Private Provision of Public Goods with Impure Altruism
Andreoni. "Giving with Impure Altruism: Applications to Charity and Ricardian Equivalence." 

Andreoni extended the BBV model to include utility directly from giving, not just from the total contribution to the public good; BBV: \( u_i(x,Y) \) vs. Andreoni: \( u_i(x,Y,g_i) \)

Result - will not have the neutrality result of the BBV model

Model -
- Utility - \( u_i(x,Y,g_i) \) is increasing, differentiable and quasi-concave \( \forall \ i = 1,2,\ldots,n \)

- Everyone Gives - \( \lim_{\varepsilon \to 0} \frac{\partial u_i}{\partial g_i} = \infty \ldots \) guarantees an interior solution \( g_i \geq 0 \)

- Voluntary Donations - individual \( i \) gives \( g_i \)

  Total voluntary giving: \( G = \sum_{i=1}^{n} g_i \); giving by everyone except individual \( i \) : \( G_{-i} = \sum_{j \neq i} g_j \)

- Forced Donations - \( \tau_i \) is exogenous lump sum tax on individual \( i \); all the tax revenue is used to supply the public good (i.e., forced donation)

  Total forced giving (taxes): \( T = \sum_{i=1}^{n} \tau_i \); taxes from everyone except individual \( i \) : \( T_{-i} = \sum_{j \neq i} \tau_j \)

- Total Giving - individual \( i \) contributes \( y_i = g_i + \tau_i \); total from everyone except \( i \) : \( Y_{-i} = \sum_{j \neq i} y_j \)

- Total Public Good - combines voluntary and forced contributions: \( Y = \sum_{i=1}^{n} (g_i + \tau_i) = G + T \)

- Two Effects - utility increases from 2 effects:
  - Altruism Effect - \( g_i \uparrow \Rightarrow Y \uparrow \Rightarrow u_i(x_i,Y,g_i) \uparrow \)
  - Egoism Effect - \( g_i \uparrow \Rightarrow u_i(x_i,Y,g_i) \uparrow \ldots \) also called "warm glow"

  Len calls it "honor’s student effect" (only do community service for recognition)

- Endowed Wealth - \( \omega_i \) (same as before)

- Everything in Dollars - public good is measured in units of the numeraire \( (x_i) \) which is measured in dollars

Nash Equilibrium - want to examine properties of contributions (and implied private consumption) taking tax policy as given (i.e., for given \( \tau_1,\tau_2,\ldots,\tau_n \))

Existence & Uniqueness - assumed in the paper (doesn’t prove them like BBV did)

Consumer Problem - \( \max_{x_i,Y,g_i} u_i(x_i,Y,g_i) \)

s.t. \( x_i + g_i = \omega_i - \tau_i \) \hspace{1cm} Consumer budget constraint \hspace{1cm} (1)

\( Y = T + G_{-i} + g_i \) \hspace{1cm} Public good constraint \hspace{1cm} (2)

Note: \( g_i \geq 0 \) not required by the everyone gives assumption

Solve (2) for \( g_i : g_i = Y - T - G_{-i} \)

Sub \( T = T_{-i} + \tau_i \) and \( Y_{-i} = T_{-i} + G_{-i} \cdot g_i = Y - (T_{-i} + \tau_i) - G_{-i} = Y - Y_{-i} - \tau_i \)

Sub this result into (1): \( x_i + (Y - Y_{-i} - \tau_i) = \omega_i - \tau_i \Rightarrow x_i = \omega_i + Y_{-i} - Y \)
Sub $g_i$ and $x_i$ into the constraint to get an unconstrained maximization problem:

$$\max_y u_i(\omega_i + Y_{-i} - Y, Y, Y - Y_{-i} - \tau_i)$$

**FOC:**

$$\frac{\partial u_i}{\partial x_i} \frac{\partial x_i}{\partial Y} + \frac{\partial u_i}{\partial Y} + \frac{\partial u_i}{\partial g_i} \frac{\partial g_i}{\partial Y} = 0 \Rightarrow -\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial Y} + \frac{\partial u_i}{\partial g_i} = 0$$

Because $\frac{\partial x_i}{\partial Y} = \frac{\partial(\omega_i + Y_{-i} - Y)}{\partial Y} = -1$ & $\frac{\partial g_i}{\partial Y} = \frac{\partial(Y - Y_{-i} + \tau_i)}{\partial Y} = 1$

**Demand** - theoretically solving the FOC for $Y$ yields the demand for public good:

$$Y = f_i(\omega_i + Y_{-i}, Y_{-i} + \tau_i)$$

Demand depends on $u_i$, $x_i$ & $g_i$ (where the two arguments satisfy the constraints of the consumer problem above)

Sub in $Y = y_i + Y_{-i}$:

$$y_i = f_i(\omega_i + Y_{-i}, Y_{-i} + \tau_i) - Y_{-i} \quad \text{(Eqn (3) in paper)}$$

Sub in $y_i = g_i + Y_{-i}$:

$$g_i = f_i(\omega_i + Y_{-i}, Y_{-i} + \tau_i) - Y_{-i} - \tau_i \quad \text{(Call this (3')... not in paper, but it'll come in handy)}$$

"Whenever you hear 'clearly,' you want to be suspicious." (Romano)

**Properties of $f_i$** - assuming $x_i, Y, g_i$ are all normal goods

**Notation** - $Y = f_i(\omega_i + Y_{-i}, Y_{-i} + \tau_i) = f_i(a, e)$ (altruism & egoism effects)

$$f_{ia} = \frac{\partial f_i}{\partial a} \text{ and } f_{ie} = \frac{\partial f_i}{\partial e}$$

**Fact i** - $g_i$ and $\tau_i$ are not seen by consumer $i$ as perfect substitutes (as they are in pure altruism model which gives neutrality result)

**Proof**: totally differentiate (3') ($g_i$ & $\tau_i$ vary; $\omega_i$ & $Y_{-i}$ are constant): $dg_i = f_{ia} d\tau_i - d\tau_i$

Neutrality requires $dg_i = -d\tau_i$ (e.g., taxes increase by $10$ so giving decreases by $10$)... this requires $f_{ia} = 0$, which results if there is no egoism effect

**Fact ii** - $f_{ie} > 0$ (formal proof requires normal good assumption; we won't do it)

**Intuitive Proof**: if no egoism effect, then $dg_i = -d\tau_i$ (neutrality); if there is an egoism effect, then $dg_i < -d\tau_i$ (i.e., consumer won't cut back private contribution by same amount as tax) :: in order for $dg_i = f_{ie} d\tau_i - d\tau_i$, we must have $f_{ie} > 0$

**Fact iii** - $f_{ia} > 0$

**Proof**: from (3'):

$$\frac{dg_i}{d\omega_i} = f_{ia} > 0 \ldots \text{because } x_i \text{ is a normal good}$$

In fact, if $x_i$ & $g_i$ are both normal, then $0 < f_{ia} < 1$

**Fact iv** - $0 < f_{ia} + f_{ie} \leq 1$ if goods are normal in social wealth ($Y_{-i}$)

**Proof**: differentiate $Y$ wrt $Y_{-i}$:

$$\frac{dY}{dY_{-i}} = f_{ia} + f_{ie} > 0 \ldots \text{because } Y \text{ is a normal good}$$

Sum is $< 1$ if everything is a normal good

**Boundary Cases**
Fact v(a) - pure altruism: \( f_{ia} = 0 \) (BBV model with \( u(x_i, Y_i) \)); get perfect crowding out (neutrality result; raise taxes by amount less than \( g_i \) and giving declines by same amount)

Fact v(b) - pure egoism: \( f_{ia} + f_{ie} = 1 \) (\( u(x_i, g_i) \)); person \( i \) doesn't care about total amount of public good, just how much he contributes to it; get no crowding out; Katie: "think of it as two private goods"

Proof: not caring about total means other people's giving doesn't affect what \( i \) gives; that means \( \frac{dg_i}{dY_{-i}} = 0 \)

Use (3') (holding \( \tau_i \) fixed) to calculate \( \frac{dg_i}{dY_{-i}} = f_{ia} + f_{ie} - 1 = 0 \Rightarrow f_{ia} + f_{ie} = 1 \)

Measure of Altruism - relative to egoism; measures to what extent social wealth (\( Y_{-i} \)) and own wealth (\( \omega_i \)) are substitutes in providing the public good (\( Y \)); i.e., if we decrease \( \omega_i \) by $1, by how much must we increase \( Y \) to get the same \( Y \)

Pure Altruist - \( Y_{-i} \& \omega_i \) are perfect substitutes (neutrality); \( \omega_i \downarrow \) by $1 \Rightarrow \omega_i \uparrow \) by $1 because consumer \( i \) doesn't reduce his contributions by $1 since he gets utility from \( g_i \) directly

Not Pure Altruist - \( \omega_i \downarrow \) by $1 \Rightarrow \omega_i \uparrow \) by less than $1 because consumer \( i \) doesn't reduce his contributions by $1 since he gets utility from \( g_i \) directly

Definition - \( \alpha_i \equiv \frac{dY_{-i}}{d\omega_i} \bigg|_{Y \text{ constant}} = \frac{f_{ia}}{f_{ia} + f_{ie}} \)

Since we want to hold \( Y \) constant, set \( dY = 0 \)

Totally differentiate \( Y = f_i(\omega_i + Y_{-i}, Y_{-i} + \tau_i) \) (with \( \tau_i \) fixed)

\( dY = f_{ia}(d\omega_i + dY_{-i}) + f_{ie} dY_{-i} = 0 \)

Rearrange terms to get \( -dY_{-i}/d\omega_i \Rightarrow f_{ia}/(f_{ia} + f_{ie}) \)

Boundary Cases

Pure Altruism - \( f_{ia} = 0 \) so \( \alpha_i = 1 \)

Pure Egoism - \( f_{ia} + f_{ie} = 1 \) so \( \alpha_i = f_{ia} \ldots \therefore f_{ia} \leq \alpha_i \leq 1 \)

Analysis of Neutrality and Crowding Out

Increase Tax - suppose we increase tax on consumer 1 (\( d\tau_1 > 0, d\tau_j = 0 \forall j \neq 1 \)); consumer 1’s wealth doesn’t change (\( \omega_1 \) constant), but other consumers can change their contributions based on consumer 1’s contribution (so \( T_{-1} \) varies)

Totally differentiate (3): \( dy_1 = f_{1a} dY_{-1} + f_{1e} (dY_{-1} + d\tau_1) - dY_{-1} \)

Sub in \( dY = dy_1 + dY_{-1} \Rightarrow dY_{-1} = dY - dy_1 \):

\( dy_1 = f_{1a} (dY - dy_1) + f_{1e} (dY - dy_1 + d\tau_1) - (dY - dy_1) \)

Multiply out the terms: \( dy_1 = f_{1a} dY - f_{1a} dy_1 + f_{1e} dY - f_{1e} dy_1 + f_{1e} d\tau_1 - dY + dy_1 \)

Combine terms: \( (f_{1a} + f_{1e}) dy_1 = (f_{1a} + f_{1e} - 1)dY + f_{1e} d\tau_1 \)

Solve for \( dy_1 \): \( dy_1 = \frac{f_{1a} + f_{1e} - 1}{f_{1a} + f_{1e}} dY + f_{1e} d\tau_1 \)
Trick: $\alpha_i = \frac{f_{ia}}{f_{ia} + f_{ie}} \Rightarrow 1 - \alpha_i = \frac{f_{ia} + f_{ie} - f_{ia}}{f_{ia} + f_{ie}} = \frac{f_{ie}}{f_{ia} + f_{ie}}$

Sub that in: $dy_1 = \frac{f_{ia} + f_{ie} - 1}{f_{ia} + f_{ie}} dY + (1 - \alpha_i) d\tau_1$  (Eqn (4) in paper)

Other consumers (recall $d \tau_j = 0$): $dy_j = \frac{f_{ja} + f_{je} - 1}{f_{ja} + f_{je}} dY$  ($j = 2,3, \ldots n$  (Eqn (5) in paper)

Add these up to get total effect on the amount of the public good:

$$dy_1 + \sum_{j \neq 1} dy_j = dY = \frac{1}{1 - \sum_{i=1}^{n} \frac{f_{ia} + f_{ie} - 1}{f_{ia} + f_{ie}}} dY + (1 - \alpha_i) d\tau_1$$

Solve for $dY$:

$$dY = \frac{1}{1 - \sum_{i=1}^{n} \frac{f_{ia} + f_{ie} - 1}{f_{ia} + f_{ie}}} (1 - \alpha_i) d\tau_1$$

Simplify by letting $c = \left(1 - \sum_{i=1}^{n} \frac{f_{ia} + f_{ie} - 1}{f_{ia} + f_{ie}} dY\right)^{-1}$  $\Rightarrow dY = c(1 - \alpha_i) d\tau_1$  (Eqn (6) in paper)

**Signing** - from the facts on p.2, we know: $f_{ia} > 0$, $f_{ia} > 0$, $0 < f_{ia} + f_{ie} \leq 1$ which means that $0 < c \leq 1$; we also know $0 < f_{ia} \leq \alpha_i \leq 1$

$\therefore$ as long as consumer 1 is not a pure altruist, we have less than perfect crowding out (i.e., $dY > 0$)

Note also that $c(1 - \alpha_i) < 1$ so we end up with the non-neutrality result: $0 < dY < d\tau_1$

**English** - total spending on the public good increases, but not by as much as the tax on consumer 1; if consumer 1 were a pure altruist, then there is perfect crowding out ($dY = 0$)

**Wealth Redistribution** - consider transferring wealth from consumer 2 to consumer 1 (paper does it in reverse); we can view this as changing their taxes so $d\tau_2 > 0$ and $d\tau_1 = -d\tau_2$

**Result** - use previous result on changing taxes to compute $dY = \frac{dY}{d\tau_1} \cdot d\tau_1 + \frac{dY}{d\tau_2} \cdot d\tau_2$

$$dY = c(1 - \alpha_1) d\tau_1 + c(1 - \alpha_2) d\tau_2$$

Sub in $d\tau_1 = -d\tau_2$:

$$dY = c(\alpha_1 - \alpha_2) d\tau_2$$

**Cases** -

- **Neutrality** - if $\alpha_1 = \alpha_2$ then $dY = 0$ (i.e., if consumers are equally altruistic, there’s no change in the public good from changing wealth... NOTE: we’re not talking about consumption of the private good which will NOT be neutral

- **Gain Y** - if $\alpha_1 > \alpha_2$ then $dY > 0$ (i.e., if we transfer wealth to a more altruistic person, there’s an increase in the public good); this results because the less altruistic person from whom the money is taken will continue to contribute (as opposed to a pure altruistic person whose contribution will decrease by the same amount of the tax)

- **Lose Y** - if $\alpha_1 < \alpha_2$ then $dY < 0$... opposite of gain case
Non-Givers - on p.1 we assumed \( \lim_{g \to 0} \frac{\partial u_i}{\partial g_i} = \infty \) so everyone gives; if we drop this assumption and tax a non-giver, we still get the non-neutrality result from redistribution; if the consumer wasn’t giving, increasing his taxes will not change his contributions, but giving that money to a contributor will increase giving overall (neutrality means \( dY = 0 \))

**Overall** - this model has a lot more givers than BBV’s pure altruism model, which makes this more realistic (pure altruism doesn’t hold empirically; see p.8 of BBV notes)

**Implications for Ricardian Equivalence**

**Ricardian Equivalence** - timing of taxation to finance a public good is irrelevant (i.e., government debt financing doesn’t matter)

**Implication** - the original RE result requires pure altruism (the equivalence of it) across generations

**Simple Model** - emphasize “simple”
- 2 generations with 2 periods (1 generation in each period: parents and offspring)
- **Public Good** - \( S \equiv t_p + d = \) public expenditure in period 1 on the durable public good (there is no public good investment in period 2)
- \( t_p \equiv \) taxes on parent (portion of public good paid by parent)
- \( d \equiv \) debt financing (portion of public good paid for by offspring)

- **Parent Utility** - \( u(C_p, G_p) + \beta u^0(C_o, G_o) \) where
  - \( u \equiv \) parent’s utility from own consumption; \( u^0 \equiv \) offspring’s own utility
  - \( C_p \equiv \) parent’s private consumption, \( G_p \equiv \) parent’s consumption of public good
  - \( C_o \equiv \) offspring’s private consumption, \( G_o \equiv \) offspring’s consumption of public good
  - \( \beta \equiv \) personal discount factor (how much parent values offspring’s wellbeing)

Note: \( G_p = G_p(S) \) with \( G_p \ '> 0 \); \( G_o = G_o(S) \) with \( G_o \ '> 0 \)

- **Bequest** - parent transfers amount \( b \) to offspring, but based on utility, parent doesn’t care about the amount of bequest, just the offspring’s total utility

- **Balanced Budget Requirement** - tax on offspring must pay off the debt and the interest accumulated at exogenous interest rate \( r : t_o = (1 + r)d \)

- **Exogenous Income** - each generation enjoys income \( I_p \) and \( I_o \), respectively

- **Personal Budget Balance** -
  - \( C_p = I_p - t_p - b \)
  - \( C_o = I_o - t_o + (1 + r)b \)

- **Government Policy** - completely defined by \( (S, d) \) ... from that we can find \( t_p, t_o, C_p, C_o, G_p, G_o, b \) (other terms; \( \beta, r, I_p, I_o \) are given)

Parent’s Problem (taking government policy as giving):

\[
\max_b u\left(I_p - t_p - b, G_p\left(t_p + \frac{t_o}{1+r}\right)\right) + \beta u^0\left(I_o - t_o + (1+r)b, G_o\left(t_p + \frac{t_o}{1+r}\right)\right)
\]

FOC: \(-u_i + \beta(1+r)u^0_i = 0 \) (assuming \( b > 0 \))
Ricardian Equivalence - given $S$ (level of public good), consumption vector and utilities are invariant to method of financing (i.e., $t_p, t_o$)

Proof: begin with initial policy $S, t_p, t_o, d$ in equilibrium... so FOC is satisfied

Now change $t_p, t_o$ while holding $S$ constant

Recall $S \equiv t_p + \frac{t_o}{1+r} \Rightarrow dS = dt_p + \frac{dt_o}{1+r} = 0$ (because $S$ is constant)

Suppose parents react by $db = -dt_p$, then:

$\begin{align*}
dC_p &= -dt_p - db = 0 \quad \text{(parent private consumption unchanged)} \\
dC_o &= -dt_o + (1+r)db = -dt_o - (1+r)dt_p = 0 \quad \text{(follows from } dS = 0)\
\end{align*}$

Since private consumption and level of public good don't change, utilities are also unchanged

Add Warm Glow - add extra utility term where parent gains utility directly from amount bequeathed to offspring: $u^p(b)$ with $u^{p'} > 0$ and $u^{p''} < 0$

$$\max_b u \left( I_p - t_p - b, G_p \left( t_p + \frac{t_o}{1+r} \right) \right) + \beta u o \left( I_o - t_o + (1+r)b, G_o \left( t_p + \frac{t_o}{1+r} \right) \right) + u^p(b)$$

**EXERCISE** - show Ricardian Equivalence doesn't hold (i.e., show if we reduce parent's tax burden ($dt_p < 0$), then parent is better off and offspring is worse off)

FOC: $-u_i + \beta(1+r)u^o_i + u^{p'} = 0$

Suppose initial tax system $t_p, t_o$ system is in equilibrium so FOC holds

Now change tax keeping $S$ (level of public good) fixed so $dt_p < 0$

$$dS = 0 = dt_p + \frac{dt_o}{1+r} \Rightarrow dt_o = -(1+r)dt_p$$

Assume neutrality so $db = -dt_p$

$$\begin{align*}
dC_p &= -dt_p - db = 0 \quad \text{(parent private consumption unchanged)} \\
dC_o &= -dt_o + (1+r)db = (1+r)dt_p - (1+r)dt_o = 0 \quad \text{(also unchanged)}\
\end{align*}$$

Because $u^{p''} < 0$, since $db = -dt_p > 0$, we must have $du^p < 0$ (i.e., $u^p$ is smaller after the tax change)

But look at the FOC: $u_i$ & $u^o_i$ are unchanged (from $dC_p = dC_o = 0$) and $u^p$ is smaller so we actually have $-u_i + \beta(1+r)u^o_i + u^{p'} < 0$

∴ the neutral bequest is too large; optimal bequest is smaller ($db^* < -dt_p$)

**Keynesian Effect** - effect of lowering tax in this case increases current (parent) consumption

$\begin{align*}
dC_p &= -dt_p - db > 0 \\
dC_o &= -dt_o + (1+r)db = (1+r)dt_p - (1+r)db < 0\
\end{align*}$
From paper: "Parents are impurely altruistic with respect to their gifts to heirs, while the heirs can be thought of as 'purely altruistic' with respect to their own consumption. As such, redistributions from children (more altruistic) to parents (less altruistic) will reduce the private supply of the public good (the consumption of the heir). Parents will be unwilling to perfectly substitute bequests for debt; hence they will keep some of their new 'wealth' for themselves." (p. 1456)