Majority Choice

Why - Government provides lots of goods (public and private); now we'll study the process by which government decides how much to provide; many models just assume efficiency (a "benevolent dictator" who makes optimal decisions based on maximizing social welfare), but we want to know if an actual decision process leads to efficiency

Majority Choice -
\[ X \subset \mathbb{R}^n \] is a potential set of policies
\[ x \in X \] is a policy vector (could be scalar if \( n = 1 \); \( n > 1 \) case has existence problems)

**Majority Equilibrium** - \( x_e \in X \) such that there exists no other \( x \in X \) that is strictly preferred by a strict majority of voters (i.e., \( x_e \) won't be defeated in pairwise comparison); \( x_e \) is also called the **Condorcet Winner**

Implicit Assumptions -
- Population of voters
- Vote to maximize their own utility (could be altruistic utility function)
- People who are indifferent vote for status quo (\( x_e \))
- Tie vote keeps status quo (\( x_e \) wins ties)
- No coalitions of voters; people vote individually
"I'm not going to write them on the board because we're not going to use them because I don't remember what they are."

Applications -
- **Committee Decisions** - smaller groups; subset of the electorate
- **General Elections** - everyone in the electorate votes
  - **Direct Democracy** - referenda; wasn't a big deal when E&H and Plott wrote their papers, but it's become a big issue now
  - **Election a Candidate** - this is subject to criticism; E&H don't like it because of presumption of pairwise comparison (can't have more than 2 candidates) and presumption that candidates can credibly commit to policy positions
- **Spatial** - E&H made a big deal about policies being defined spatially... they're political scientists so they didn't take basic economic principles as given

Single Dimensional Case - \( x \in X \subset \mathbb{R}^1 \); \( x \) is scalar; it may or may not be continuous

Utility - \( u_i = u_i(x) \) for \( i = 1, \ldots, n \) (number of voters); Note: this is a reduced form indirect utility function (only looks at policy issue)

**Median Voter Theorem** - Black (Econometrica, 1958) & Downs (JPE, 1957); assumptions:
(i) Unidimensional policy choice
(ii) \( x_i^* \) is voter \( i \)'s ideal point: \( u_i(x_i^*) > u_i(x_i) \) \( \forall \) feasible \( x_i \neq x_i^* \)
(iii) Single peaked preferences so \( x_i^* \) is unique: let \( y \) & \( z \) be two points along the \( x \) dimension such that either \( y, z \geq x_i^* \) or \( y, z \leq x_i^* \) (i.e., they're on the same side of \( x_i^* \)), voter \( i \)'s preferences are single peaked if \( u_i(y) > u_i(z) \iff |y - x_i^*| < |z - x_i^*| \) (i.e., prefer \( y \) to \( z \) iff \( y \) is closer to the peak)
Median Policy - let $N_R$ be the number of ideal points with $x_i^* \geq x_m$ (i.e., to the right of $x_m$) and let $N_L$ be the number of ideal points with $x_i^* \leq x_m$ (i.e., to the left of $x_m$); $x_m$ is a median policy iff $N_R \geq n/2$ and $N_L \geq n/2$.

Median Voter Theorem - $x_e$ is a majority voting equilibrium iff $x$ is a median policy ($x_m$).

Proof: Sufficient ($x_m$ => equilibrium)

Consider policy alternative $x > x_m$
Fact that $N_L \geq n/2$ implies a strict majority do not strictly prefer $x$
That means $x$ does not defeat $x_m$
Analogously, $x < x_m$ does not defeat $x_m$
\[\therefore x_m \text{ is an equilibrium}\]

Necessary (equilibrium => $x_m$)
Suppose $x_e$ is not a median policy $x_m$
Either $N_R < n/2$ or $N_L < n/2$
Consider $N_R < n/2$, then $N_L > n/2$ (because $N_R + N_L \geq n$)
That means $\exists$ a point to the left of $x_e$ that would defeat $x_e$ (because a majority strictly prefers it to $x_e$)
Analogous for $N_L < n/2$ case
\[\therefore x_e \text{ is a median policy}\]

Example - assume for simplicity that the $n$ ideal points are distinct

Odd - if $n$ is odd, then $x_m$ is unique and equal to the ideal point of the "median voter" (also called pivotal voter)
Any point left of $x_e$ will be defeated because 3, 4 & 5 prefer $x_e$

Even - if $n$ is even, then there is a continuum of equilibria between the two median voters

Aside - many models assume continuum of types/voters; results in distribution of ideal points ($x^*$) and will have a unique $x_m$; if the cumulative distribution function is $F(x^*)$ then the unique voting equilibrium is $x_m$ which solves $F(x_m) = 0.5$

Single Peak Assumption - it is sufficient, but not necessary for $x_e$ to exist; that is, if preferences are single peaked, then $x_e$ exists, but if preferences are not single peaked, $x_e$ may not exist, but it might (e.g., single crossing preferences)

Public Good Example - consider majority choice of public good financed by proportional income tax ($t \in [0,1]$); assumptions:
- Voters have same preferences with constant elasticity of substitution (CES):
  \[u = [x^\rho + G^\rho]^{1/\rho}\]
  where $x$ = amount of private good; $G$ = amount of public good; $\rho < 1$ is required for quasiconcavity (also determines the CES)
- Voters differ by endowed income $y$
- $F(y)$ is the cumulative distribution function of $y$; it can be discrete or continuous
- Government budget constraint is $G \leq tY$, where $Y = \sum y =$ aggregate income
  Efficient allocation has $G = tY$ so government doesn’t waste funds
- Normalize population size to 1; simplification so we don’t need to multiply everything by the number of voters; also results in mean income $\bar{y} = Y$
- Private budget constraint is $x \leq (1-t)y$ (again efficient allocation has $x = 0$ so consumer doesn’t throw money away)

**Majority Choice Policy** - has 2 elements $(t, G)$, but reduces to one dimensional when imposing Pareto efficient government budget constraint ($G = t\bar{y}$); this is not an unreasonable assumption because any $(t, G)$ with $G < t\bar{y}$ would be unanimously defeated by $(t, G')$ with $G' = t\bar{y}$ or $(t', G)$ with $G = t'\bar{y}$ (i.e., get more for the same taxes or pay less for the same amount of public good)

**One Dimension** - can view problem as voting over $t$ or over $G$; get the same result; we’ll look at voting over $t$

**Indirect Utility** - "With some abuse of notation"... incorporate Pareto efficiency of individual and government budgets to get $V(t) = [(1-t)^\rho y^\rho + t^\rho \bar{y}^\rho]^{1/\rho}$

**Single Peaked** - want to show $V(t)$ is single peaked so we can apply median voter theorem

**Definition** - convenient way to show single peak:

wherever $V''(t^*) = 0$ we have $V''(t^*) < 0$

"You can always do that. It’s a good trick"

"This is just straight forward algebra"

$$V' = \frac{1}{\rho} \left[ (1-t)^{\rho-1} y^\rho + t^\rho \bar{y}^\rho \right] - \rho(1-t)^{\rho-1} y^\rho + \rho t^{\rho-1} \bar{y}^\rho$$

$$\Rightarrow \frac{V' - \rho(1-t)^{\rho-1} y^\rho}{V^\rho} = \frac{1}{\rho} \left[ (1-t)^{\rho-1} y^\rho + t^\rho \bar{y}^\rho \right]$$

Solve $V'(t^*) = 0$ for $t^*$ (only need numerator):

$$-(1-t)^{\rho-1} y^\rho + t^{\rho-1} \bar{y}^\rho = 0 \Rightarrow t^{\rho-1} \bar{y}^\rho = (1-t)^{\rho-1} y^\rho \Rightarrow \left[ \frac{t}{1-t} \right]^{\rho-1} = \left[ \frac{y}{\bar{y}} \right]^\rho$$

On the left, the exponent is always $> 1$ because $\rho < 1$, that means the left side is decreasing in $t$ (from $\infty$ when $t = 0$ to 0 when $t = 1$); since right side if fixed value there is a unique solution

"Easy to Verify"

If $\rho \in (0,1)$, then $t^*$ is decreasing in $y$

If $\rho < 0$, then $t^*$ is increasing in $y$

If $\rho = 0$, then $t^*$ is the same for all $y$; "trivial" case where CES reduces to Cobb-Douglas and there is unanimous preference for $t^*$

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\[ V'' = \frac{V^{\rho-1}[(\rho - 1)(1-t)^{\rho-2} y^\rho + (\rho - 1) t^{\rho-2} \bar{y}^\rho] - \left[-(1-t)^{\rho-1} y^\rho + t^{\rho-1} \bar{y}^\rho\right] (\rho - 1)V^{\rho-2} V'}{[V^{\rho-1}]^2} \]

\[ = \frac{1}{V^{\rho-1}} (\rho - 1)[(1-t)^{\rho-2} y^\rho + t^{\rho-2} \bar{y}^\rho] < 0 \]

\[ \therefore V(t) \text{ is single peaked} \]

**Apply MVT** - since \( V(t) \) is single peaked, we can apply median voter theorem so equilibrium is a median policy; further we know (for the non-trivial case), \( t^*(y) \) is monotonic so the unique equilibrium is \( t^*(y_{\text{median}}) \)

**Different Perspective** - look at same model from individual voter’s view:

\[
\max_{t,G} u = \left[ x^\rho + G^\rho \right]^{1/\rho}
\]

\[
\text{s.t. } x = (1-t)y \quad \text{Individual budget constraint (technically } \leq)
\]

\[
G = t\bar{y} \quad \text{Government budget constraint (technically } \leq)
\]

**Indirect Utility** - \( V(t,G) \equiv \max_x \left[ x^\rho + G^\rho \right]^{1/\rho} \) s.t. \( x \leq (1-t)y \) \( \Rightarrow \)

\[
V(t,G) \equiv \max_x \left[ (1-t)^\rho y^\rho + G^\rho \right]^{1/\rho}
\]

**Equivalent Problem** - \( \max_{t,G} V(t,G) \) s.t. \( G = t\bar{y} \)

This is quasiconcave preferences with a convex budget set so it’s equivalent to single peaked (see graphs)

**Indifference Curves** - \( V(t,G) = \text{constant} \); "easy to show" shape with derivatives:

\[
\text{Totally differentiate: } V_t dt + V_G dG = 0 \quad \Rightarrow \quad \left. \frac{dt}{dG} \right|_{V \text{ constant}} = -\frac{V_G}{V_t}
\]

Work out \( V_G \) & \( V_t \) : (recall \( V(t,G) = [(1-t)^\rho y^\rho + G^\rho]^{1/\rho} \))

\[
\left. \frac{dt}{dG} \right|_{V \text{ constant}} = -\frac{V_G}{V_t} = \frac{1}{\rho}(1-t)^{\rho-2} y^\rho + G^\rho \left[ \frac{\rho G^{\rho-1}}{(1-t)^{\rho-1} y^\rho} \right] = \frac{G^{\rho-1}}{(1-t)^{\rho-1} y^\rho} > 0
\]

(so indifference curves are upward sloping)

\[
\left. \frac{dt}{dG^2} \right|_{V \text{ constant}} = \frac{1}{\rho(1-t)^{\rho-2} y^\rho} + G^{\rho-1} (1-t)^{\rho-1} \left[ \frac{dt}{dG} \right]_{V \text{ constant}} < 0 \quad \text{(concave)}
\]

**Failure of Single Peak** - “Some of them are truly less important than others”

**Example with No Equilibrium** - consider example with 3 voters shown here:

Voting equilibrium does not exist; proof by brute force:

\[
x \leq A \quad \Rightarrow \quad 2 \text{ & } 3 \text{ prefer } B
\]

\[
x \in \{A,B\} \quad \Rightarrow \quad 1 \text{ & } 3 \text{ prefer } D
\]

\[
x \in \{B,C\} \quad \Rightarrow \quad 1 \text{ & } 2 \text{ prefer } B \text{ (or } 1 \text{ & } 3 \text{ prefer } C)
\]

\[
x \in \{C,D\} \quad \Rightarrow \quad 1 \text{ & } 2 \text{ prefer } A
\]

\[
x \geq D \quad \Rightarrow \quad 2 \text{ & } 3 \text{ prefer } C \quad \therefore \text{ no equilibrium}
\]
**Proportional Tax Model** - return to model we just studied (starts at bottom of p.2); suppose public good is not produced according to \( G = \bar{r} \), but instead is \( G = \bar{G}(r\bar{y}) \) which is not well behaved (i.e., not concave); best tax \( t \) is still the minimum required to get \( G \); there are no single peaked preference, but \( V(t, G) \) has single crossing so voting equilibrium still exists

**Equilibrium** - if \( V(t, G) \) has single crossing, the form of the "production function" \( G(t) \) doesn't matter (as long as some points exist); there will be a voting equilibrium (formal theorem will be proved below)

**Single Crossing** - indifference curves (i.e., \( V(t, G) \) = constant) among different agents (i.e., with different incomes) cross at most once; effectively this means there is an unambiguous ordering of preferences (it's used in lots of other areas of economics)

**Verify** - good way to verify single crossing is to show one set of indifference curves is always steeper (or flatter) than the other

**Slopes** - already showed slope on previous page:

\[
\frac{dt}{dG} = -\frac{G}{V} \left( \frac{G^{\rho-1}}{1-t} \right) y^{-\rho} > 0
\]

Changing type means changing income, so check how slope changes wrt \( y \):

\[
\frac{d^2 t}{dG dy} = -\rho \left( \frac{G}{1-t} \right)^{\rho-1} y^{-\rho-1}
\]

If \( \rho = 0 \), this derivative = 0 (indifference curves are the same for all types)
If \( \rho \in (0,1) \), this derivative < 0 (as \( y \uparrow \), indifference curves get flatter; Epple & Romano called this Slope Decreasing in Income, SDI)
If \( \rho < 0 \), this derivative > 0 (as \( y \uparrow \), indifference curves get steeper; E&R called this Slope Rising in Income, SRI... higher income types have greater willingness to bear tax increase in exchange for increasing in \( G \))

**Single Crossing Theorem** - with single crossing preferences, voting equilibrium exists and will be at the most preferred point of the median individual (in the discrete case all points between the two median individuals are equilibria)

**Proof**: "Nice, simple geometric proof"; proof by construction

Consider graph on the right; point shown is the most preferred point for the median individual (i.e., 50% have higher income and 50% have lower income that this guy)

Define the four regions as shown; now for each region show there are no points majority preferred to \( (t^*, G^*) \)

Region A: \( \{ (t, G) : t \geq t^*, G \leq G^* \text{, at least 1 strict inequality} \} \)

\( (t^*, G^*) \) is Pareto preferred to any point in A (it has the lowest tax with the biggest \( G \))

Region B: set of points strictly preferred to \( (t^*, G^*) \) (doesn't include \( V_{\text{median}} \))

These points are not feasible (if any of them were, that would contradict \( (t^*, G^*) \) being the most preferred point for the median person)

Region C: set of points with \( G > G^* \) that lie above (& including) \( V_{\text{median}} \)
By single crossing, half the voters do not prefer any such points because they're strictly worse off in C; these are the guys with flatter indifference curves through \((t^*, G^*)\); depends on SRI or SDI whether they're higher or lower income than median

**Region D:** analogous to C for the steeper indifference curves
∴ no majority of individuals wants to deviate from \((t^*, G^*)\) to it's the voting equilibrium

**Multi-Dimensional Voting**

**Example** - \(u_i(y_i - t_1 - t_2, G_1, G_2)\); 2 head taxes (i.e., everyone pays same amount) to pay for two public goods; tax 1 finances one so \(G_1 = t_1 N\); tax 2 finances the other so \(G_2 = t_2 N\)

(where \(N\) is the number of voters)

Indirect Utility is \(V(y_i, t_1, t_2)\)

"Very hard to show equilibrium."
"The analysis of these issues is much much much more difficult."

**Two-Dimensional Problem**
- Let \(x = (x_1, x_2)\) denote a policy choice and assume any \(x \in R^2\) is feasible
- Let \(x^*_i = (x^*_1, x^*_2)\) denote voter \(i\)'s ideal point (\(i = 1, 2, \ldots, n\))

**Circular Utility** - \(u_i(x_1, x_2) = -\left[\left(x_1 - x^*_1\right)^2 + \left(x_2 - x^*_2\right)^2\right]^{1/2}\) = negative of Euclidean distance from the voter's ideal point; gives circular indifference curves centered in the ideal point

**Distinguishing Voters** - in this case, all we need to identify an individual is his ideal point; from that, we can determine which policies he prefers based on their distance from his ideal point

When do we get equilibrium? "The short answer is not very often."

**3 Voters** - voting equilibrium exists iff ideal points lie in a straight line in \((x_1, x_2)\) plane

**Proof:** Necessary (Equilibrium \(\Rightarrow\) straight line)

Prove contrapositive: not straight line \(\Rightarrow\) not equilibrium

Pick any three ideal points that are not in a straight line (so they form a triangle)

**Contract Curve** between any 2 individuals is the straight line between them (where their indifference curves are tangent); this line minimizes the sum of the distances of each point to the 2 ideal points; we can't make 1 of these individuals better off without making the other worse off

**Pareto Set** among all 3 individuals is the triangle (for any point outside the triangle, all 3 can be made better off); "We're going to be short on proofs" (i.e., "clearly")

Any point inside the triangle is defeated (any combination of 2 individuals can find points they prefer)

Any point on the triangle is defeated (there exist points where 2 people are strictly better off
∴ no equilibrium exists
Sufficient (Straight line $\Rightarrow$ Equilibrium)
Consider ideal point in the middle
Person 2 is against any deviations (because it's his ideal point)
Anything 1 favors is opposed by 3 (they have diametrically opposed preferences)
$\therefore$ Ideal point is the middle is the voting equilibrium
"May not rise to the level of a proof"

For more pictures and explanations for the 3 voter case, see ECO 7120 (General Equilibrium) notes "Introduction to Welfare and Equilibrium" (p.5)

5 Voters - assume distinct ideal points; there are only 2 cases where we get equilibrium:
- **Trivial Case** - "stars are aligned"; same as 3 voter case
- **Intersection Case** - center guy is at intersection of contract curves from opposite corners

**Proof**: (sort of) first show this is an equilibrium:
Person 5 is against any deviations (it's his ideal point)
Anything 1 favors is opposed by 4 (and vice versa)
Anything 2 favors is opposed by 3 (and vice versa)
$\therefore$ anything one person favors is opposed by the other four and anything two people favor is opposed by the other three

Now show no other points are an equilibrium:
- **Paretian Set** - equilibrium has to be in the Paretian set for every three voters (otherwise 3 people can vote against it). $\therefore$ equilibrium must be in the intersection of these Paretian sets

There are a total of $\frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3}{3!} = 10$ sets

Look at the four Paretian sets shaded here; the only intersection is 5's ideal point
Now look at the Paretian set for (1, 3, 4); this set does not contain 5's ideal point
$\therefore$ the intersection of all the Paretian sets is empty

More Voters - doesn't help; equilibrium is less likely
Even Numbers - gets more messy
Key - pair people up so we get diametrically opposed preferences in equilibrium

Plott - THE paper on majority voting (and difficulty of finding equilibrium); didn't bother with proof of sufficiency because necessity is hard enough to satisfy

**Assumptions** -
(a) more general preferences (quasi-concave)
(b) any dimension of policy space ($\geq 2$)
(c) $n$ is odd; Plott said something like "straight forward to extend to even number" to which Romano said, "That's one of those 'Maybe it's straight forward for someone like Ken Arrow.'"

**Necessary Conditions** - only cover this intuitively; a majority voting equilibrium must...
- Be an ideal point of one voter (there are other equilibria for even number)
- Be such that other voters can be split into pairs such that their contract curves go through the equilibrium points with their ideal points on opposite sides
$\therefore$ equilibrium doesn't exist generically; if we randomly select utility functions, equilibrium is not very likely
Other Issues - potential resolutions of existence problem under multi-dimensional voting:

Agenda Control - McKelvey (JET, 1976)... "not an easy paper"; usually only applies to committees; adds more structure and makes equilibrium possible

Status Quo - start with initial policy $x_1$

Agenda Setter - picks policies $x_2, x_3, \ldots, x_T$ (what they are, how many there are, and in what order they're considered)

Votes - electorate votes on two policies at a time (e.g., consider $x_1$ vs. $x_2$, then consider the winner there vs. $x_1$, etc.)

Result - can get to any outcome (even non-Paretian); agenda setter has tremendous power; he'll likely rig it to get his ideal point

Big Assumption - myopic voting (voters express preferences honestly; they don't consider agenda setter's motives)
"I don't know squat about that literature"

Vote Trading - also called logrolling; gets Paretian outcome; effectively creates a market for votes
"I don't know anything about the theory or the legality."
"I'm not at all interested in this stuff."

Uncertainty - smoothes things out to equilibrium exists

"Are people waiting to get in here yet?"
"I only have a minute so it'll be brief."

Restricted Policy Choice Space -

Discrete Policies - Epple & Romano; public provision of private good through proportional income tax where individuals can supplement (i.e., income tax pays for certain amount of public good per person, then people can go to private market and buy more of it on their own); all guys above mean income want zero tax, but if we make supplementation illegal, preferences of the "rich" guys change drastically (they now prefer a large income tax)

Multidimensional voting problem over $(t, S)$ ($S = 1$ means supplementation allowed)
E&R show equilibrium exists and has $S = 1$

Representative Democracy - Besley & Coate (QJE, 1997); more general than the Citizen-Candidate Model by Osborne & Slivinski (unidimensional); people can only credibly commit to what they really want; only so many potential candidates $\therefore$ feasible space is restricted to candidates' ideal points

Sequential Voting - reduces to single dimension in each state; "it may or may not make any economic sense"