \( p \) were independent (so we could do 2 normalizations); if \( p \cdot y = 0 \) (no profit like we assumed above) or \( \tau = 1 \) (100% tax) there's no problem, but otherwise we'll have \( q \) and \( p \) not independent

Consumer Demands - \( x(q, I) = x(q, (1 - \tau)p \cdot y) = \hat{x}(q, p) \) ... depends on both consumer and producer prices; only get homogeneity of degree zero if we change \( q \) and \( p \) together

Solution - set \( p_1 \) so \( I = 0 \) (effectively a 100% profits tax)... this is not possible if there is some good that is not taxed

Complexities - need to worry about demand effects of taxes and producer responses (supply curves); Munk, Stiglitz, and Dasgupta cover "a lot of gory detail"

Other Restrictions - other types of tax restrictions covered by Munk, Stiglitz, and Dasgupta; "no clean results"

Multiconsumer Economy - all previous work used a 1 consumer economy; could model with multiple consumer by using indirect utility in social welfare function:

\[
V(q) = W(v^1(q), v^2(q), \ldots, v^n(q));
\]

\[
\frac{\partial V}{\partial q_k} = \sum_{h=1}^{n} \frac{\partial W}{\partial v^h} \frac{\partial v^h}{\partial q_k} = \sum_{h=1}^{n} \frac{\partial W}{\partial v^h} (-\alpha^h x_k^h) = -\sum_{h=1}^{n} \beta^h x_k^h
\]

where \( \frac{\partial v^h}{\partial q_k} = -\alpha^h x_k^h \) (middle of p.2)

\( \alpha^h \) = private marginal utility of “income” for person \( h \); (say “income” because we’re really measuring consumption in terms of good 1)

\( x_k^h \) = person \( h \)’s consumption of good \( k \)

\( \beta^h = \alpha^h \frac{\partial W}{\partial v^h} = social MU of consumption by person \ h \)

Correlations - Diamond-Mirrless Eqn 77 is a “real mess”; with multiple consumers demand reductions at optimal taxes depend on more factors (“correlations”):

\( \beta^h, x_k^h \) - is consumption concentrated around different MU of consumption?

\( \beta^h, \frac{\partial}{\partial t} \sum_{i=1}^{n} t_i x_i^h \) - how change in income affects amount of commodity tax paid

\( \beta^h, \left( \sum_{i=1}^{n} t_i x_i^h \right) \frac{\partial x_k^h}{\partial t^h} \) - product of tax paid and change in demand for good \( k \) wrt income
Pole Subsidy - Diamond (1975) cleaned up results from the "correlations" paper by adding a policy rule (pole subsidy) allowing government to return some of the commodity tax (same amount, I, for everyone); changes budget constraint: \( q \cdot x^h = I \)

Commodity vs. Income Tax - a commodity tax with same rate on all goods and \( q \cdot x = 0 \) is similar to an income tax

Add Public Goods - \( e \) (can be single good or many; same amount consumed by all consumers)

Consumer Problem - \( \max_{x} u^h(x^h, e) \)

Maximized Value - since \( x^h(q, I, e), \) max value (indirect utility) is \( v^h(q, I, e) \)

Aggregate Consumption - \( x = \sum_{h=1}^{H} x^h; x_i = \sum_{h=1}^{H} x_i^h \)

Production Constraint - \( F(x^h(q, I, e), e) = 0 \)

Social Problem - \( \max_{q,I,e} W(v^1, v^2, ..., v^H) \) s.t. \( F = 0 \)

Producer prices (p) are hidden in constraint

Lagrangian - \( L = W - \lambda F \)

FOC wrt \( q_k \) - \( \frac{\partial L}{\partial q_k} = \sum_{h=1}^{H} \frac{\partial W}{\partial v^h} \frac{\partial v^h}{\partial q_k} - \lambda \sum_{i=1}^{n} F_i \frac{\partial x_i}{\partial q_k} = 0 \) (using \( F_i = \frac{\partial F}{\partial x_i} \))

Tricks:

Normalizations - \( q_1 = p_1 = 1 = F_i \)

Private Production Objective - \( p_i = p_iF_i \forall i = 2, ..., n \) (top of p.3)

Aggregate consumption - \( x_i = \sum_{h=1}^{H} x_i^h \)

Social MU of Consumption - \( \beta^h = \alpha^h \frac{\partial W}{\partial v^h} \) (from p.13)

\( \frac{\partial v^h}{\partial q_k} = -\alpha^h x_k^h \) (middle of p.2)

FOC becomes: \( \sum_{h=1}^{H} \beta^h x_k^h = -\lambda \sum_{h=1}^{H} \sum_{i=1}^{n} p_i \frac{\partial x_i^h}{\partial q_k} \)

Replace \( p_i = q_i - t_i \): \( \sum_{h=1}^{H} \beta^h x_k^h = -\lambda \sum_{h=1}^{H} \sum_{i=1}^{n} (q_i - t_i) \frac{\partial x_i^h}{\partial q_k} \)

Multiply out the second term: \( \sum_{h=1}^{H} \beta^h x_k^h = -\lambda \sum_{h=1}^{H} \left( \sum_{i=1}^{n} q_i \frac{\partial x_i^h}{\partial q_k} - \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial q_k} \right) \)

Trick: differentiate budget constraint \( q \cdot x = I \) wrt \( q_k \): \( x_k^h + \sum_{i=1}^{n} q_i \frac{\partial x_i^h}{\partial q_k} = 0 \)

\( \sum_{i=1}^{n} q_i \frac{\partial x_i^h}{\partial q_k} = -x_k^h \) ... sub this into the FOC: \( \sum_{h=1}^{H} \beta^h x_k^h = -\lambda \sum_{h=1}^{H} -x_k^h - \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial q_k} \)
Slutsky Equation - \[
\frac{\partial x_i^h}{\partial q_k} = S_{ik}^h - x_k^h \frac{\partial x_i^h}{\partial I} \]
... sub this into the FOC:

\[
\sum_{h=1}^{H} \beta^h x_i^h = \lambda \sum_{h=1}^{H} \left( x_i^h + \sum_{i=1}^{n} t_i \left( S_{ik}^h - x_k^h \frac{\partial x_i^h}{\partial I} \right) \right)
\]

Move terms around:

\[
\sum_{h=1}^{H} \beta^h x_i^h = \sum_{h=1}^{H} \lambda x_i^h + \lambda \sum_{h=1}^{H} \sum_{i=1}^{n} t_i S_{ik}^h - \sum_{h=1}^{H} \lambda \sum_{i=1}^{n} x_i^h t_i \frac{\partial x_i^h}{\partial I}
\]

Social MU of Income to Consumer \( h \) - \( \gamma^h \equiv \beta^h + \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial I} \)

Give income to an individual; direct impact is social MU of consumption (\( \beta^h \)) plus social effect on government budget (i.e., tax revenue); the effect on the budget constraint equals the social value of relaxing the budget constraint (\( \lambda \)) times change in taxes paid by consumer's extra income; sometimes refer to \( \gamma^h \) as a reduction in cost of giving extra income to consumers (government gets some of it back in form of commodity tax revenue)

Now FOC becomes:

\[
\sum_{h=1}^{H} (\gamma^h - \lambda) x_i^h = \lambda \sum_{h=1}^{H} \sum_{i=1}^{n} t_i S_{ik}^h
\]

Right side is change in compensated demand for good \( k \) due to taxes (similar to what we had for Ramsey Rule (p.8), but here the left side is different):

\[
\sum_{h=1}^{H} (\gamma^h - \lambda) x_i^h = \lambda \Delta x_i^c
\]

FOC wrt \( I \) - \[
\frac{\partial L}{\partial I} = \sum_{h=1}^{H} \frac{\partial W}{\partial v^h} \frac{\partial v^h}{\partial I} - \lambda \sum_{i=1}^{n} F_i \frac{\partial x_i}{\partial I} = 0
\]

Tricks:

Same normalizations, etc. from p. 14 makes the right term:

\[
\lambda \sum_{h=1}^{H} \sum_{i=1}^{n} p_i \frac{\partial x_i^h}{\partial I}
\]

For left term use \( \beta^h = \alpha^h \frac{\partial W}{\partial v^h} \) and \( \frac{\partial v^h}{\partial I} = \alpha^h : \sum_{h=1}^{H} \beta^h (\alpha^h) \)

FOC becomes:

\[
\sum_{h=1}^{H} \frac{\beta^h}{\alpha^h} (\alpha^h) = \lambda \sum_{h=1}^{H} \sum_{i=1}^{n} p_i \frac{\partial x_i^h}{\partial I}
\]

Replace \( p_i = q_i - t_i : \sum_{h=1}^{H} \beta^h = \lambda \sum_{h=1}^{H} \sum_{i=1}^{n} \left( q_i \frac{\partial x_i^h}{\partial I} - t_i \frac{\partial x_i^h}{\partial I} \right) \)

Differentiate budget constraint \( q \cdot x^h = I \) wrt \( I \) - \[
\sum_{i=1}^{n} q_i \frac{\partial x_i^h}{\partial I} = 1
\]

FOC becomes:

\[
\sum_{h=1}^{H} \beta^h = \lambda \sum_{h=1}^{H} 1 - \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial I}
\]
\[ \sum_{h=1}^{H} \left( \beta^h + \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial I} \right) = \sum_{h=1}^{H} \gamma^h = \lambda H \]

1. social benefit of giving everyone a dollar = social cost of giving everyone a dollar
2. \( \lambda = \frac{1}{H} \sum_{h=1}^{H} \gamma^h \) = mean of \( \gamma^h \)

Combine FOCs - from interpretation (2) above, \( \sum_{h=1}^{H} (\gamma^h - \lambda) = 0 \) \( \therefore \) \( \bar{x}_k \sum_{h=1}^{H} (\gamma^h - \lambda) = 0 \) ;

add this to the FOC wrt \( q_k \) : \( \sum_{h=1}^{H} (\gamma^h - \lambda) x_k^h = \lambda \Delta x_k^i \) (from p.15)

\[ \sum_{h=1}^{H} (\gamma^h - \lambda) x_k^h - \bar{x}_k \sum_{h=1}^{H} (\gamma^h - \lambda) = \lambda \Delta x_k^i \]

Factor: \( \sum_{h=1}^{H} (\gamma^h - \lambda) (x_k^h - \bar{x}_k) = \lambda \Delta x_k^i \)

The left side is similar to covariance (just missing the denominator)

Results -
(a) if \( \text{Cov}(\gamma^h, x_k^h) > 0 \) (good concentrated among people with high \( \gamma \) ) \( \Rightarrow \) want to increase consumption (based on compensated demand) of good \( k \)
(b) if \( \text{Cov}(\gamma^h, x_k^h) < \text{Cov}(\gamma^h, x_j^h) \) \( \Rightarrow \) reduce consumption of good \( k \) more than that of good \( j \)

Public Goods - already added them in the previous section; this section will get the equivalent of the Samuelson rule in the presence of commodity taxes

\[ V(q, I, e) = W(v^1(q, I, e), v^2(q, I, e), \ldots, v^H(q, I, e)) \]

Recall from previous section:

\[ \beta^h = \text{social MU of consumption by person } h \quad \beta^h = \frac{\partial W}{\partial v^h} \frac{\partial v^h}{\partial q_k} \]

\[ \gamma^h = \text{social MU of income to consumer } h \quad \gamma^h = \beta^h + \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial I} \]

Public Good - now we'll look at how expenditure on public goods affects welfare:

Directly - how change in consumer's utility from change in public good affects welfare

Indirectly - how tax revenues change based on change in consumer demands from change in public goods

Social Marginal Benefit of Public Good for Consumer \( h \) - \( \delta^h = \frac{\partial W}{\partial v^h} \frac{\partial v^h}{\partial e} + \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial e} \)

Lagrangian - \( L = W - \lambda F \)

FOC wrt \( e \) - \( \frac{\partial L}{\partial e} = \sum_{h=1}^{H} \frac{\partial W}{\partial v^h} \frac{\partial v^h}{\partial e} - \lambda \sum_{i=1}^{n} F_i \frac{\partial x_i}{\partial e} - \lambda F_e = 0 \) (using \( F_i = \frac{\partial F}{\partial x_i} \); \( F_e = \frac{\partial F}{\partial e} \))

Use \( F_i = p_i - q_i - t_i \cdot \sum_{h=1}^{H} \frac{\partial W}{\partial v^h} \frac{\partial v^h}{\partial e} - \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial e} + \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial e} = \lambda F_e \)
Now, change in public goods can change demands \((x_i)\), but it has no effect on the budget constraint:

\[
\sum_{i=1}^{n} q_i \frac{\partial x_i}{\partial e} = 0
\]

Can break out consumers so

\[
\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial e} = \sum_{i=1}^{n} \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial e}
\]

FOC becomes:

\[
\sum_{h=1}^{H} \left( \frac{\partial W}{\partial v^h} \frac{\partial v^h}{\partial e} + \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial e} \right) = \sum_{h=1}^{H} \delta^h = \lambda F_e
\]

**Samuelson Rule** - says \(\sum MRS = MRT\) for public goods

**Modified** - \(MRT = F_e = \frac{1}{\lambda} \sum_{h=1}^{H} \delta^h\)

From previous section \(\lambda = \frac{1}{H} \sum_{h=1}^{H} \gamma^h\) : \(MRT = F_e = \frac{\sum_{h=1}^{H} \delta^h}{\sum_{h=1}^{H} \gamma^h} = \frac{\text{Sum of social gain from public good}}{\text{Sum of social gain from private good (income)}}\)

**Difference** - Samuelson assumed lump sum tax paid for the public goods so \(t_i = 0\); in this case, we can't combine terms; end up with \(MRT = \text{weighted sum of } MRS + \text{"external gain" from public good (i.e., additional commodity taxes raised)}\)