Endogenous Timing

First half based on Hamilton & Slutsky. "Endogenizing the Order of Moves in Matrix Games." Theory and Decision. 1993
(These were originally written as one paper, but they were published separately.)

Endogenous Timing - give players choice of when they move (e.g., period 1 or period 2); players don't choose role (leader/follower), but when they move; if both choose same time, it's a simultaneous game; if player A chooses first period and player B chooses second period then A is the leader and B is the follower

Endogenous Decision Type - give players choice on whether to set price or quantity (or for international: choose whether to use tariff or quota)

2 Player, 2 Strategy Game

Strictly Ordered - players have no ties so \( a_1 \neq a_2 \neq a_3 \neq a_4 \) and
\( b_1 \neq b_2 \neq b_3 \neq b_4 \); only have 3 types to worry about:
(i) unique pure strategy equilibrium (e.g., prisoners' dilemma)
(ii) unique mixed strategy equilibrium (e.g., matching pennies)
(iii) multiple equilibria; will be 2 pure and 1 mixed (e.g., coordination problem)

Pure Strategy Lemma - there exists a unique pure strategy equilibrium if and only if at least one player has a dominant strategy

Proof: (a) Dominant strategy \( \Rightarrow \) unique PSE
One player having a dominant strategy means we can eliminate one row (column) from consideration in finding an equilibrium
Since the payoffs are strictly ordered, the column (row) player will have a definite choice when facing two payoffs in the single row (column)
\( \therefore \) a single pure strategy equilibrium exists
(b) Unique PSE \( \Rightarrow \) dominant strategy
Proof with contra positive (no dominant strat \( \Rightarrow \) no unique PSE
We can always swap rows and columns as necessary to ensure \( a_1 \) is the biggest payoff to player A
Assume there is no dominant strategy
That means \( a_1 > a_3 \) (because we just said \( a_1 \) is biggest payoff) and \( a_4 > a_2 \) (otherwise U would be a dominant strategy)
There are 2 possibilities for player B:
(i) \( b_1 > b_2 \Rightarrow b_3 < b_4 \) (assumed no dominant strategies)
\( \therefore \) multiple pure strategy equilibria
(ii) \( b_1 < b_2 \Rightarrow b_3 > b_4 \) (assumed no dominant strategies)
\( \therefore \) no pure strategy equilibrium
Mixed Strategy Lemma - in mixed strategy equilibrium, each player’s expected payoff is greater than two of his payoffs and less than the other two payoffs.

**Proof:** For A and B to use a mixed strategy it must be the case that

\[ pa_1 + (1-p)a_2 = pa_3 + (1-p)a_4 \]

\[ qb_1 + (1-q)b_2 = qb_3 + (1-q)b_4 \]

Just consider player A (similar argument for player B).

Expected payoff is

\[ a^s = q\left[pa_1 + (1-p)a_2\right] + (1-q)\left[pa_3 + (1-p)a_4\right] \]

Using fact that \( pa_1 + (1-p)a_2 = pa_3 + (1-p)a_4 \), we can say

\[ a^s = pa_1 + (1-p)a_2 = pa_3 + (1-p)a_4 \]

That is, Player A’s expected payoff is the average of two of player A’s payoffs; since we assumed strictly ordered payoffs, either \( a_1 \) or \( a_2 \) must be greater than \( a^s \) and the other is less than \( a^s \) (there are two equations so it turns into two payoffs greater and two less).

2 Games - for endogenous timing we have two options: announce timing or don’t announce.

**Announce Timing** - players have simultaneous choice to pick whether the move in first round or second round; the players see the result of their choice and then play the two strategy game from previous page using the timing they picked.

**Subgame Perfect** - use refinement (we haven’t studied formally yet); look at lower trees (two-stage game) and get equilibria for each one.

**Simultaneous Move** - for leftmost and rightmost trees, players have simultaneous game; this has the three options we covered earlier (unique PSNE, unique MSNE, multiple equilibria)... we’ll assume a unique equilibrium: \( a^s, b^s \).

**Sequential Move** - the middle games are Stackelberg games (unique equilibria); in first one A is leader and B is follower so label equilibrium \( a^f, b^f \); in second one B is leader and A is follower so label equilibrium \( a^l, b^l \).

**Reduced Extensive Form** - using subgame perfect refinement.

**Reduced Normal Form** - using subgame perfect refinement.

**Pure Strategy Equilibrium** - assume the unique equilibrium in the simultaneous move game is a pure strategy; further, since the pure strategy lemma says one of the players has a dominant strategy, assume it’s the row player and row 1 is dominant.

\[ a_1 > a_3 \] and \[ a_2 > a_4 \]
Column Player - we can always label the columns so that \( b_1 > b_2 \) so the simultaneous equilibrium \((a^*, b^*) = (a_1, b_1)\)

Four Cases - based on relationship between \( b_3 \) & \( b_4 \), \( a_1 \) & \( a_4 \), and \( b_1 \) & \( b_4 \)

1. \( b_3 > b_4 \) - both players have dominant strategy; timing doesn't matter

\[
\begin{array}{c|cc}
\text{Player A} & \text{L} & \text{R} \\
\hline
\text{U} & a_1, b_1 & a_2, b_2 \\
\text{D} & a_3, b_3 & a_4, b_4 \\
\end{array}
\Rightarrow
\begin{array}{c|cc}
\text{Player A} & \text{F} & \text{S} \\
\hline
\text{F} & a_1, b_1 & a_1, b_1 \\
\text{S} & a_1, b_1 & a_1, b_1 \\
\end{array}
\]

2. \( b_3 > b_4 \) & \( a_1 > a_4 \) - timing doesn't matter

- **A First, B Second** - if he plays U, player B plays L; results in \((a_1, b_1)\); if he plays D, player B plays R; results in \((a_4, b_4)\); since \( a_4 > a_1 \), A will play U... get \((a_1, b_1)\)

- **B First, A Second** - since A will play U for either of B’s strategies and \( b_1 > b_2 \), B plays L... get \((a_1, b_1)\)

\[
\begin{array}{c|cc}
\text{Player A} & \text{L} & \text{R} \\
\hline
\text{U} & a_1, b_1 & a_2, b_2 \\
\text{D} & a_3, b_3 & a_4, b_4 \\
\end{array}
\Rightarrow
\begin{array}{c|cc}
\text{Player A} & \text{F} & \text{S} \\
\hline
\text{F} & a_1, b_1 & a_1, b_1 \\
\text{S} & a_1, b_1 & a_1, b_1 \\
\end{array}
\]

3. \( b_3 > b_4 \), \( a_4 > a_1 \), & \( b_1 > b_4 \) - players choose simultaneous

- **A First, B Second** - if he plays U, player B plays L; results in \((a_1, b_1)\); if he plays D, player B plays R; results in \((a_4, b_4)\); since \( a_4 > a_1 \), A will play D... get \((a_4, b_4)\)

- **B First, A Second** - same as case (2)

\[
\begin{array}{c|cc}
\text{Player A} & \text{L} & \text{R} \\
\hline
\text{U} & a_1, b_1 & a_2, b_2 \\
\text{D} & a_3, b_3 & a_4, b_4 \\
\end{array}
\Rightarrow
\begin{array}{c|cc}
\text{Player A} & \text{F} & \text{S} \\
\hline
\text{F} & a_1, b_1 & a_4, b_4 \\
\text{S} & a_1, b_1 & a_1, b_1 \\
\end{array}
\]

2 pure strategy Nash equilibrium; \((S,F)\) is weakly dominated

Timing Game - S is weakly dominated for player A (and for B)... technically we’d have to show that with the full extensive form tree, but it’s weakly dominated in the reduced form (Slutsky got grief for that in the paper)... result is both players choosing the first period (simultaneous)

4. \( b_3 > b_4 \), \( a_4 > a_1 \), & \( b_4 > b_1 \) - player A chooses first round, B chooses second

Sequential - same results as case (3)

\[
\begin{array}{c|cc}
\text{Player A} & \text{L} & \text{R} \\
\hline
\text{U} & a_1, b_1 & a_2, b_2 \\
\text{D} & a_3, b_3 & a_4, b_4 \\
\end{array}
\Rightarrow
\begin{array}{c|cc}
\text{Player A} & \text{F} & \text{S} \\
\hline
\text{F} & a_1, b_1 & a_4, b_4 \\
\text{S} & a_1, b_1 & a_1, b_1 \\
\end{array}
\]

2 pure strategy Nash equilibrium; \((S,F)\) is weakly dominated

Timing Game - same weakly dominated argument from case (3)
Result - if there is a payoff that Pareto dominates (like \((a_4, b_4)\) in case (4)), players will prefer to choose timing to get away from simultaneous equilibrium.

Mixed Strategy Equilibrium - assume the unique equilibrium in the simultaneous move game is a mixed strategy; there are two cases with no pure strategy equilibrium, we'll look at arrows going clockwise (symmetric argument for arrows going counter-clockwise);

assumptions:
- \(a_1 > a_3\) and \(a_4 > a_2\)
- \(b_2 > b_1\) and \(b_3 > b_4\)

\((a_1, b_1)\) Pareto dominates \((a^*, b^*)\) (the mixed strategy expected payoff); requires 2 things:
(i) \(b_1 > b_1 > b_1 > b_4\)... Mixed Strategy Lemma (p.1) says \(b_1\) has to be one of top two payoffs
(ii) \(a_1 > a_4\)... Mixed Strategy Lemma again; don't know how \(a_3\) & \(a_4\) are related

A First, B Second - if A chooses U, B prefers R... results in \((a_2, b_2)\);
if A chooses D, B prefers L... results in \((a_2, b_3)\); A's choice in U or D depends on relationship between \(a_2\) & \(a_3\), but it won't matter when we look at the timing game below (B's prefers to move to sequential)

B First, A Second - if B chooses L, A prefers U... results in \((a_1, b_1)\); if B chooses R, A prefers D... results in \((a_1, b_3)\); since \(b_2 > b_1 > b_3 > b_4\), B will choose L

Timing Game - it's clear that \((F,F)\) and \((S,S)\) will not be equilibria because from \((F,F)\), player A prefers to move down and from \((S,S)\), player B prefers to move left; to find equilibrium we have to look closer at \((F,S)\):
If it's \((a_2, b_2)\), we know \(a_1 > a_4 > a_2\). a* > a2, so player A prefers to move down... \((F,S)\) is not an equilibrium
If it's \((a_3, b_3)\), we know \(b_2 > b_1 > b_3 > b_4\). b* > b3, so player B prefers to move left... \((F,S)\) is not an equilibrium
We end up with one of the players having a dominant strategy and \((S,F)\) is the unique pure strategy equilibrium

Result - if the expected payoff from the mixed strategy equilibrium is Pareto dominated (and players don't both have a dominant strategy), the players will prefer to choose timing to get away from the mixed strategy

Plausibility of Mixed Strategies - Slutsky says this lends credibility to using mixed strategies if the expected payoff is not Pareto dominated

2 x 2 Only - this rule works for 2 x 2 because the Pareto dominant payoff will be one player's best reply; for 3 x 3 this isn't always the case:
\((a^*, b^*) = (a^1, b^1) = (a^1, b^2) = (10, 10)\)
**Don't Announce Timing** - also called "action commitment"; there's ambiguity in first period as to who will do what, but any firm that moves in the second period will know that it is either the follower or will be competing simultaneously.

One Dominant Strategy - assume only 1 player has a dominant strategy so there's a unique pure strategy equilibrium; assume it's the row player so \( a_i > a_j \) and \( a_2 > a_4 \); we can arrange columns so that \( b_1 > b_2 \); since column player doesn't have dominant strategy we also know \( b_4 > b_3 \).

Simultaneous Equilibrium - by construction \((a^*, b^*) = (a_1, b_1)\) (using subgame perfection).

Weak Dominance - now apply another refinement:
- **Column Player** - W weakly dominates L and R
- **Row Player** - W is the same as (weakly dominates) U

Result - if \( a_i > a_4 \) timing has no effect; if \( a_4 > a_1 \), row player moves first and plays the dominated strategy.

Summary -
- **With Announcement** - players only use timing choice if they have a Pareto dominant payoff (that's a best reply)
- **Without Announcement** - don't need a Pareto dominant payoff to use the timing choice, just need \( a_4 > a_1 \)

Gator's Example - why announce firing Coach Zook in mid-season and let him finish the season? UF committed to hire a new coach.
Continuous Strategy Spaces - basically doing the same thing using continuous strategy spaces instead of 2 x 2 game; firms choose to move in first or second period simultaneously; then they choose their output (or price) in their designated period; (this starts the second paper)

Subgame Perfect - apply refinement:

Simultaneous - if firms move in same period, result is Cournot equilibrium

Sequential - if firms move in sequential periods, result is Stackelberg equilibrium

No Ties - unlike 2 x 2 game, we don’t have to worry about ties in this case (makes it easier): 

Never (S, S) - because leader can always choose simultaneous level, \( a^1 > a^* \) and \( b^1 > b^* \)

Payoff Cases -

If \( a^* > a^1 \) and \( b^* > b^1 \), then (F, F) is unique equilibrium

If \( a^* > a^1 \) and \( b^* > b^1 \), then (S, F) and (F, S) are multiple equilibria (also have mixed strategy equilibrium; similar to coordination game)

If \( a^* > a^1 \) and \( b^* > b^1 \), then (S, F) is unique equilibrium (Pareto dominates \( (a^*, b^*) \))

If \( a^* > a^1 \) and \( b^* > b^1 \), then (F, S) is unique equilibrium (Pareto dominates \( (a^*, b^*) \))

∴ players use timing choice if there is a Pareto improvement... this will be more clear (hopefully) using some specific cases

Specific Cases - the choice on timing will depend on these two things: whether the best replies slope in the same direction and which direction the indifference curves face

Best Replies - can slope up or down so there are four cases: (up, up), (up, down), (down, up), (down, down)

Indifference Curves - can face left or right for player on vertical axis; up or down for player on horizontal axis; again there are four cases:

Both Upward Sloping -

(a) Indiff Curves up and right - both best replies enter the Pareto improvement set; multiple equilibria: (S, F) and (F, S)... both are better than simultaneous equilibrium

(b) Indiff curves down and left - same result as (a)
(c) Indiff curves down and right - neither best reply enters the Pareto improvement set; leader is better off, but follower is worse off so (F, F) is unique equilibrium
(d) Indiff curves up and left - same result as (c)

Both Downward Sloping - same result as both upward sloping: 2 cases with multiple sequential equilibria and 2 cases with unique simultaneous equilibrium
(a) Indiff Curves up and right - neither best reply enters the Pareto improvement set; leader is better off, but follower is worse off so (F, F) is unique equilibrium
(b) Indiff curves down and left - same result as (a)
(c) Indiff curves down and right - both best replies enter the Pareto improvement set; multiple equilibria: (S, F) and (F, S)... both are better than simultaneous equilibrium
(d) Indiff curves up and left - same result as (c)

Best Replies Have Opposite Slopess - in all cases only 1 best reply goes into the Pareto improvement set so there is always a unique sequential move equilibrium
(a) Indiff Curves up and right - only one best reply enters the Pareto improvement set; player 1 is better off as the leader and 2 is better off as the follower so (F, S) is unique equilibrium
(b) Indiff curves down and left - same result as (a)
(c) Indiff curves down and right - only one best reply enters the Pareto improvement set; player 1 is best off as the follower and 2 is best off as the leader so (S, F) is unique equilibrium
(d) Indiff curves up and left - same result as (c)

Result -

**Qualitatively Symmetric** - best replies slope in same direction and \( \partial u_i / \partial s_j \) and \( \partial u_j / \partial s_i \) have same sign (i.e., Pareto improvement set is up-right or down-left), then

**Slope Up** - (S, F) and (F, S) are multiple equilibria (strategic complements)

**Slope Down** - (F, F) is unique equilibrium (strategic substitutes)

(see ECO 7938 Product Differentiation, "Strategic Moves" notes for more details on strategic substitutes and complements)

**Endogenous Stackelberg** - only way to get unique leader-follower equilibrium is to have best replies slope in opposite directions

**Point** - make some factors endogenous by adding a prior stage; can make almost anything a decision variable in an earlier stage... sometimes get interesting results