Refinements

Intro from ECO 7404 Notes

**Refinements in Strategic Form:**
- **Eliminate Weakly Dominated Strategies** - throwing out strictly dominated strategies (even iteratively) never eliminates a Nash equilibrium (because Nash equilibrium never has player using a strictly dominated strategy); goal is to eliminate weakly dominated strategies to get a better defined equilibrium
- **Trembling Hand Perfect** - consider possibility that players make random errors; look at limit of probabilities of errors to get subset of original Nash equilibria

**Theorem** - finite, 2 player game, trembling hand perfect and eliminating weakly dominated strategies are the same

**Refinements in Extensive Form:**
- **Tree Rules** - a little more in depth on the details behind extensive form
  - **Successor** - nodes that can be reached from a given node by following arrows
  - **Immediate Successor** - node that's at end of any arrow leading away from a given node
  - **Predecessor** - analogous to successors except we trace backward through the tree; also applies to **immediate predecessor**
  - **Path** - sequence of nodes that (1) starts with the initial node, (2) ends with a terminal node, and (3) has the property that successive nodes in the sequence are immediate successors of each other
    - **Rule 1** - every node is a successor of the initial node, and the initial node is the only one with this property
    - **Rule 2** - each node except the initial node has exactly one immediate predecessor; the initial node has no predecessors
    - **Rule 3** - multiple branches extending from the same node have different action labels
    - **Rule 4** - each information set contains decision nodes for only one of the players; if this weren't the case, at some point in the game, the players won't know who is to make a decision
    - **Rule 5** - all nodes in a given information set must have the same number of immediate successors and they must have the same set of action labels on the branches leading to these successors; if this weren't the case, the player would be able to distinguish between the nodes
- **Equilibrium Path** - what we observe if we watch the game
- **Subgame** - can't break up an information set and has to start from a single node
- **Subgame Perfect** - if game comes to initial node of a subgame, the game will progress as if it were the subgame so we should find Nash equilibria to all subgames (don't believe people will do irrational things); closely related to throwing out weakly dominated strategies
- **Sequential Rationality** - players ought to demonstrate rationality whenever they are called on to make decisions; optimal strategy for a player should maximize his or her expected payoff, conditional on every information set at which this player has the move; closely related to trembling hand

New notes follow on next page
**Sequential Equilibrium** - builds on subgame perfection

**Nash Equilibrium** - only addresses the equilibrium path

**Subgame Perfection** - says all subgames have equilibrium play

**Kreps & Wilson** - apply equilibrium play to all other areas (including those where there are no subgames); conditional on getting to any point in the tree, players proceed with optimizing behavior

**Actions & Beliefs** - have to specify actions and beliefs at each node; actions are optimizing with respect to beliefs; beliefs must be consistent with structure of the game

**Finding Nash Equilibria** - easier to do in strategic form (for finite games); it’s easy to overlook a Nash equilibrium in extensive form

**Example 1** - Consider this simple, 2-player game: Put it in strategic form and we see there are 2 Nash equilibria: LL and RR

Which is better?

In strategic form, player 2 has R weakly dominated by L so we toss the RR equilibrium

In extensive form, RR is not subgame perfect (if player 2 finds himself at the decision after player 1 chooses L, player 2 would be irrational to pick R over L)

**Commitment** - if player 2 can figure out a way to commit to R, he can force player 1 to choose R (instead of L) so player 2 would get a payoff of 3 instead of 2; views on commitment:

(a) some say it’s independent of the game tree
(b) commitment must be built into the game tree... we’ll use this

**Example 2** - modify example 1 by allowing player 2 to post a bond for commitment prior to player 1’s decision (i.e., player 2 decides whether or not to post a bond; if he does (C), he pays a small fee (ε) and incurs a large penalty for choosing L... effectively, the bond allows player 2 to change his payoffs for his treat of playing R is credible)

Put this tree in strategic form to find all the equilibria

Note 1: player 1’s strategies are listed as XY, where X is the strategy played if player 2 chooses C and Y is the strategy played if player 2 chooses N

Note 2: we’re ignoring ε in the payoffs to save space

There are 5 of Nash equilibria
"Best Equilibrium" (i.e., the most likely or most logical, not necessarily the Pareto optimal)

In strategic form, we can use **iterated weak dominance** to find that (LR, CR) is the best

<table>
<thead>
<tr>
<th></th>
<th>NL</th>
<th>NR</th>
<th>CL</th>
<th>CR</th>
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<tbody>
<tr>
<td>LL</td>
<td>2,2</td>
<td>0,0</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>LR</td>
<td>2,2</td>
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Extensive form again uses subgame perfection to get the same result

**Complication** - if 2 can commit, can 1 also commit? If so, game tree should show it; we can also have multiple levels of commitment so this method can get very complicated

**Commitment in Game Tree** - once the game tree is written, don't talk about actions that aren't reflected in the tree (i.e., if a player doesn't have commitment devices shown in the tree, don't talk about the player using commitment to get to his preferred equilibrium)

**Problem with Subgame Perfection** - (we're leading up to sequential equilibrium), subgame perfect doesn't apply if there are no proper subgames as in example 3

**Example 3** - there are no subgames so every Nash equilibrium is subgame perfect

**Strategic form** - Slutsky did this on the fly and admitted that his choice was not the best to work with (i.e., making player 3 the row player, 2 the column player and 1 the matrix player); there are two equilibria:

\[(a_1, b_2, a_3)\] and \[(b_1, b_2, b_3)\]

**Kreps & Wilson** - argue that not all Nash equilibria are equally compelling; look for players doing things off the equilibrium path that are irrational

\[(a_1, b_2, a_3)\] **Equilibrium** - player 2 isn't on the equilibrium path, so check if he’s rational playing \(b_2\) given his belief that player 1 plays \(a_i\) and player 3 plays \(a_3\); the only way player 2 gets to make a decision is if player 1 deviates (i.e., plays \(b_1\) instead of \(a_1\)); there's no reason to think player 3 changes (he doesn't know player 1 deviated), so given player 2 gets his choice and he expects player 3 to play \(a_3\), player 2 is best off playing \(a_2\) (payoff of 4 instead of 1)
Trembling Hand - the strategic form version of what Kreps & Wilson argued (sort of)

(a_1, b_2, a_3) Equilibrium - we should be able to eliminate this; define:

\[ \varepsilon = \Pr[\text{player 1 deviates}] \] (i.e., plays \( b_1 \) instead of \( a_1 \))

\[ \delta = \Pr[\text{player 3 deviates}] \] (i.e., plays \( b_3 \) instead of \( a_3 \))

Assume \( \varepsilon \) and \( \delta \) are small

Consider gain to player 2 by playing deviating (i.e., plays \( a_2 \) instead of \( b_2 \)):

(Note: "deviation" for player's 1 & 3 are random errors, but for player 2 it's a choice)

<table>
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<tr>
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<tr>
<td>1 &amp; 3 don't deviate</td>
<td>((1-\varepsilon)(1-\delta))</td>
<td>2 - 2 = 0</td>
</tr>
<tr>
<td>1 deviates (3 doesn't)</td>
<td>(\varepsilon(1-\delta))</td>
<td>4 - 1 = 3</td>
</tr>
<tr>
<td>3 deviates (1 doesn't)</td>
<td>((1-\varepsilon)\delta)</td>
<td>0 - 0 = 0</td>
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<tr>
<td>1 &amp; 3 deviate</td>
<td>(\varepsilon\delta)</td>
<td>0 - 1 = -1</td>
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Expected Gain: \(\varepsilon(1-\delta)3 - \varepsilon\delta = 3\varepsilon - 3\varepsilon\delta - \varepsilon\delta = 3\varepsilon - 4\varepsilon\delta = \varepsilon(3 - 4\delta)\)

:\. player 2 gains if \(3 - 4\delta > 0 \Rightarrow \delta < 3/4\)

We assumed \( \varepsilon \) and \( \delta \) are small so player 2 will prefer to deviate and \((a_1, b_2, a_3)\) is not a trembling hand perfect equilibrium

Note: 1 & 3 deviate is "second order small" (\(\varepsilon\delta\)) so that loss isn't a big deal

(\(b_1, b_2, b_3\)) Equilibrium - define:

\[ \varepsilon = \Pr[\text{player 1 deviates}] \] (i.e., plays \( a_1 \) instead of \( b_1 \))

\[ \delta = \Pr[\text{player 2 deviates}] \] (i.e., plays \( a_2 \) instead of \( b_2 \))

Assume \( \varepsilon \) and \( \delta \) are small

Consider gain to player 2 by playing deviating (i.e., plays \( a_2 \) instead of \( b_2 \)):

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Expected Gain: \(- (1-\varepsilon)(1-\delta) + (1-\varepsilon)\delta 3 = (1-\varepsilon)[-1+\delta+3\delta] = (1-\varepsilon)(-1+4\delta)\)

:\. player 2 gains if \(-1+4\delta > 0 \Rightarrow \delta > 1/4\)

We assumed \( \varepsilon \) and \( \delta \) are small so player 2 will not prefer to deviate

Note: we only have to find one player that wants to deviate so for the \((a_1, b_2, a_3)\) equilibrium we were able to stop; for this equilibrium we still have to check if player 1 or player 3 want to deviate

Check player 3; define:

\[ \varepsilon = \Pr[\text{player 1 deviates}] \] (i.e., plays \( a_1 \) instead of \( b_1 \))

\[ \delta = \Pr[\text{player 2 deviates}] \] (i.e., plays \( a_2 \) instead of \( b_2 \))

Assume \( \varepsilon \) and \( \delta \) are small

Consider gain to player 3 by playing deviating (i.e., plays \( a_3 \) instead of \( b_3 \)).
Condition | Prob | ΔPayoff
--- | --- | ---
1 & 2 don’t deviate | $(1-\varepsilon)(1-\delta)$ | $1 - 1 = 0$
1 deviates (2 doesn’t) | $\varepsilon(1-\delta)$ | $2 - 0 = 2$
2 deviates (1 doesn’t) | $(1-\varepsilon)\delta$ | $0 - 1 = -1$
1 & 2 deviate | $\varepsilon\delta$ | $2 - 0 = 2$

Expected Gain

\[
\begin{align*}
\varepsilon(1-\delta)2 - (1-\varepsilon)\delta + \varepsilon\delta 2 &= 2\varepsilon - 2\varepsilon\delta - \delta + \varepsilon\delta + \varepsilon\delta 2 = 2\varepsilon - \delta + \varepsilon\delta \\
\therefore \text{ player 3 gains if } 2\varepsilon - \delta > 0 &\Rightarrow \varepsilon > \delta / 2
\end{align*}
\]

So if player 2 is more than twice as likely to make an error as player 1 then player 3 will gain by deviating... not exactly concrete; need to know player 3's beliefs about $\varepsilon$ and $\delta$

**Trembling Hand Perfect** - basic idea is introducing a little noise to the game; different ways to do it (strategic vs. extensive form); players have beliefs in what opponents will do based on Nash equilibrium, but they also consider the chance of mistakes

**Error** - define error as a mixed strategy $E^i$ where some positive probability is assigned to every strategy of player $i$; by assumption this error does not depend on the player’s equilibrium strategy; examples where errors are dependent:
- **Football** - can choose strategy to pass or run; errors are different (incomplete pass or interception if player passes; fumble if player runs)
- **Darts** - aim at 20 and probability of hitting 1 (error) is more than if player aims at 19

**Chance of Error** - with some small probability $\varepsilon$, player $i$ makes an error (i.e., plays the mixed strategy $E^i$ instead of his Nash equilibrium strategy $X^i$)

**Sequence** - look at error as a sequence $\varepsilon^n$ with $\lim_{n\to\infty} \varepsilon^n = 0$

**Not Serious** - not taking errors seriously; not modeling it and don’t care what causes it because $\varepsilon$ is small

**Overall Mixed Strategy** - pick an $\varepsilon$ (or some $n$ in the sequence); combining player’s equilibrium strategy and error results is the mixed strategy $(1 - \varepsilon^n)X^i + \varepsilon^n E^i$ (note limit as $n \to \infty$ is equal to $X^i$)

**Example** - assume player 1 has three strategies $x_1, x_2, x_3$ and player has pure strategy equilibrium $x_i$; overall mixed strategy is:

\[
(1-\varepsilon) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} E^i_1 \\ E^i_2 \\ E^i_3 \end{bmatrix} = \begin{bmatrix} 1 - \varepsilon + \varepsilon E^1_i \\ \varepsilon E^2_i \\ \varepsilon E^3_i \end{bmatrix}
\]

**New Equilibrium** - this overall mixed strategy game that results from incorporating errors satisfies the assumptions of the Nash theorem so we can find a new Nash equilibrium using errors for all the players $NE(\varepsilon^n, E)$, where:

- $\varepsilon^n$ is matrix of all error probability sequences for each player
- $E$ is matrix of all mixed strategy vectors for each player

**General Case** - this formulation allows errors to affect everyone; Zelton’s original work had player’s own errors affect rivals only (not himself)
**Trembling Hand Perfect Equilibria** - for some $n$ in the sequence $\lim_{n \to \infty} \text{NE}(\varepsilon^n, E) \subseteq \text{NE}(0)$

$\therefore$ set of trembling hand perfect equilibria is $\bigcup_{E} \lim_{n \to \infty} \text{NE}(\varepsilon^n, E) \subseteq \text{NE}(0)$

**Subgame Perfect** - since everything has some positive probability in the error, trembling hand perfect equilibria looks at player decisions at every node; that means these equilibria are subgame perfect

**Sequential Equilibrium** - similar to trembling hand, but not exactly the same; start with Nash equilibrium and use equilibrium strategies to define beliefs at all nodes (including those not on the equilibrium path and those with zero probability of occurring); there are lots of rules for how to do this

**Changing Beliefs** - once deviation is observed (i.e., end up on non-equilibrium path) consider how a player's beliefs change

**Consistency** - beliefs must be consistent with equilibrium strategies, including subgame perfect equilibrium strategies (if they get you to that information set); if player is on a node that's not part of any equilibrium path, then beliefs can be arbitrary

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**Consistency** - beliefs must be consistent with equilibrium strategies, including subgame perfect equilibrium strategies (if they get you to that information set); if player is on a node that's not part of any equilibrium path, then beliefs can be arbitrary

**Mixed Strategies** - if a player is using a mixed strategy in equilibrium, beliefs must be consistent with that strategy

**Difference** -

Trembling hand puts in errors, finds NE, then takes limit of equilibria
Sequential puts in errors, puts in beliefs, takes limit of beliefs, the find equilibrium
Their both very similar, but examples can be constructed where they're different
Trembling hand imposes more restrictions so $\text{TH} \subset \text{Seq} \subset \text{NE}$

**Benefit of Seq** - emphasizes using actions and beliefs (TH does too, but not explicitly); this is a good bridge to incomplete information

**Example** - shows the difference between sequential and trembling hand

**Trembling Hand** - effectively generates a new payoff matrix based on what strategies the players intend to play (in this case we have to check $(D,L)$ and $(U,R)$)
**Expected Payoff** - define \( \pi(x, y) \) as the expected payoff vector when player 1 plays mixed strategy \( x \) and player 2 plays mixed strategy vector \( y \)

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Aim at U</strong></td>
<td>( \pi((1-\delta)U + \delta D, (1-\varepsilon)L + \varepsilon R) )</td>
</tr>
<tr>
<td><strong>Aim at D</strong></td>
<td>( \pi((1-\delta)D + \delta U, (1-\varepsilon)L + \varepsilon R) )</td>
</tr>
</tbody>
</table>

In this case, we don’t really need to solve all the boxes (it gets very tedious in larger problems); we can just examine each Nash equilibrium to see if it’s still an equilibrium.

**\((U,R)\)** - determine if player 2 really prefers playing \( R \):

\[
EV^L = (1-\delta)(1) + \delta(0) = 1 - \delta \\
EV^R = (1-\delta)(1) + \delta(-1) = 1 - 2\delta \\
EV^L > EV^R \text{ so player 2 actually prefers L... } (U,R) \text{ is not trembling hand perfect}
\]

**\((D,L)\)** - determine if player 2 really prefers playing \( L \):

\[
EV^L = \delta(1) + (1-\delta)(0) = \delta \\
EV^R = \delta(1) + (1-\delta)(-1) = 2\delta - 1 \\
EV^L > EV^R \text{ (recall } \delta \in [0,1] \text{ because it's a probability, but it's assumed to be small)} \\
\text{so player 2 does prefer L; now check for row player:} \\
EV^U = (1-\varepsilon)(1) + \varepsilon(1) = 1 \\
EV^D = (1-\varepsilon)(2) + \varepsilon(-1) = 2 - 3\varepsilon \\
EV^D > EV^U \text{ (as long as } \varepsilon < 1/3 \text{) so player 1 does prefer D} \\
\therefore (D,L) \text{ is trembling hand perfect}
\]

**Limit of Equilibria** - effectively took limit (wrt error probabilities) of equilibria

**Gets Hard** - with more than two strategies, need to look at union of all distributions of errors

**Sequential** - takes limit of beliefs (usually easier than TH)

**Belief** - let \( \pi_n(x) \) be sequence of beliefs of column player about row player choosing strategy \( x \), and \( \theta_n(y) \) be sequence of beliefs of row player about column player choosing strategy \( y \)

**Consistent, Not Equilibrium** - beliefs to do not have to be equilibrium beliefs, but have to be consistent in the limit (i.e., as \( n \rightarrow \infty \), the beliefs have to agree with equilibrium)

For this case, consider:

\[
\pi_n(U) = 1 - 1/n \quad \theta_n(R) = 1 - 1/n \\
\pi_n(D) = 1/n \quad \theta_n(L) = 1/n
\]

Limits as \( n \rightarrow \infty \):

\[
\pi_\infty(U) = 1, \pi_\infty(D) = 0 \quad \theta_\infty(R) = 1, \theta_\infty(L) = 0
\]

This is consistent because \((U,R)\) is a Nash equilibrium

**Sequential Equilibrium** - specifies two things: actions and beliefs:

\( \text{SE} = \{ (U,R); \pi(U) = 1, \theta(L) = 0 \} \) ... action: \((U,R)\); beliefs \( \pi_n(U) = 1 \) & \( \theta_n(L) = 0 \)

**Rationality** - given belief (taken at the limit), is action a Nash equilibrium

**Consistency** - limit of beliefs could result from fully mixed errors
In this case, sequential does nothing because (D,L) and (U,R) are sequent equilibria.

**Kreps & Wilson Theorem** - paraphrased by Slutsky: typically, sequential and trembling hand equilibria are the same; for small sets of games ("measure zero") they’re not the same; in those cases, trembling hand eliminates more than sequential (TH ⊂ Seq ⊂ Nash); theorem is very hard to prove.

**Summary** -

For 2 players: IWD ⊂ TH = WD ⊂ Seq ⊂ Nash
- TH is exactly equivalent to single round of eliminating weakly dominated strategies
- Iteratively eliminating weakly dominated strategies eliminates more than TH

More than 2 players: TH ⊂ WD ⊂ Nash; TH ⊂ Seq ⊂ Nash; can't compare WD & Seq or anything with IWD
- TH eliminates more equilibria than a single round of eliminating weakly dominated strategies

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**Aside** - classic game theory movies:
- **Dr. Strangelove** - credibility
- **The Maltese Falcon** - the book by Dashiell Hammett has a sub-plot about husband disappearing (not in the movie) that deals with trembling hand (error doesn't mean game is changed)