1.1. In the Bertrand model with increasing marginal cost, each firm would prefer to charge a price greater than \( p^* \) where \( p^* \) satisfies \( S_1(p^*) + S_2(p^*) = D(p^*) \). Prove this for the proportional-rationing case.

**Notation**

- \( p_i \): price charged by firm \( i \)
- \( D(p) \): market demand at price \( p \)
- \( C_i(q_i) \): total cost to firm \( i \) to produce \( q_i \)
- \( ATC = C_i(q_i)/q_i \): average total cost for firm \( i \) to produce \( q_i \) units
- \( MC = C_i'(q_i) \): marginal cost for firm \( i \) to produce the \( q_i \)th unit; by assumption \( C_i'(q_i) > 0 \)
- \( p^* \): price in "competitive outcome"; given by:
  - \( p^* = MC = C_i'(q_i) = C_2'(q_2) \) (Tirole p.214), or
  - \( S_1(p^*) + S_2(p^*) = D(p^*) \)

**Proportional-rationing**

Assume (without loss of generality) \( p_1 < p_2 \) (so firm 1 determines how much demand to satisfy and how much residual demand is left for firm 2)

- \( D_2(p_2) \): firm 2's residual demand given it charges \( p_2 \)

Define \( \bar{q}_1 \equiv S_1(p_1) \), firm 1's supply at price \( p_1 \) and assume \( \bar{q}_1 < D(p_1) \) so firm 1 doesn't satisfy all of the demand

The probability of not being able to buy from firm 1 is \( \frac{D(p_1) - \bar{q}_1}{D(p_1)} \) (Tirole p.214)

Residual demand facing firm 2 is \( D_2(p_2) = D(p_2) \left( \frac{D(p_1) - \bar{q}_1}{D(p_1)} \right) \) (Tirole p.214)

**Graphically**

The assumption is that \( p_1 = p_2 = p^* \) is not a Nash equilibrium

Assume \( p_1 = p^* \)

Given firm 1's price, firm 2 will maximize profit by behaving like a monopolist in on the residual demand; the graph clearly shows that \( p_2 > p_1 \)
Mathematically
Assume \( p_1 = p^* \)
That means profit for firm 2 is \( \pi_2(p_1, p_2) = p_2D_2(p_2) - C(D_2(p_2)) \)
Substitute for the residual demand:
\[
\pi_2(p^*, p_2) = p_2D(p_2)\left( \frac{D(p^*) - \bar{q}_1}{D(p^*)} \right) - C(D(p_2)\left( \frac{D(p^*) - \bar{q}_1}{D(p^*)} \right))
\]
Now look at the slope of firm 2's profit at the point where \( p_2 = p^* \)
\[
\frac{\partial \pi_2(p^*, p_2)}{\partial p_2} = D(p_2)\left( \frac{D(p^*) - \bar{q}_1}{D(p^*)} \right) + p_2D'(p_2)\left( \frac{D(p^*) - \bar{q}_1}{D(p^*)} \right) -
C\left( D(p_2)\left( \frac{D(p^*) - \bar{q}_1}{D(p^*)} \right) \right)D'(p_2)\left( \frac{D(p^*) - \bar{q}_1}{D(p^*)} \right)
\]
\[
\frac{\partial \pi_2(p^*, p_2)}{\partial p_2}\bigg|_{p_2=p^*} = D(p^*)\left( \frac{D(p^*) - \bar{q}_1}{D(p^*)} \right) + p^*D'(p^*)\left( \frac{D(p^*) - \bar{q}_1}{D(p^*)} \right) -
C\left( D(p^*) - \bar{q}_1 \right)\frac{D'(p^*)}{D(p^*)}(D(p^*) - \bar{q}_1)
\]
Factor \( \frac{D'(p^*)}{D(p^*)}(D(p^*) - \bar{q}_1) \): \[
\frac{\partial \pi_2(p^*, p_2)}{\partial p_2}\bigg|_{p_2=p^*} = D(p^*) - \bar{q}_1 + (p^* - C'(D(p^*) - \bar{q}_1))\frac{D'(p^*)}{D(p^*)}(D(p^*) - \bar{q}_1)
\]
Since we're evaluating this at \( p_2 = p^* \), the second term goes away because price equals marginal cost:
\( p^* - C'(D(p^*) - \bar{q}_1) = 0 \)
We assumed \( D(p^*) - \bar{q}_1 > 0 \) (so firm 1 doesn't satisfy all the demand at \( p^* \))
Therefore,
\[
\frac{\partial \pi_2(p^*, p_2)}{\partial p_2}\bigg|_{p_2=p^*} > 0
\]
Firm 2 can increase profit by raising price above \( p^* \)
1.2. Assume that the demand curve is linear and that the cost function for each firm is 
\( C_i(q_i) = q_i^2 \). With efficient rationing, derive a firm’s best reply. In particular, show that, for values of \( p_2 \) near \( p^* \), \( p_1(p_2) > p_2 \), while for higher values of \( p_2 \), \( p_1(p_2) < p_2 \). If you can, identify the point where firm 1’s best reply jumps down.

\[
C_i(q_i) = q_i^2 \Rightarrow MC = C_i'(q_i) = 2q_i \text{ and } ATC = \frac{C_i(q_i)}{q_i} = q_i
\]

Efficient rationing means that high price firm behaves like a monopolist on the residual demand. The low price firm serves as many customers as it wants (quantity based on where price crosses MC curve).

Since both firms are identical, we can just look at the best reply for firm 1

Firm 1’s best reply determines the price at which firm 1 maximizes profit, given firm 2's price: 
\[
p_1(p_2) = \arg \max_{p_1} \pi_1(p_1, p_2)
\]

Firm 1’s profit depends on the relationship between \( p_1 \) and \( p_2 \) (assume \( p_1 > p_2 \))

Assume firm 2 sells \( \bar{q}_2 \)

Since firm 2 is the low seller, it produces at \( p_2 = MC = 2\bar{q}_2 \)

Profit for firm 1 is determined by:

\[
\pi_1(p_1, p_2) = p_1\left(D(p_1) - \bar{q}_2\right) - C(D(p_1) - \bar{q}_2) = p_1\left(D(p_1) - \frac{p_2}{2}\right) - C\left(D(p_1) - \frac{p_2}{2}\right)
\]

\[
p_1\left(D(p_1) - \frac{p_2}{2}\right) - \left(D(p_1) - \frac{p_2}{2}\right)^2
\]

Assume linear demand so let \( D(p_1) = a - bp_1 \)

\[
\pi_1(p_1, p_2) = p_1\left(a - bp_1 - \frac{p_2}{2}\right) - \left(a - bp_1 - \frac{p_2}{2}\right)^2
\]

Now look at the first order condition:

\[
\frac{\partial \pi_1(p_1, p_2)}{\partial p_1} = a - 2bp_1 - \frac{p_2}{2} - 2\left(a - bp_1 - \frac{p_2}{2}\right)(-b) =
\]

\[
a - 2bp_1 - \frac{p_2}{2} + 2ab - 2b^2 p_1 - 2b \frac{p_2}{2}
\]

Set this equal to zero and solve for \( p_1 \) as a function of \( p_2 \)

\[
2bp_1 + 2b^2 p_1 = a - \frac{p_2}{2} + 2ab - 2b \frac{p_2}{2}
\]

\[
\left(2b + 2b^2\right)p_1 = a(1 + 2b) - (1 + 2b) \frac{p_2}{2}
\]

\[
p_1 = \frac{a(1 + 2b) - (1 + 2b) \frac{p_2}{2}}{2b(1 + b)}
\]
This is firm 1's best reply to firm 2's price of $p_2$, but only for the region where $p_1 > p_2$. (There are other instances where $p_1 = p_2$ or $p_1 < p_2$. Given the rationing rule, these are more complicated to solve.)

We can graphically show for $p_2$ near $p^*$, $p_1(p_2) > p_2$
Firm 1 acts as a monopolist on its residual demand When $p_2$ is near $p^*$, firm 2 is serving roughly half the available demand; the residual demand curve for firm 1 will cross near the marginal cost curve (as shown) so it guarantees that a portion of residual demand will be above $p_2$.
Firm 1's marginal revenue will cross the marginal cost curve before the level $\bar{q}_2$, so $q_1 < \bar{q}_2$.
Following this level up to the residual demand curve, we have $p_1(p_2) > p_2$.

We can graphically show higher values of $p_2$, $p_1(p_2) < p_2$
The graph shows that we can select a value of $p_2$
sufficiently high so that the residual demand curve is completely below $p_2$.
Firm 1 will respond as a monopolist on this curve so it must have $p_1(p_2) < p_2$.

The point at which firm 1's best reply jumps down occurs when the formula above (which assumed $p_1 > p_2$) returns a value of $p_1$ which violates the assumption

\[
p_2 = \frac{a(1+2b) - (1+2b) \frac{p_2}{2}}{2b(1+b)}
\]

\[
2b(1+b) p_2 = a(1+2b) - (1+2b) \frac{p_2}{2}
\]

\[
2bp_2 + 2b^2 p_2 = a + 2ab - (1+2b) \frac{p_2}{2}
\]

\[
2b^2 p_2 + \frac{4b}{2} \frac{p_2}{2} + (1+2b) \frac{p_2}{2} = a + 2ab
\]

\[
2b^2 p_2 + \frac{(1+6b)}{2} p_2 = a + 2ab
\]

\[
p_2 = \frac{a + ab}{2b^2 + (1+6b)} = \frac{2a + 2ab}{4b^2 + 6b + 1}
\]
1.3. In the special no rationing Bertrand model (i.e., firms post a price and must then sell to all customers who want to buy), there is a continuum of pure strategy Nash equilibria when both firms choose the same price. When \( p_1 = p_2 = \hat{p} \), both firms split the market and produce where \( MC > MR = ATC \) so \( \pi_i = 0 \). When \( p_1 = p_2 = \overline{p} \), neither firm has an incentive to change because raising price would result in zero sales and lowering price yields the same profit. That is:

\[
\pi_i(\overline{p}, \overline{p}) = \overline{p} \frac{D(\overline{p})}{2} - C\left(\frac{D(\overline{p})}{2}\right) = \overline{p}D(\overline{p}) - C(D(\overline{p}))
\]

Let \( p^* \) be the pure strategy Nash equilibrium price in the no-rationing Bertrand model (i.e., \( p^* = MC = C_1(q_1) = C_2(q_2) \)). Prove that \( \hat{p} < p^* < \overline{p} \) for the no-rationing model.

\( \hat{p} \) is determined by the point where \( MR = ATC \)
\( p^* \) is determined by the point where \( MR = MC \) so clearly \( \hat{p} < p^* \)

Find \( \overline{p} \) is a little more difficult. Looking at the second graph, the profit for serving the whole market is the sum of the blue and gray areas. Mathematically:

\[
\pi^*(\overline{p}) = [\overline{p} - ATC(\hat{q})]\hat{q}
\]

The profit for serving half the market with price \( \overline{p} \) is the sum of the pink and gray areas.

\[
\pi_H(\overline{p}) = [\overline{p} - ATC(\hat{q}/2)]\frac{\hat{q}}{2}
\]

Since we want these to be equal to each other, we only need to worry about the blue and the pink areas being equal:

\[
\pi^*(\overline{p}) = \pi_H(\overline{p}) \iff [\overline{p} - ATC(\hat{q})]\frac{\hat{q}}{2} = [ATC(\hat{q}/2) - ATC(\hat{q})]\frac{\hat{q}}{2}
\]

This can’t really be solved without knowing the form of the firm’s cost curve.

Instead focus on how \( p^* \) affects profits for serving the whole market versus half the market

\[
\pi^*(p^*) = 0 \quad \text{(by definition)}
\]

\[
\pi_H(p^*) > 0
\]

In order to get these profit levels equal to each other (preferably not at zero profit), we need to increase the profit of serving the whole market. If we lower price below \( p^* \), profit goes negative. Therefore, the price at which the firm is indifferent between serving the whole market and half the market (i.e., \( \overline{p} \)), must be greater than \( p^* \).

\[
\therefore \hat{p} < p^* < \overline{p}
\]
1.4. Vives established that when there are only two players each with a single strategic variable, if the best replies are downward sloping everywhere they are continuous and all discontinuities (jumps) are downward, then there must exist a fixed point (i.e., pure strategy Nash equilibrium where the best replies intersect). Draw examples to convince yourself that this result holds.
1.5. a) Suppose that, at \( \hat{x}_2 \), firm 1's best reply jumps up, and that it is increasing in \( x_2 \) everywhere else. What must be true about \( \frac{\partial \pi_1}{\partial x_1 \partial x_2} \) over the interval \([x_1^a, x_1^b]\) for \( x_2 = \hat{x}_2 \)? Explain.

b) Suppose that, at \( \hat{x}_2 \), firm 1's best reply jumps down, and that it is decreasing in \( x_2 \) everywhere else. What must be true about \( \frac{\partial \pi_1}{\partial x_1 \partial x_2} \)? Explain.

a) Consider both parts of player 1’s best reply as separate functions:

\[ x_1^a(x_2) \text{ and } x_1^b(x_2) \]

At \( \hat{x}_2 \), profit for player 1 is the same so we have:

\[ F = \pi_1(x_1^b(\hat{x}_2), \hat{x}_2) - \pi_1(x_1^a(\hat{x}_2), \hat{x}_2) = 0 \]

Differentiate wrt \( x_2 \) (still evaluating at \( \hat{x}_2 \)):

\[ \frac{\partial F}{\partial x_2} \bigg|_{x_2 = \hat{x}_2} = \frac{\partial \pi_1}{\partial x_1} \frac{\partial x_1^a}{\partial x_2} + \frac{\partial \pi_1}{\partial x_2} (x_1^b(\hat{x}_2), \hat{x}_2) - \frac{\partial \pi_1}{\partial x_1} \frac{\partial x_1^a}{\partial x_2} - \frac{\partial \pi_1}{\partial x_1} \frac{\partial x_1^b}{\partial x_2} \]

Since \( \pi_1 \) is a local max at \( \hat{x}_2 \), we know \( \frac{\partial \pi_1}{\partial x_1} = 0 \) (drops terms 1 and 3)

\[ \frac{\partial F}{\partial x_2} \bigg|_{x_2 = \hat{x}_2} = \frac{\partial \pi_1}{\partial x_2} (x_1^b(\hat{x}_2), \hat{x}_2) - \frac{\partial \pi_1}{\partial x_2} (x_1^a(\hat{x}_2), \hat{x}_2) \]

Given the upward jump, this difference must be positive. Below \( \hat{x}_2 \), the best reply is \( x_1^a(x_2) \) so we know \( \pi_1(x_1^a(x_2), x_2) > \pi_1(x_1^b(x_2), x_2) \) (i.e., \( F < 0 \)). Above \( \hat{x}_2 \), the opposite is true: the best reply is \( x_1^b(x_2) \) so we know \( F > 0 \). So in the neighborhood around \( \hat{x}_2 \), as we increase \( x_2 \), the function \( F \) goes from being negative to positive. That means it is increasing so

\[ \frac{\partial F}{\partial x_2} \bigg|_{x_2 = \hat{x}_2} > 0 \]

By the fundamental theorem of calculus we can rewrite this as an integral:

\[ \frac{\partial F}{\partial x_2} \bigg|_{x_2 = \hat{x}_2} = \int_{x_1^a}^{x_1^b} \frac{\partial^2 \pi_1}{\partial x_1 \partial x_2} \, dx_1 > 0 \]

Since we know \( x_1^b > x_1^a \), the above statement implies \( \frac{\partial^2 \pi_1}{\partial x_1 \partial x_2} > 0 \)

b) The same math as part (a) gets us

\[ \frac{\partial F}{\partial x_2} \bigg|_{x_2 = \hat{x}_2} = \frac{\partial \pi_1}{\partial x_2} (x_1^b(\hat{x}_2), \hat{x}_2) - \frac{\partial \pi_1}{\partial x_2} (x_1^a(\hat{x}_2), \hat{x}_2) \]

Given the downward jump, this difference must be negative. Below \( \hat{x}_2 \), the best reply is \( x_1^b(x_2) \) so we know \( \pi_1(x_1^b(x_2), x_2) > \pi_1(x_1^a(x_2), x_2) \) (i.e., \( F > 0 \)). Above \( \hat{x}_2 \), the opposite is true: the best reply is \( x_1^a(x_2) \) so we know \( F < 0 \). So in the neighborhood around \( \hat{x}_2 \), as we increase \( x_2 \), the function \( F \) goes from being positive to negative. That means it is decreasing so...
By the fundamental theorem of calculus we can rewrite this as an integral:

\[
\frac{\partial F}{\partial x_2} \bigg|_{\tilde{x}_2 = x_2} = \frac{\partial \pi_1(x_1^b(\tilde{x}_2),\tilde{x}_2)}{\partial x_2} - \frac{\partial \pi_1(x_1^a(\tilde{x}_2),\tilde{x}_2)}{\partial x_2} < 0
\]

Since we know \( x_1^b > x_1^a \), the above statement implies \( \frac{\partial^2 \pi_1}{\partial x_1 \partial x_2} < 0 \)

**Documentation**

Guille referred me to the footnote in Tirole p.214; I reviewed the math with Guille and Josh; Prof Hamilton said it was sufficient to use pictures.

Prof Hamilton said pictures were sufficient for the first part of 1.2 (showing best reply of \( p_1 \) in response to high and low \( p_2 \)). He double checked my guess on \( \pi_1 \) and pointed out that at \( p_2 = p^* \), we have \( p_2 = p^* \) so \( \bar{q}_2 = p_2 / 2 \). I reviewed my answer with Guille and Jean.

I tried to get help on 1.3 from anyone who would speak to me... to no avail. Guille, Jean, and I worked through all the areas in the graph until we finally came up with focusing the argument around \( p^* \) rather than finding \( \overline{p} \). BTW, I have nightmares about p-hats, p-stars, p-bars, p-etc.

Prof Hamilton caught a mistake I made in 1.4 where I had an upward jump (instead of downward).

Prof Hamilton reviewed 1.5a in class. In his office, he further explained the logic behind why we used the \( F \) function and why we know the function changes signs at \( \tilde{x}_2 \).