Product Differentiation

Solutions to Bertrand "Paradox" (i.e., \( p = MC \) with only 2 firms)

**Capacity Constraints** - proposed by Edgeworth; result in Bertrand-Edgeworth Cycles - for some prices you want to undercut and for others (i.e., low price firm near capacity) you want to charge a higher price on residual demand

Kreps & Scheinkman - using efficient rationing, showed you get Cournot outcome
Osborne & Pitchik (JET, 1986) - showed Cournot outcome with other demand curves
Davidson & Deneckere (RAND, 1985) - showed Cournot-like result (\( p > MC \)) with other rationing rules; proportional rationing does NOT result in Cournot outcome (but it's close)

**Spatial Competition** - Hotelling (1929) - moved away from capacity constraint; talked about product differentiation based on location

**Monopolistic Competition** - models proposed by Chamberlin (1933)

**Key** - a price cut doesn't capture the entire market (as it does in Bertrand model)

**General Differentiated Products Model** - Cheng; Vives

Consider a duopoly with demand functions: \( q_i = F_i(p_1, p_2) \) (\( i = 1, 2 \))

**Assumptions:**

1. the two goods are gross substitutes
2. \( F_i \) is twice continuously differentiable with bounded derivatives
3. \( \frac{\partial F_i}{\partial p_i} \leq 0 \), strictly \( < 0 \) for \( q_i > 0 \) (demand slopes downward)
4. \( \frac{\partial F_i}{\partial p_j} \geq 0 \) (\( i \neq j \)), strictly \( > 0 \) if \( q_i > 0 \) and \( q_j > 0 \) (goods are substitutes)
5. \( \frac{\partial F_i}{\partial p_i} \frac{\partial F_j}{\partial p_j} > \frac{\partial F_i}{\partial p_j} \frac{\partial F_j}{\partial p_i} \) (own effects are larger than cross effects)

**Profit Function:** \( \pi_i(p) = (p_i - c_i) F_i(p) \) (\( c_i \) = firm \( i \)'s constant marginal cost)

6. \( \frac{\partial^2 \pi_i}{\partial p_i^2} < 0 \), strictly \( < 0 \) for \( q_i > 0 \) (profit function is concave)

7. \( \frac{\partial^2 \pi_i}{\partial p_i \partial p_j} \geq 0 \) (\( i \neq j \)), strictly \( > 0 \) if \( q_i > 0 \) and \( q_j > 0 \) (marginal profitability is increasing in rival's price)

8. \( \frac{\partial^2 \pi_i}{\partial p_i^2} \frac{\partial^2 \pi_j}{\partial p_j^2} > \frac{\partial^2 \pi_i}{\partial p_i \partial p_j} \frac{\partial^2 \pi_j}{\partial p_i \partial p_j} \) if \( q_i > 0 \) and \( q_j > 0 \) (own effects are larger than cross effects)

**Bertrand Model** - firms choose prices; assumptions guarantee equilibrium exists and is unique; (6) & (7) say best replies are upward sloping; (8) guarantees relative slopes appear as in picture (isoprofit 1 is increasing to the left of equilibrium; isoprofit 2 is increasing above... this will show where the Cournot equilibrium is relative to Bertrand)

**Equilibrium** - (\( p_1^*, p_2^* \)) s.t.
\[ \pi_i(p_1^*, p_2^*) \geq \pi_i(p_1^*, p_2^*) \text{ (i.e., } p_i^* \text{ is best reply to } p_2^*) \]
\[ \pi_2(p_1^*, p_2^*) \geq \pi_2(p_1^*, p_2^*) \text{ (i.e., } p_2^* \text{ is best reply to } p_1^*) \]

**Best Replies** - \( R_i(p_j) = \left\{ p \mid p \geq 0, \frac{\partial \pi_i}{\partial p_i} = 0 \right\} \)

This says profit for firm \( i \) is maximized given \( p_j \); that means isoprofit curves for firm 1 must be horizontal (slope 0) when they cross firm 1’s best reply (including equilibrium); for firm 2, the isoprofit curves cross vertically.

**Cournot** - firms choose quantity: \( q_i = F_i(p_1, p_2) \quad (i = 1, 2) \)

**Inverse Demand** - \( p_i = G_i(q_1, q_2) \) ... not only would this be "a lot of work", it would put the solution in quantity space (different than the Bertrand solution in price space).

Cheng - showed how to plot Cournot model in price space

Assume Cournot equilibrium exists and is unique

**Isoquantity Curve** - \( Q_i(q_i) = \{ p \mid p \geq 0, F_i(p) = q_i \} \) ... all pairs of prices so firm \( i \)'s demand equals \( q_i \)

**Upward Sloping** - (3) & (4) say if we raise \( p_1 \) we have to raise \( p_2 \) to keep firm 1's output equal to \( q_i \)

**Equilibrium** - choose \( q_i \) to max \( \pi_i \) given \( q_j \) ... firm will choose isoprofit curve tangent to rival's \( Q_i(q_i) \); this determines \( p_1 \) and \( p_2 \) indirectly selecting \( q_j \)

**Comparison** - we assumed isoprofit curves are well behaved;
looking at how isoprofit curves intersect at the two equilibria, we get:
(i) Bertrand prices are less than Cournot prices: \( p_1^B < p_1^C \) and \( p_2^B < p_2^C \)
(ii) At least 1 firm's profit is higher in Cournot model (if symmetric, both firms will have higher profit)

**No Decision** - although more profit in Cournot model sound since, firm's don't get to choose whether their industry is Bertrand or Cournot; it is a function of technology, not a decision variable

Aside: **Low Price Guarantees** - manufacturers use different model numbers for different retailers so they can relax price competition

**Horizontal and Location Models**

**Result** - price > marginal cost

**Address Models** - each firm and consumer has a location in geographic or "product space"

**Horizontal** - at equal prices consumers are not unanimous in rankings; Hotelling's original beach example falls into this category; if both firms charge \( p = MC \), the market splits so some consumers prefer product 1 and others prefer product 2

**Vertical** - associated with product quality (e.g., computer parts: faster is better); at equal prices consumers have unanimity in rankings; will still see product differentiation because people's willingness to pay varies

**Chamberlinian Models** - symmetric; \( D_i(p_1, p_j) \) but not all demand goes to low price firm

**Problem** - Chamberlinian models don't allow us to endogenize differentiation

**In Between Models** - based on random utility (preferences with noise)
Our Focus - we'll look at one dimension (product) for simplicity; results generalize to many dimensions

Hotelling -
Two firms locate on [0,1]
Consumers uniformly distributed on [0,1]
Perfect information (consumers know prices)
Transport cost \( t \) per unit distance
Inelastic demand for product (everyone buys at lowest cost)

**Mill Pricing** - firms quote prices; consumer pay transport cost (also called FOB: “free on board”)
Assume \( L_1 \leq L_2 \) (i.e., \( a \leq 1 - b \) or \( a + b \leq 1 \)); just relabeling so firm on the left is 1 and firm on the right is 2

**Consumer** location is \( x \in [0,1] \)
Delivered price from firm 1: \( p_1 + t|x-a| \)
Delivered price from firm 2: \( p_2 + t|1-b-x| \)
\( \bar{x} \) is location of consumer who is indifferent
For linear case (with assumption/labeling putting firm 1 on left):
\[
p_1 + t(\bar{x} - a) = p_2 + t(1 - b - \bar{x})
\]
\[
2\bar{x} = p_2 - p_1 + t(1 - b + a)
\]
\[
\bar{x} = \frac{p_2 - p_1 + 1 - b + a}{2t}
\]
If \( p_1 = p_2 \), then \( \bar{x} \) = midpoint between locations

If a price changes, \( \bar{x} \) moves at rate \( 1/2t \)
In graph, **red** marks consumers who buy from firm 1; **blue** marks consumers who buy from firm 2
Note, it’s possible for prices to be different enough (or transport cost to be low enough) for one firm to serve the entire market
Firm 2 gets entire market if \( p_1 + t(x - a) \geq p_2 + t(1 - b - x) \)
\( x = a \) (i.e., the delivered cost from firm 2 is less than the cost from firm 1 at firm 1’s location)
Firm 1 gets entire market if \( p_1 + t(x - a) \leq p_2 + t(1 - b - x) \)
\( x = 1 - b \)
In between these extremes, the firms split the market with firm 1 getting \( \bar{x} \) customers and firm 2 getting \( 1 - \bar{x} \) customers; note: because of linear structure, we guarantee \( a < \bar{x} < 1 - b \)

**Firm's Demand** -
\[
D_1(p_1, p_2; a, b) = \begin{cases} 
0 & \text{if } p_1 \geq p_2 + t(1 - b - a) \\
\bar{x} & \text{if } |p_1 - p_2| \leq t(1 - b - a) \\
1 & \text{if } p_1 \leq p_2 - t(1 - b - a)
\end{cases}
\]
\[
D_2 = 1 - D_1
\]
This is linear in firm's own price and increasing in rival's price

**Nonlinear Costs** - transport cost depend on distance in nonlinear way
\[ a < \bar{x} < 1-b \quad (\text{i.e.,} \quad \bar{x} \text{ in middle}) \quad \text{if} \quad p_1 + T(\bar{x} - a) = p_2 + t(1-b - \bar{x}) \]
\[ \bar{x} < a \quad (\text{i.e.,} \quad \bar{x} \text{ left of} \ a) \quad \text{if} \quad p_1 + T(a - \bar{x}) = p_2 + t(1-b - \bar{x}) \]
\[ 1-b < \bar{x} \quad (\text{i.e.,} \quad \bar{x} \text{ right of} \ b) \quad \text{if} \quad p_1 + T(\bar{x} - a) = p_2 + t(\bar{x} - 1 + b) \]

**Convex** - cost of travel increases at increasing rate (walking or driving when factoring leisure); market areas may not include stores themselves (i.e., someone living at location of firm 2 will buy from firm 1)

**Concave** - cost of travel increases at decreasing rate (airlines); market areas aren’t connected sets, but person at store location will always buy at that store (if anyone buys from that store)

**Firm Decisions**
- Assume linear transport cost
  \[ m = \text{unit product cost (same for both firms)} \]
  \[ \pi_1(p_1, p_2; a, b) = (p_1 - m)D_1(p_1, p_2; a, b) \]
  \[ \pi_2(p_1, p_2; a, b) = (p_2 - m)D_2(p_1, p_2; a, b) \]

**Subgame Perfect Nash Equilibrium** - not exactly the way Hotelling did it (Nash came well after), but it’s basically what he did because Hotelling (correctly) had firms choose location first, then prices; SPNE means we solve backwards: find best replies for prices, then find best replies for locations

Assume \( p_2 > m \)

**Problem** - demand is not continuous (recall three regions on previous page)
- Initially, firm 1 gets entire market so \( \pi_1 = p_1 - m \quad (D_1 = 1) \)
- In middle region, \( \pi_1 = (p_1 - m)\bar{x} \)
  \[ \pi_1 = (p_1 - m)\left( \frac{p_2 - p_1}{2t} + \frac{1-b+a}{2} \right) = \frac{p_1p_2 - p_1^2}{2t} + \frac{p_1(1-b+a)}{2} - \frac{p_2m - p_1m}{2t} - m(1-b+a) = \]
  \[ = \frac{-1}{2t}p_1^2 + \left[ \frac{p_2 + m}{2t} + \frac{(1-b+a)}{2} \right]p_1 + \left[ \frac{-p_2m}{2t} - \frac{m(1-b+a)}{2} \right] \]
  \[ = Ap_1^2 + Bp_1 + C \quad \text{... this is a quadratic form with} \ A < 0 \quad (\text{concave}) \]
  - Hotelling wanted to rule out the linear portion arguing a rival firm wouldn’t tolerate zero demand; he only focused on the middle portion, but notice second graph indicates that firm’s best reply could be to undercut rival (i.e., capture entire market)

**Concave Region** - focus only on middle segment for now and find the best reply (later we’ll compare this to the top of the linear region to see which firm would prefer)
- Find best reply for firm 1:
  \[ \pi_1 = (p_1 - m)\bar{x} \]
- Take partial wrt own price and set equal to zero
  \[ \frac{\partial \pi_1}{\partial p_1} = \bar{x} + (p_1 - m)\frac{\partial \bar{x}}{\partial p_1} = 0 \]
- Sub for \( \bar{x} = \frac{p_2 - p_1}{2t} + \frac{1-b+a}{2} \) (from previous page)
\[
\left( \frac{p_2 - p_1}{2t} + \frac{1 \cdot b + a}{2} \right) + (p_1 - m) \frac{\partial}{\partial p_1} \left( \frac{p_2 - p_1}{2t} + \frac{1 \cdot b + a}{2} \right) = 0
\]

Take derivative
\[
\left( \frac{p_2 - p_1}{2t} + \frac{1 \cdot b + a}{2} \right) + (p_1 - m) \frac{-1}{2t} = 0
\]

Multiply second term by \( t/t \) so we can cancel out the \( 2t \) in the denominators
\[
p_2 - p_1 + t(1 \cdot b + a) - (p_1 - m) = 0
\]

Solve for \( p_1 \) as function of \( p_2 \) (this is firm 1's best reply to firm 2's price for given locations \( a \) and \( 1 \cdot b \))
\[
p_1(p_2) = \frac{p_2 + t(1 \cdot b + a) + m}{2}
\]

Best reply for firm 2:
\[
\pi_2 = (p_2 - m)(1 - \bar{x})
\]

Take partials wrt own price and set equal to zero
\[
\frac{\partial \pi_2}{\partial p_2} = 1 - \bar{x} \cdot (p_2 - m) \frac{\partial \bar{x}}{\partial p_2} = 0
\]

Sub for \( \bar{x} = \frac{p_2 - p_1}{2t} + \frac{1 \cdot b + a}{2} \) (from previous page)
\[
1 - \left( \frac{p_2 - p_1}{2t} + \frac{1 \cdot b + a}{2} \right) - (p_2 - m) \frac{\partial}{\partial p_2} \left( \frac{p_2 - p_1}{2t} + \frac{1 \cdot b + a}{2} \right) = 0
\]

Take derivative
\[
1 - \left( \frac{p_2 - p_1}{2t} + \frac{1 \cdot b + a}{2} \right) - (p_2 - m) \frac{1}{2t} = 0
\]

Simplify the first part of the equation \((1 - \bar{x})\)
\[
\left( \frac{p_1 - p_2}{2t} + \frac{1 \cdot a + b}{2} \right) - (p_2 - m) \frac{1}{2t} = 0
\]

Note symmetry with firm 1's derivative... this is why we used the \( a \) and \( b \)
\[
p_2(p_1) = \frac{p_1 + t(1 \cdot a + b) + m}{2}
\]

Let \((\tilde{p}_1, \tilde{p}_2)\) be the intersection of best replies; this point is a "candidate Nash equilibrium" (still need to check that linear portion)

Sub \( \tilde{p}_2(\tilde{p}_1) = \frac{\tilde{p}_1 + t(1 \cdot a + b) + m}{2} \) into \( \tilde{p}_1(\tilde{p}_2) \)
\[
\tilde{p}_1 = \frac{\tilde{p}_1 + t(1 \cdot a + b) + m}{2} + t(1 \cdot b + a) + m
\]

Break it up
\[
\tilde{p}_1 = \frac{p_1}{4} + \frac{t}{4} - \frac{at}{4} + \frac{bt}{4} + \frac{m}{4} + \frac{t}{2} + \frac{at}{2} - \frac{bt}{2} + \frac{m}{2}
\]
Move the \( \tilde{p}_1 \) to the left side and get a common denominator
\[
\frac{3}{4} p_1 = \frac{3t}{4} + \frac{at}{4} - \frac{bt}{4} + \frac{3m}{4}
\]
Solve for \( \tilde{p}_1 \)
\[
\tilde{p}_1 = m + t \left(1 + \frac{a-b}{3}\right)
\]
From the symmetry we found earlier, all we have to do is swap the \( a \) and \( b \) to get
\[
\tilde{p}_2 = m + t \left(1 + \frac{b-a}{3}\right)
\]
**True Best Reply** - now evaluate profit at \((\tilde{p}_1, \tilde{p}_2)\) and compare it to profit for firm 1 when firm 2 charges \( \tilde{p}_2 \) and firm 1 undercuts (i.e., charges \( p_1 = \tilde{p}_2 - t(1-b-a) \))

**Concave Portion** - \( \pi_1(p_1, \tilde{p}_2) = (\tilde{p}_1 - m)\bar{x}(\tilde{p}_1, \tilde{p}_2) \)
\[
\bar{x}(\tilde{p}_1, \tilde{p}_2) = \frac{\tilde{p}_2 - \tilde{p}_1 + 1-b+a}{2t}
\]
Sub values for \( \tilde{p}_1 \) and \( \tilde{p}_2 \) we just found
\[
\bar{x}(\tilde{p}_1, \tilde{p}_2) = \frac{\left[ m + t \left(1 + \frac{b-a}{3}\right)\right] - \left[ m + t \left(1 + \frac{a-b}{3}\right)\right]}{2t} + \frac{1-b+a}{2}
\]
Work out the first term
\[
\bar{x}(\tilde{p}_1, \tilde{p}_2) = \frac{b-a}{3} + \frac{1-b+a}{2}
\]
Finish simplifying... why we factor out the 1/2 will become obvious
\[
\bar{x}(\tilde{p}_1, \tilde{p}_2) = \frac{3+a-b}{6} = \frac{1}{2} \left(1 + \frac{a-b}{3}\right)
\]
Now sub this value of \( \bar{x}(\tilde{p}_1, \tilde{p}_2) \) into \( \pi_1(p_1, \tilde{p}_2) \)
\[
\pi_1(p_1, \tilde{p}_2) = (\tilde{p}_1 - m)\frac{1}{2} \left(1 + \frac{a-b}{3}\right)
\]
Sub value for \( \tilde{p}_1 \)
\[
\pi_1(p_1, \tilde{p}_2) = \left[ m + t \left(1 + \frac{a-b}{3}\right)\right] - m \frac{1}{2} \left(1 + \frac{a-b}{3}\right)
\]
Simplify
\[
\pi_1(p_1, \tilde{p}_2) = \frac{t}{2} \left(1 + \frac{a-b}{3}\right)^2
\]
From the symmetry we found earlier, all we have to do is swap the \( a \) and \( b \) to get
\[
\pi_2(p_1, \tilde{p}_2) = \frac{t}{2} \left(1 + \frac{b-a}{3}\right)^2
\]
**Linear Portion** - find \( \pi_1(p_1, \tilde{p}_2) \) at \( p_1 = \tilde{p}_2 - t(1-b-a) \)
In this case firm 1 captures entire market so
\[
\pi_1 = (p_1 - m) = [\tilde{p}_2 - t(1-b-a)] - m
\]
Sub value for \( \tilde{p}_2 \) we found earlier
\[
\pi_i = \left[ m + t \left( 1 + \frac{b - a}{3} \right) \right] - t(1 - b - a) - m
\]

Simplify
\[
\pi_i = t \left( \frac{2a + 4b}{3} \right)
\]

From the symmetry we found earlier, all we have to do is swap the \( a \) and \( b \) to get
\[
\pi_2 = t \left( \frac{2b + 4a}{3} \right)
\]

**Compare** - we’re trying to determine the true best reply to the rival firm charging the candidate Nash equilibrium price (\( \tilde{p}_1 \) and \( \tilde{p}_2 \)); here’s the summary:

\[
\pi_i(\tilde{p}_1, \tilde{p}_2) = \pi_i(\tilde{p}_1, \tilde{p}_2) = \pi_i(\text{undercut})
\]

**Firm 1**
\[
\pi_1(\tilde{p}_1, \tilde{p}_2) = \frac{t}{2} \left( 1 + \frac{a - b}{3} \right)^2
\]

**Firm 2**
\[
\pi_2(\tilde{p}_1, \tilde{p}_2) = \frac{t}{2} \left( 1 + \frac{b - a}{3} \right)^2
\]

This part is difficult (Hamilton: "a bit of a mess") because of the quadratics, but if we assume \( a = b \) (i.e., firms are equidistant from the ends)
we’re only looking at \( t/2 \) vs. \( 2ta \) (this is the same comparison for both firms); in order for the firms to be in the concave portion of their profit functions, we must have \( t/2 \geq 2ta \) which simplifies to \( a \leq 1/4 \)

If firms locate towards the ends, we have a pure strategy Nash equilibrium defined by (\( \tilde{p}_1, \tilde{p}_2 \)), note, however that
\[
\frac{\partial \pi_1(\tilde{p}_1, \tilde{p}_2)}{\partial a} = \frac{\partial}{\partial a} \left[ \frac{t}{2} \left( 1 + \frac{a - b}{3} \right)^2 \right] = t \left( 1 + \frac{a - b}{3} \right) > 0
\]

\( \therefore \) firm 1 wants to move towards the center

**Principle of Minimum Differentiation** - this was Hotelling’s (incorrect) conclusion; he didn’t account for the linear portion of the demand curve so he concluded that the equilibrium would be each firm locating in the middle (avoiding differentiation)

**Hotelling Summary** - linear transport costs; no existence theorem because best replies are not quasiconcave; we found candidate equilibrium (\( \tilde{p}_1, \tilde{p}_2 \) (max on concave portion) and tested it; assuming firms equidistant from ends (\( a = b \)) to simplify the math, we found this is indeed a Nash equilibrium as long as firms are less than 1/4 from the end

**Pure Strategy** - solving the quadratic would actually yield that (\( \tilde{p}_1, \tilde{p}_2 \)) in an equilibrium as long as \( a^2 + b^2 \leq 1/8 \)

**Problem** - these are equilibrium based on choosing price; when firms get to choose location, however, their profits are increasing as they approach the boundary of the pure strategy region; that means profits are probably higher outside that region
Mixed Strategy in Prices - what's basically happening is one firm undercutting, but then second firm cuts price further to get some of the market back; as prices drop, eventually one firm will be on the concave portion of the best reply so it will charge a higher price

Osborne & Pitchik (1987) - show that there exists an equilibrium in mixed strategies where firms locate at about \( a = b = 0.27 \) and choose from a continuous price distribution (despite discontinuity of payoffs); this wasn't easy as they had to solve a differential equation numerically; solution rejects Hotelling's minimum differentiation result

After Hotelling - variations

Smithies - one stage game so firms choose locations and prices simultaneously

Get best replies: take partials of \( \pi_1(p_1, p_2, a, b) \) wrt \( p_1 \) & \( a \); \( \pi_2(p_1, p_2, a, b) \) wrt \( p_2 \) & \( b \)

Gives 4 equations with 4 unknowns

Solution yields \( \tilde{a} + \tilde{b} < 1 \) (firms aren't located together) and \( \tilde{p}_1, \tilde{p}_2 > 0 \)

Is \( (\tilde{p}_1, \tilde{a}) \) best reply to \( (\tilde{p}_2, \tilde{b}) \)?

If firm 1 locates at firm 2 and charges \( \tilde{p}_2 - \epsilon \), firm 1 gets all its original profit plus all of firm 2's profit (minus some \( \epsilon \)) :: splitting the market will never be a best reply in 1 stage game

What if located together; 1 firm can move away and capture with positive price :: \( \tilde{a} = \tilde{b} \) is not an equilibrium either

Result - no pure strategy Nash equilibrium; mixed strategy equilibrium not known

D'Aspremont & Thissee - quadratic transport cost \( ad^2 \) (undercutting doesn't cause discontinuities in the profit function like the linear case)

To split market in the middle, must have \( p_1 + t(x - a)^2 = p_2 + t(1 - b - x)^2 \)

Solution is \( x = \frac{p_2 - p_1}{2t(1 - b - a)} + \frac{1 - b + a}{2} \) (similar to what we did on p.3)

Price equilibrium exists for all location pairs

\[
p_1^*(a, b) = t(1 - a - b) \left( 1 + \frac{a - b}{3} \right) \quad \text{and} \quad p_2^*(a, b) = t(1 - a - b) \left( 1 + \frac{b - a}{3} \right)
\]

If firms located together \( (a + b = 1) \), then get Bertrand result \( (p_1 = p_2 = 0) \)

For \( a + b < 1 \), \( p_1 > p_2 \) if \( a > b \) (follows from \( p_1^* \) and \( p_2^* \))

But \( \frac{\partial \pi_1^*(a, b)}{\partial a} < 0 \) and \( \frac{\partial \pi_2^*(a, b)}{\partial b} < 0 \quad \forall \quad a, b \in [0,1] \)

:: firms don't want to move closer, but want to be at the extremes (solution is \( a = b = 0 \))... maximal differentiation

Misleading - why are firms are restricted to \([0,1]\)?

it makes sense for Hotelling's beach example, but what about Main street?

PS 2.1 - find \( L_1 \) and \( L_2 \) outside \([0,1]\) to maximize profit... further out means more differentiation

PS 2.2 - suppose restricted to \([0,1]\) but add third firm; maximal differentiation suggests location would be \( 0, 1/2, 1 \), but that's not the equilibrium; solve by assuming 1 firm is at 1/2 and figure out where other two firms want to be
Two Dimensions - real world has cross streets

Euclidian Distance - location problem on a grid is too hard so we'll go "as the crow flies" and assume transport costs are proportional to using

Euclidean distance: \( \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \)

Equal Prices - with firms at arbitrary locations means they split market; mark territory by looking at customers who are equidistant from firms

\[ p_1 < p_2 \] - if firm 1 drops price and firm 2 doesn't the boundary is determined by

\[ p_1 + \sqrt{(x-x_1)^2 + (y-y_1)^2} = p_2 + \sqrt{(x-x_2)^2 + (y-y_2)^2} \]

so it's not customers who are equidistant, but constant difference in distances

Continuous - if person located at firm 2 buys from firm 1 then only line segment behind firm 2 will buy from 2; this line segment is really a "zero measure" when integrating \( \therefore \) undercutting means firm captures market slowly; once firm captures customer at rival’s location there’s only the line segment to go (a "zero measure") so profit is no longer discontinuous...

note also that firms don't have the same incentive to undercut as in the one dimensional model

Aside - still get interesting results from one dimensional model if we fix location (e.g., \( a = b = 0 \))

Other Pricing Models

Mill Pricing - firm sells product at uniform price (customer pays shipping); this is what we've been using

Delivered Pricing - "cost including freight" (CIF); firms deliver good and charge different net price \( (p - td) \) to different consumers; every location can be charged a different price

Assumptions:

- \( c_1 = c_2 \) (equal, constant marginal cost)
- Linear transport cost... so total cost for firms are
  \[ c_1 + t \left| x - a \right| \text{ and } c_2 + t \left| x - 1 + b \right| \]
- Delivered good is homogeneous product
- Completely inelastic demand; each consumer buys 1 unit at lowest price
- Consumers can't ship good so firms can treat \( \hat{x} \) and \( \bar{x} \) independently... firms can look at consumers in isolation

Pick arbitrary location \( \hat{x} \) (firms will compete on price schedules)

If firm 1’s total cost is less than firm 2, then firm 1 will maximize profit by charging as close to firm 2's cost as possible:

\[ p_1^*(\hat{x}) = c_2 + t \left| \hat{x} - 1 + b \right| - \varepsilon \]

\( \therefore \) profit is difference between own cost and rival's cost for customers served

General - with \( n \) firms:

\[ p_i^*(x) = 2\text{nd lowest delivered cost} \]

\[ q_i^*(x) = 1 \]

Bertrand - problem at each location boils down to Bertrand with unequal cost

Split Market - with \( c_1 = c_2 \), firms split market evenly
\[ \pi_1(p_1^*(x, a, b), p_2^*(x, a, b); a, b) = \int_0^{a+b} \left\{ t(1-b-x) - t \mid x-a \right\} \, dx \]

(note, since \( c_1 \approx c_2 \), these terms cancel when we do \( p_1^* - c_1 \))

**Independent Price** - price firm charges customer is independent of firm’s location (it’s based on rival’s location) \( \therefore \) maximizing profit is equivalent to minimizing transport cost to locations being served; result in \( a^* = b^* = 1/4 \)

**Efficiency** - if a social planner were doing this, he’d pick locations for the two firms to minimize total transport costs; we get the same result: \( a^* = b^* = 1/4 \) \( \therefore \) in this model the Nash equilibrium is efficient (note price and profit don’t matter because of inelastic demand)

**Why** - in this model we coupled profits with other firm’s location; price was based on rival’s location and cost based on firm’s location (i.e., uncoupled revenue and cost) so we ended up with max profit being the same as min cost

**Fixed Cost to Entry** - adding more firms is still efficient; firms maximize profit by minimizing cost

**Fixed Cost to Entry** - assume firm 2 is located at 3/4 and already paid entry cost; if firm 1 enters at 1/4; before paying the entry cost, profits to firm 1 are equal to the transport cost savings \( \therefore \) firm 1 enters if the transport cost savings of entry > fixed cost... that’s the criteria for efficient entry

**Two Dimensions** - these results don’t generalize; assume \( L_{H} > L_{V} \) if transport costs are proportional to Euclidian distance we get two equilibrium pairs; only the blue pair is efficient (split firms along \( L_{H} \))

**Location Models Without Price Competition**

**Eaton & Lipsey** - Review of Economic Studies, 1975;

**Fixed Prices** - assume same for both firms; could be regulated environment or manufacturer dictating price to retailer

**Inelastic Demand** - each consumer buys 1 unit at lowest cost (same as everything so far)

**Constant MC** - assume \( P - MC = 1 \)

**Profits** - \( \pi_1 = \frac{a+1-b}{2} \) and \( \pi_2 = 1 - \frac{a+1-b}{2} = \frac{1+b-a}{2} \)

**Result** - firm 1 wants to locate next to firm 2; then 2 wants to leapfrog (jump to the other side of 1); then 1 jumps (etc.) so Nash equilibrium is \( a^* = b^* = 1/2 \)

**Minimum Differentiation** - Hotelling’s result, but it’s an actual equilibrium this time

**3 Firms** - no pure strategy Nash equilibrium (special case)

**4 Firms** - Nash equilibrium: \( L_1 = L_2 = 1/4 \) and \( L_3 = L_4 = 3/4 \)

**Inefficient** - outside firms are not helping to minimize total transportation cost; minimum differentiation “force” always driving outside firms to co-locate with next furthest outside firm
Elastic Demand - Hamilton, Thisse and Weskamp (RSUE, 1989); demand at each location depends on delivered price

Reservation Price - easy case; 1 consumer at each location who buys 1 unit as long as the delivered price is \( \leq r \) (reservation price)

Pricing - compare \( r \) to second lowest delivered cost (charge the lower of the two)

Result - still have price (and total revenue) based on opponent's location so max profit is same as min transport cost... still efficient

Demand Curves - harder; demand curve at each location; assume linear case: quantity demanded at location \( x \) is linear function of delivered cost:

\[
q(x) = \alpha - \beta p(x)
\]

Pricing - still have price equal to second lowest delivered cost (or monopoly price if it's lowest... we'll assume monopoly price is never lowest for our analysis)

2 Firm Case - firm's price (hence quantity) still doesn't depend on it's own location; firm trying to minimize transport cost so it locates at median of sales distribution (just inside the quartiles)

Inefficient - efficient location is based on MC + shipping cost so it's still at 1/4 and 3/4

Low Price - not at firm's location, but at boundary of sales territory

Hosed Customers - for linear transport cost, all customers on the non-rival side bring the same level of profit (and don't benefit much from competition like those between the firms)

Quantity Competition - with elastic demand, we can also look at firms competing with quantities at each location (rather than price); basically end up with Cournot competition at each point (with cost asymmetry for almost all customers except customer in middle or no cost asymmetry if \( L_1 = L_2 \))

Equilibrium Quantities - exist at all locations

Location Equilibrium - for moderate transport costs is minimum differentiation result:

\[
L_1 = L_2 = 1/2 \quad \text{with} \quad q_i(x) > 0 \quad \text{(each firm sells a positive amount at each location)}
\]

Problem - Kreps & Scheinkman argued that justification for Cournot is based on Bertrand with capacity constraints (firms choose capacity, then price); in this case there would have to be a capacity constraint at each location which is not very realistic

Cournot with Mill Pricing - Hamilton & Slutsky; resolves the Kreps/Schenkman complaint by using aggregate quantity constraints (rather than a constraint at each location); more realistic, but very hard to work with

Result - if firms close together there's a mixed strategy in quantities and firms locate at edge of mixed strategy region (i.e., as far apart as possible), but that's still close to the center so result is similar to Hotelling's minimum differentiation

Bertrand vs. Cournot -

No Differentiation - i.e., don't worry about location; Bertrand is more competitive than Cournot; "Bertrand Paradox" means adding 1 firm gets to competitive (efficient) result of price = MC

With Differentiation - i.e., with location; Cheng and Vives showed Bertrand is more competitive for fixed firm characteristics; Cournot firms compete more aggressively for
location, but get inefficient result (locate together); Bertrand firms compete more aggressively after less competitive location... Bertrand firms know if they compete aggressively on location too, they'll end up with zero profit

**Extending Horizontal Differentiation Models** - "probably not much new and elegant to solve"; now being used to answer other questions by feeding into other models

**Product Differentiation vs. Transportation** - delivered pricing is the equivalent of firms tailoring products for consumers; mill pricing is the equivalent of consumers modifying the product or simply suffering a utility loss from not consuming their most preferred varieties

### Summary

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<th>Bertrand</th>
<th>Cournot</th>
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<td>(price competition)</td>
<td>(quantity competition)</td>
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<td><strong>Mill Pricing</strong></td>
<td>Hotelling (orig but incorrect)</td>
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<td>(same price + different</td>
<td>- Mixed strat in prices</td>
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<td>delivered cost)</td>
<td>- Pure strat in location</td>
<td>- &quot;agglomeration&quot; almost</td>
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<td>(a* = b* = 1/2)</td>
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<td>quadratic transport cost</td>
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<td>- Pure strat in prices and</td>
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<td>location (a* = b* = 0)</td>
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<td><strong>Delivered Pricing</strong></td>
<td>- Pure strat in prices (2nd</td>
<td>- Pure strat in location</td>
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<td>(unique price + shipping</td>
<td>lowest price) and location</td>
<td>(a* = b* = 1/2)</td>
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<td>for each consumer)</td>
<td>(a* = b* = 1/4)</td>
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