Applications of Nash Equilibrium

Price Setting Models - look at Bertrand model
Assume perfect substitutes (homogeneous goods); product differentiation mitigates the discontinuity
Firms simultaneously choose price
Low price firm gets entire demand
Same price means firms split market (usually assume each gets 1/2)
High price firm gets zero demand
\[
d(p_1) = \begin{cases} 
  D(p_1) & q_1 < q_2 \\
  D(p_1)/2 & q_1 = q_2 \\
  0 & q_1 > q_2 
\end{cases}
\]

Payoff Discontinuity - profit: \( \pi^1 = p_1d(p_1) - C(d(p_1)) \); standard model is shown above; payoff function is not continuous

Possible Solution - if firms have same marginal costs, sum of payoffs is continuous (which implies upper semi continuous); this is one of Dasgupta & Maskin's "extremely obscure conditions"

(This section will show how to go about finding a mixed strategy equilibrium)

Search - Salop & Stiglitz studied low and high cost of search; later Varian presented model where some search and some don't (equivalent to low cost = 0 and high cost infinite); we'll look at Varian's approach (as presented in Golding & Slutsky, "Equilibrium Price Distributions in an Asymmetric Duopoly")

Marginal Cost - assume constant \( c \): actual number of customers doesn't matter so we'll focus on proportions:

Homogeneous Product - consumers indifferent between products (except for price); only consumers who search only care about price
\( \lambda \) = fraction of consumers who search (only buy from low price seller); evenly split if firms charge same price
\( 1 - \lambda \) = fraction of consumers who do not search

Split - assume \( \delta \) is fraction of no search consumers who buy from the low price firm (\( 1 - \delta \) go to firm 2; usually use \( \delta = 1/2 \))

Labeling - low cost is firm 1 so marginal costs: \( c_1 < c_2 \)

Demand - Varian looked at using reservation price \( r \) with consumers buying at most one unit; could make consumers indifferent at \( r \) or assume they buy at \( r \) as shown in picture; Varian used this type of demand thinking it would be easier to work with; turns out it doesn't matter

Independent - if we assume demand is independent of search and is identical for all consumers, we can use a standard, downward sloping demand curve \( d(p) \) for each consumer

Profit per Customer - \( \pi^1(p) = \pi(p_1, c_1) = (p_1 - c_1)d(p_1) \) ... assume this is concave
Number of Customers - three cases for firm profit based on who has lowest price:

\[
N^1 = \begin{cases} 
\delta(1-\lambda) + \lambda & p_1 < p_2 \\
\delta(1-\lambda) + \frac{1}{2}\lambda & p_1 = p_2 \\
\delta(1-\lambda) & p_1 > p_2
\end{cases} \quad \text{and} \quad N^2 = \begin{cases} 
(1-\delta)(1-\lambda) + \lambda & p_2 < p_1 \\
(1-\delta)(1-\lambda) + \frac{1}{2}\lambda & p_2 = p_1 \\
(1-\delta)(1-\lambda) & p_2 > p_1
\end{cases}
\]

Result - profit function is not continuous; not quasiconcave; none of the theorems we’ve studies apply, but we’ll see an equilibrium still exists (mixing over continuum of strategies)

Pure Strategy - \( p_1^* = p^m(c_1) \) and \( p_2^* = p^m(c_2) \) is only possible pure strategy equilibrium iff

\[
\pi(p^m(c_2), c_2) \geq [1 + \lambda/(1-\lambda)(1-\delta)]\pi(p^m(c_1), c_2)
\]

Cases where there is no pure strategy Nash equilibrium:

1. \( p_1 \neq p_2 \) and at least one \( p_i \neq p^m(c_i) \)
2. \( p_1 = p_2 \)
3. \( p_1 = p_2 = c_i \) (firm \( i \) gains by raising price)
4. \( c_1 = c_2 \) (i.e., \( p^m(c_1) = p^m(c_2) \))
5. high cost firm has very small share of locked (non-searching) customers (\( \delta \) near 1)
6. almost all customers are informed (\( \lambda \) near 1) & costs not too far apart (\( p^m(c_1) > c_2 \))

Proof of \( p_1^* = p^m(c_1) \) and \( p_2^* = p^m(c_2) \) being pure strategy equilibrium

(a) \( p_1^* = p^m(c_1) \) is best reply to \( p_2^* = p^m(c_2) \)

Know \( p^m(c_1) < p^m(c_2) \) because \( c_1 < c_2 \) (labeling convention)

\( \therefore \) doesn’t pay to change \( p_1^* \) because firm 1 already has all the customers it can get (no change in price will attract firm 2’s share of the non-searching customers); changing price away from \( p^m(c_1) \) only lowers profit for firm 1

(b) \( p_2^* = p^m(c_2) \) is best reply to \( p_1^* = p^m(c_1) \)

Based on firm 2’s profit function, there are only two points to check: \( p^m(c_2) \) or \( p^m(c_1) - \epsilon \)

Pick \( p^m(c_2) \) if profit here is \( \geq \) profit at \( p^m(c_1) - \epsilon \):

\[
\pi(p^m(c_2), c_2) \geq [1 - \delta](1-\lambda)\pi(p^m(c_1), c_2)
\]

Mixed Strategy - showed in some situations a pure strategy Nash equilibrium exists, but now we’ll show that a mixed strategy equilibrium always exists (despite not satisfying Nash theorem assumptions); mixed strategy can be over an interval (continuum) of prices and/or on fixed points (atoms)

Atom - specific price (rather than interval) used in mixed strategy; single price with positive probability (vs. zero probability for any price in an interval)

Probability Density Function - not good to work with because it’s never uniquely defined; there’s zero probability for any given price so PDF can always change

Cumulative Distribution Function - easier to see atoms; “more tractable than PDF”

\[
F(p) = \int_0^p f(p) dp
\]

Nondecreasing, Right-hand continuous, Bounded ([0,1])

\( F(p) \) for firm 1; \( G(p) \) for firm 2
\( \hat{F}(p) \) to denote continuous region
\( p_j, \ j = 1,2,\ldots \) to denote atoms (at most a countably infinite number of jumps)
\[ f^o(p_j) = F(p_j) - \lim_{p \to p^+} F(p_j) \] = height of jump (\( p \to p_j^- \) means limit from below/left)

**Support** - assume lowest price used in mixed strategy for firm 1 is \( m \) (i.e., \( F(m) = 0 \) & \( F(p) > 0 \ \forall \ p > m \)) and highest price is \( s \) (i.e., \( F(s) = 1 \) & \( F(p) < 1 \ \forall \ p < s \)); for firm 2 let the bounds be \( n \) and \( t \)

**Homogeneous Product** - means firm only cares whether opponent's price is <, =, or > its own price (not by how much they differ); number of customers are same as shown on top of previous page (\( N^1 \) & \( N^2 \)), but now consider probabilities that those occur:

<table>
<thead>
<tr>
<th>Case</th>
<th>Prob</th>
<th>( N^1 )</th>
<th>Case</th>
<th>Prob</th>
<th>( N^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1 &lt; p_2 )</td>
<td>( 1 - G(p_1) )</td>
<td>( \delta(1 - \lambda) + \lambda )</td>
<td>( p_2 )</td>
<td>( f^o(p_2) )</td>
<td>( (1 - \delta)(1 - \lambda) + \frac{1}{2} \lambda )</td>
</tr>
<tr>
<td>( p_1 = p_2 )</td>
<td>( g^o(p_1) )</td>
<td>( \delta(1 - \lambda) + \frac{1}{2} \lambda )</td>
<td>( p_2 )</td>
<td>( f^o(p_2) )</td>
<td>( (1 - \delta)(1 - \lambda) + \frac{1}{2} \lambda )</td>
</tr>
<tr>
<td>( p_1 &gt; p_2 )</td>
<td>( G(p_1) - g^o(p_1) )</td>
<td>( \delta(1 - \lambda) )</td>
<td>( 1 - F(p_2) )</td>
<td>( (1 - \delta)(1 - \lambda) + \lambda )</td>
<td></td>
</tr>
</tbody>
</table>

**Risk Neutrality** - assuming expected dollar values equivalent to expected utility

**Expected Profits** - have to account for expected number of consumers

Expected \# Consumers for firm 1 charging \( p_1 \):
\[
[1 - G(p_1)]\delta(1 - \lambda) + \lambda(1 - G(p_1)) + \frac{1}{2} \lambda g^o(p_1) \]

\( \delta(1 - \lambda) + \lambda G(p_1) + \frac{1}{2} \lambda g^o(p_1) \)
\[
G(p_1)\delta(1 - \lambda) + \frac{1}{2} \lambda g^o(p_1) \delta(1 - \lambda) \]
\[
\delta(1 - \lambda) + \lambda(1 - G(p_1)) + \frac{1}{2} \lambda g^o(p_1) \]
\[
\therefore \text{Expected profit for firm 1 charging } p_1 : \]
\[
\pi(p_1, c_1)\left[ \delta(1 - \lambda) + \lambda (1 - G(p_1)) + \frac{1}{2} \lambda g^o(p_1) \right] \]

Expected profit for firm 1 playing CDF \( F \) against CDF \( G \):
\[
\pi^1(F, G) = \int m \left[ \pi(p, c_1) \left[ \delta(1 - \lambda) + \lambda (1 - G(p)) \right] \right] \ dF + \]
\[
\sum_j \pi(p_j, c_1) f^o(p_j) \left[ \delta(1 - \lambda) + \lambda (1 - G(p_j)) + \frac{1}{2} g^o(p_j) \right] \]

Expected profit for firm 2 playing CDF \( G \) against CDF \( F \) (derive same way)
\[
\pi^2(F, G) = \int_n \left[ \pi(p, c_2) \left[ (1 - \delta)(1 - \lambda) + \lambda (1 - F(p)) \right] \right] \ dF + \]
\[
\sum_k \pi(p_k, c_2) g^o(p_k) \left[ (1 - \delta)(1 - \lambda) + \lambda (1 - F(p_k)) + \frac{1}{2} f^o(p_k) \right] \]

**Properties of Equilibrium Distribution Functions (i.e. Mixed Strategy)**

1. \( s < p^m(c_1) \) and \( t < p^m(c_2) \) ... neither firm ever charges above it's monopoly price; firm could always lower price to the monopoly price, raising profit per customer and not decreasing (may even gain) the number of customers
2. “flat in \( F \)” only if “flat in \( G \)” ... firm 1 doesn’t price in any open interval below \( p^m(c_1) \) iff firm 2 doesn’t price there

<table>
<thead>
<tr>
<th>( G(p) )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 2 can increase profit by putting more weight on higher price ( p_j ) without risk of losing customers because firm 1 places no probability on these prices</td>
<td></td>
</tr>
</tbody>
</table>
(3) \( m = n > c \) if both firms place positive probabilities on prices other than their monopoly prices

(4) at most 1 firm at an atom (i.e., \( g^*(\hat{p}) = 0 \) or \( f_2^*(\hat{p}) = 0 \) but not both)... if both occurred then there’s a finite probability of a tie and both firms would want to undercut (i.e., not an equilibrium)

(5) \( F \) and \( G \) are strictly increasing (no "flat" regions); in order to put positive probability on a strategy (price), there must be equal expected payoff for those strategies \( \therefore \) \( E[\pi^2(p_1)] = E[\pi^2(p_2)] \); we know from (2) that if there is a flat region, they must both have a flat region; also from (4) there can only be one firm with an atom at \( p \); assume it’s firm 2 (i.e., in \( G \)); since player 1 has a flat region (no probability), firm 2 can increase price (raising profit per customer) without losing customers \( \therefore E[\pi^2(p_1)] > E[\pi^2(p_2)] \) which is a contradiction

(6) Firms will never price between \( p^m(c_1) \) and \( p^m(c_2) \); (1) already argued this for firm 1; any time firm 2 is above \( p^m(c_1) \) it only gets the non-searching customers and it will not lose these customers no matter how much it charges \( \therefore \) it will charge its maximum profit per customer price: \( p^m(c_2) \)

(7) Only atoms at \( p^m(c_1) \) and \( p^m(c_2) \); each firm can have at most 1 atom and it will be at that firm's monopoly price; assume firm 2 has an atom at \( \hat{p} \); from mixed strategy rules (equal expected payoff) we must have \( E[\pi^2(\hat{p} - \epsilon)] = E[\pi^2(\hat{p} + \epsilon)] \); for small \( \epsilon \) firm 2 will have a small loss in profit per customer, but \( \hat{p} - \epsilon \) gives firm 2 a finite increase in customers so \( E[\pi^2(\hat{p} - \epsilon)] > E[\pi^2(\hat{p} + \epsilon)] \) which is a contradiction (i.e., firm 2 can't have atoms below \( p^m(c_1) \)); (same argument holds for firm 1 having an atom below \( p^m(c_1) \)); consider firm 2 having an atom at \( p^m(c_1) \)... this can't be the case if firm 1 has an atom here (from (4)); if firm 1 doesn't have an atom there, then firm 2 raises profit per customer without losing customers by moving the atom to \( p^m(c_2) \)

(8) \( p^m(c_1) < p^m(c_2) \; \Rightarrow \; \text{firm 1 has atom at } p^m(c_1) \); If stores have different monopoly prices then firm with lower price will have an atom at that price

**Problem with Mixed Strategy** - just like with discrete case, firm 1 picks its probabilities to make firm 2 indifferent... why would firm 1 make firm 2 indifferent if it's not improving firm 1’s profit (common criticism of mixed strategy)

**Need to Find** - properties above characterize the cumulative distributions which greatly narrow down the work needed to find an equilibrium (remember before we were faced with the potential for a countably infinite number of atoms and now it’s narrowed down to 1 atom for firm 1 and at most 1 for firm 2); need to solve for (5 unknowns):
- Exact value of lower bound \( m \)
- Magnitudes of atoms at \( p^m(c_1) \) and \( p^m(c_2) \)
- Shape of CDF between the bounds for \( F \) and \( G \)
Continuous Part -
- From (3) and (7) we know $F(m) = G(m) = 0$
- Because strategies with positive probability have equal expected payoff we know:
  
  \[ E_G[\pi^1(p,G)] = K \text{ (constant) for } m \leq p \leq p^m(c_i) \]

  \[ E_G[\pi^1(p,G)] = \pi^1(p,c_i)[\delta(1-\lambda) + \lambda(1-G(p))] = K \]

  Note: this is similar to the expected profit formula in middle of p.3 except $g^o(p)$ term is missing because (7) says there’s no atom here

Profit per customer  # of Non- Expected # of
(bottom of p.1) searchers searchers

Solve for $G(p)$:

\[ \left[ \delta(1-\lambda) + \lambda(1-G(p)) \right] = \frac{K}{\pi^1(p,c_i)} \]

\[ G(p) = 1 - \frac{K}{\lambda \pi^1(p,c_i)} + \frac{\delta(1-\lambda)}{\lambda} \]

Solve for $K$ using $G(m) = 0$: $\pi^1(m,c_i)[\delta(1-\lambda) + \lambda] = K$

Plug that in to $G(p)$:

\[ G(p) = 1 - \frac{\pi^1(m,c_i)[\delta(1-\lambda) + \lambda]}{\lambda \pi^1(p,c_i)} + \frac{\delta(1-\lambda)}{\lambda} = 1 + \frac{\delta(1-\lambda)}{\lambda} - \frac{\pi^1(m,c_i)}{\pi^1(p,c_i)} \left[ 1 + \frac{\delta(1-\lambda)}{\lambda} \right] \]

\[ \therefore G(p) = \left[ 1 + \frac{\delta(1-\lambda)}{\lambda} \right] \left[ 1 - \frac{\pi^1(m,c_i)}{\pi^1(p,c_i)} \right] \]

Repeat the procedure to get $F(p) = \left[ 1 + \frac{(1-\delta)(1-\lambda)}{\lambda} \right] \left[ 1 - \frac{\pi^2(m,c_i)}{\pi^2(p,c_i)} \right]$

Lower Bound - there are lots of cases to check in order to find $m$, but the basic procedure is to take advantage of the fact that expected payoffs are equal at $m$ (where firm serves all searchers) and at the monopoly price (where firm only serves its share of the non-searchers): $E[\pi^1(m)] = E[\pi^1(p^m(c_i))]$

The actual method used in the paper is very specific to this model; in fact if we switch to three or more firms, many of the properties derived earlier don’t apply so even the continuous portions we just solved for would be different

Summary - important thing to get out of this long example is the reasoning (not the actual results)

Theorem vs. Actual - usually nice to know existence theorem holds so we know equilibrium exists before trying to find it; in cases like this example, the theorems don’t hold, but it’s still possible to find an equilibrium (although it may not always exist)... finding an equilibrium is just as good as showing an existence theorem holds

Properties - try to develop properties that characterize the CDF in order to narrow down the list of possible solutions; this part uses the structure of the game and the property of mixed strategies which says all strategies with positive probability must have same expected payoff

Second - use properties to solve for the continuous portion of the CDF using the fact that expected payoff at each point with positive probability must be equal
Discrete vs. Continuous - Sometimes we have to solve a continuous problem by using a discrete approximation and solving numerically; other times it may be easier to use a continuous approximation of a discrete game

Jeopardy - trivia game with rounds where players accumulate money by answering questions correctly; we're interested in Final Jeopardy, the last round of the game
- 3 players enter final round with $S_1$, $S_2$, $S_3$
- If $S_i < 0$, player $i$ doesn't play in the final round
- Players see the question topic then decide on how much to wager ($0 \leq B_i \leq S_i$) before seeing the actual question; players are not timed in this decision and can ask for help from the game host and judges (so there will not be any mistakes on how much players need to bed to beat another player)
- Players get 30 seconds to answer the question
- If player $i$ answers correctly, he ends the game with $S_1 + B_1$; if he answers incorrectly, he ends with $S_1 - B_1$
- Player with highest total at end of Final Jeopardy keeps his total and comes back to play again; 2nd place gets $2,000; 3rd place gets $1,000
- Tie Rules -
  Tie for 2nd - player with higher total entering the round gets 2nd place and the other player gets 3rd place
  Tie for 1st - both players keep their winnings and both players come back to play again

Game Theory - this is a relatively controlled real-world experiment with serious results (real, significant monetary rewards for players)

Irrational Behavior - actual results show that players are "almost never rational"

Last Place -
Pr[Correct] - history shows that probability of getting the last question right is close to 50% for all players; player in first place entering Final Jeopardy is slightly above 50% and third place is slightly below, but there is significant correlation between probabilities so if the first and second place players get the question wrong, the probability that the third place player answers correctly is significantly less then 50%

Shouldn't Bet - typical scores entering Final Jeopardy are:
$S_1 = 10,000$, $S_2 = 7,000$, $S_3 = 6,200$
Usually have bets like $B_1 = 4,000$, $B_2 = 7,000$
The expected score for the third place player betting everything is $p(12,400) + (1 - p)(0)$
Compare this to his certain winnings of not betting anything: 6,200
Given the data on probabilities above, the only way the third place player can win is if both opponents get the question wrong, but that means $p < 1/2$ so the player's expected score is less than his certain score... player 3 should bet zero
Real world - third place player almost never bets zero

Tie Rule - Using the same example above, the first place player is more likely to bet 4,001 so he won't tie; that is irrational based on the tie breaking rule (both keep their winnings and both come back); the first place player already beat the second place player so knowing this player will come back is better than having another player of unknown skill; also, there's a chance for repeated games (player 2 may cooperate in the next game and bet just enough for the players to tie); it's irrational for player 1 to go for an extra dollar and ignore the probability of increased additional winnings from the tie
Example - here’s an example of how we’ll analyze the game: suppose there are only two players who enter the final round with 10,000 and 7,000 points; the probability that player 1 answers the question correctly is \( p \); player 2 gets it right with probability \( q \).

Empirical Evidence - suggests \( p = q = 1/2 \) (actually slightly better than 1/2)

Bet 4,000 or All? - this is referring to player 1

\[
E(\text{Payoff if bet all}) = 20,000p \\
E(\text{Payoff if bet 4,000}) = 14,000p + 6,000(1-p)(1-q) \\
\]

So player 1 should bet it all if \( 20,000p > 14,000p + 6,000(1-p)(1-q) \) ⇒ \( 6,000p > 6,000(1-p)(1-q) \) ⇒ \( p > (1-p)(1-q) \)

Using empirical evidence this is true: \( \frac{1}{2} > \frac{1}{2} \frac{1}{2} = \frac{1}{4} \)

Problem - didn’t incorporate repeated game; assume player expects to win an additional amount \( R \) if he returns:

\[
E(\text{Payoff if bet all}) = (20,000 + R)p \\
E(\text{Payoff if bet 4,000}) = (14,000 + R)p + (6,000 + R)(1-p)(1-q) \\
\]

So player 1 should bet it all if \( (20,000 + R)p > (14,000 + R)p + (6,000 + R)(1-p)(1-q) \) ⇒ \( 20,000p + Rp > 14,000p + Rp + 6,000(1-p)(1-q) + R(1-p)(1-q) \) ⇒ 6,000 \( p - (1-p)(1-q) - R(1-p)(1-q) > 0 \)

Using empirical evidence for probabilities, first term is > 0, but second term is < 0; instead consider \( p = q = 1/2 : 6,000/4 - R/4 > 0 \) ⇒ \( R < 6,000 \)

∴ if expected future earnings are less than 6,000, player 1 should bet it all, but if he expects a greater payoff for returning, he should play more conservatively (puts more probability on winning rather than emphasizing bigger payoff in current game)

Formal Model - now we'll complicate it and make it ugly

Discrete game - Jeopardy uses dollar amounts; we may have an equilibrium in mixed strategies but a 10,001 x 7,001 matrix is tough to work with so we'll assume it's continuous

2 Players - 3 is too complicated

Tied - players are tied with \( S \) entering Final Jeapordy

Strategy - \( b_i \) = amount wagered by player \( i \)

Subjective Probability - \( p = \Pr[1 \text{ gets it right}] \); \( q = \Pr[2 \text{ gets it right}] \); these are the probabilities the players put on themselves... could be different for different subjects, but we'll work with the empirical evidence: \( p \geq 1/2 \) & \( q \geq 1/2 \)

** we'll also assume they are independent

End Amount - \( T_i \) = amount player has at the end of the round:

\[
T_i = \begin{cases} 
S + b_i & \text{if player answers correctly (i.e. with probability } p \text{ for player 1, } q \text{ for 2)} \\
S - b_i & \text{if player answers incorrectly (} 1 - p \text{ for player 1, } 1 - q \text{ for 2)} 
\end{cases}
\]

Payoffs - no second prize (or normalize to zero); this is an unusual payoff function because we're writing it in terms of the end amount, not the strategy choice (bid)

\[
u' = \begin{cases} 
T_i + R & \text{if } T_i \geq T_j \text{ and } T_i > 0 \\
0 & \text{if } T_i < T_j \text{ or } T_i = T_j = 0 
\end{cases}
\]

This is "not well behaved" as the graphs on the next page show
Existence Theorems - for the general case (first graph), the payoff function appears to be upper semi continuous (probably satisfies Dasgupta & Maskin's second approach, p.1 of "Weaken Assumptions" section); but when we look at \( T_i = T_j = 0 \) (second graph) the payoff function is not USC; but Dasgupta & Maskin talked about the sum of the payoffs being USC; in this case (third graph), a tie lets both players win so in addition to the discontinuity at zero, there is another one where the players tie (i.e., sum of payoffs is not USC so Dasgupta & Maskin's theorem doesn't apply

Mixed Strategies - there's no existence theorem to apply, but if we can find a mixed strategy equilibrium, that's just as good; as in previous section we'll work with the cumulative distribution function (CDF) for each player's strategy: \( F^1(b_1) \) & \( F^2(b_2) \) where \( \hat{F}^1(\hat{b}_1) = \Pr[b_1 \leq \hat{b}_1] \)

Properties - same as bottom of p.2 (true for all CDFs):
- Nondecreasing
- Right hand continuous
- Closed and Bounded on [0,1]
- Countable number of discontinuities

Problem - payoffs based on end amount \( (T_i) \), not strategy so define \( G^1(T_i) \) & \( G^2(T_2) \)

Deriving \( G^1 \) - \( G^1(\hat{T}_i) = \Pr[T_i \leq \hat{T}_i] \) = the probability that the end amount is less than or equal to a specific end amount \( \hat{T}_i \) (i.e., the red area in the graphs below); there are three cases for the end amount depending on where \( \hat{T}_i \) is with respect to \( S \):

\( \hat{T}_i > S \) : there are two cases: (a) don't be enough to get to \( \hat{T}_i \) or (b) bet more than enough and get the question wrong
   (a) Not betting enough doesn't matter whether the player gets the question right or wrong (fills in the second two parts of the red area in the graph)
   Math: \( F^1(\hat{T}_i - S) \)
   (b) The first part of the red area in the graph comes from betting more then enough to get to \( \hat{T}_i \) and getting the question wrong
   Math: \( (1 - p)(1 - F^1(\hat{T}_i - S)) \)

\( \hat{T}_i = S \) : again there are two cases: (a) bet nothing or (b) get question wrong is a positive bet
   (a) Betting nothing doesn't matter if player gets question right or wrong
   Math: \( f^1(0) = F^1(0) = F^1(\hat{T}_i - S) \)
   Trick 1 - since there are no bets less than zero, PDF = CDF
   Trick 2 - \( \hat{T}_i = S \Rightarrow \hat{T}_i - S = 0 \)
   (b) Any positive bet
   Math: \( (1 - p)(1 - F^1(0)) = (1 - p)(1 - F^1(\hat{T}_i - S)) \)

Note: the math for \( \hat{T}_i = S \) can be expressed the same as the math for \( \hat{T}_i > S \)
\( \hat{T}_1 < S \): there are two cases, both require player to get the question wrong (have to end up with less money than he starts with: (a) bet exactly enough to get to \( \hat{T}_1 \) or (b) bet more than enough

(a) Math: \((1 - p)f^1(S - \hat{T}_1)\)

(b) Math: \((1 - p)(1 - F^1(S - \hat{T}_1))\)

Put it all together:

\[
G^1(\hat{T}_1) = \begin{cases} 
F^1(\hat{T}_1 - S) + (1 - p)(1 - F^1(\hat{T}_1 - S)) & \text{if } \hat{T}_1 \geq S \\
(1 - p)\left[l - F^1(S - \hat{T}_1) + f^1(S - \hat{T}_1)\right] & \text{if } \hat{T}_1 < S
\end{cases}
\]

Can repeat this for player 2, but only change is swap 1 for 2 and \( q \) for \( p \)

\[
G^2(\hat{T}_2) = \begin{cases} 
F^2(\hat{T}_2 - S) + (1 - q)(1 - F^2(\hat{T}_2 - S)) & \text{if } \hat{T}_2 \geq S \\
(1 - q)\left[l - F^2(S - \hat{T}_2) + f^2(S - \hat{T}_2)\right] & \text{if } \hat{T}_2 < S
\end{cases}
\]

**Expected Payoff** - look at player 1 given \( G^2 \) we just derived; two cases:

(a) player 1 gets it right and player 2 ends up with less (or equal) money than what player 1 ends up with (i.e., \( T_2 \leq S + b \))

(b) player 1 gets it wrong with player 2 ends up with less (or equal) money than what player 1 ends up with (i.e., \( T_2 \leq S - b \))

\[
E[u^1(b, G^2)] = p(S + b + R)G^2(S + b) + (1 - p)(S - b + R)G^2(S - b)
\]

Plug in values for \( G^2 \)

\[
E[u^1(b, G^2)] = p(S + b + R)[F^2(b) + (1 - q)(1 - F^2(b))] + (1 - p)(S - b + R)[1 - F^2(b) + f^2(b)]
\]

Can simplify first term in brackets:

\[
[F^2(b) + (1 - q)(1 - F^2(b))] = F^2(b) + 1 - F^2(b) - q(1 - F^2(b)) = 1 - q(1 - F^2(b))
\]

Update the equation:

\[
E[u^1(b, G^2)] = p(S + b + R)[1 - q(1 - F^2(b))] + (1 - p)(S - b + R)(1 - q)[1 - F^2(b) + f^2(b)]
\]

Factor out \((S + R)\)... nasty algebra coming up!

\[
E[u^1(b, G^2)] = (S + R)[p(1 - q(1 - F^2(b))) + (1 - p)(1 - q)(1 - F^2(b))] + b[p(1 - q(1 - F^2(b))) - (1 - p)(1 - q)(1 - F^2(b))]
\]

Simplify first term in brackets:

\[
[p(1 - q(1 - F^2(b))) + (1 - p)(1 - q)(1 - F^2(b))] = p - pq + pqF^2(b) + 1 - p - q + pq - F^2(b) + pF^2(b) + qF^2(b) - pqF^2(b) = 1 - q - F^2(b) + pF^2(b) + qF^2(b) = 1 - q + (p + q - 1)F^2(b)
\]

Simplify second term in brackets:
\[ p(1-q(1-F^2(b)))- (1-p)(1-q)(1-F^2(b)) = \]
\[ p-pq+pqF^2(b)-1+p+q-pq+\frac{q}{2} - F^2(b) - pF^2(b) - qF^2(b) + pqF^2(b) = \]
\[ 2p - 2pq - 1 + q + (1 + 2pq - p - q)F^2(b) = \]
\[ (2p-1)(1-q) + ((1 - p)(1-q) + pq)F^2(b) \]
Plug those back:
\[
E[u^1(b, G^2)] = (S + R)[1-q + (p + q - 1)F^2(b)] + b\left[(2p-1)(1-q) + ((1 - p)(1-q) + pq)F^2(b)\right]
\]

**Atoms Only** - consider \( b'' > b' > 0 \) where neither player has an atom (i.e., drop the \( f^i(\bullet) \) term) so we're looking at the continuous part to the CDF
Define \( \Delta(b'', b') \equiv E[u^1(b'', G^2)] - E[u^1(b', G^2)] \)
In order for both \( b'' \) and \( b' \) to be in the continuous mixing, we must have \( \Delta(b'', b') = 0 \)
\[
\Delta(b'', b') = (S + R)[1-q + (p + q - 1)F^2(b'')] + b\left[(2p-1)(1-q) + ((1 - p)(1-q) + pq)F^2(b'')\right] - \]
\[
(S + R)[1-q + (p + q - 1)F^2(b')] - b\left[(2p-1)(1-q) + ((1 - p)(1-q) + pq)F^2(b')\right]
\]
\[
\Delta(b'', b') = (S + R)(F^2(b'') - F^2(b'))(p + q - 1) + (b'' - b')(2p-1)(1-q) + \]
\[
b''F^2(b'') - b'F^2(b') \]
\[
\left[(1-p)(1-q) + pq\right] \quad \text{(recall } 1/2 \leq p < 1 \text{ & } 1/2 \leq q < 1 \}
\]
\[
\Rightarrow \Delta(b'', b') > 0 \quad \text{... player can't mix over these bets in the continuous part of the CDF}
\]
(i.e., all bets played are mass points)

**Best Replies** - now \( \max E[u^1] \) for a given value of \( b_2 \)

Consider \( b_2 = S \) - look at three options: \( b_1 = S \), \( 0 < b_1 < S \), \( b_1 = 0 \)

(a) \( b_1 = S \) - player 1 only wins if he gets it right (regardless of what player 2 does); if he gets it wrong, he gets nothing (even if player 2 also gets it wrong... they can't win if they tie with zero dollars) \( \Rightarrow \)
\[ E[u^1] = p(2S + R) \quad \text{(prob of getting it right times double the money + R)} \]

(b) \( 0 < b_1 < S \) - only way player 1 wins if it player 2 gets it wrong (with probability \( 1 - q )\); player 1’s end amount depends on whether he gets it right or wrong:
\[ E[u^1] = (1-q)[p(S+b_1 + R) + (1-p)(S-b_1 + R)] \]

Player 1 wants to maximize this wrt \( b_1 \) so look at FOC:
\[
\frac{\partial E[u^1]}{\partial b_1} = (1-q)[p - (1-p)] = (1-q)(2p-1) > 0 \quad \Rightarrow \text{ make } b_1 \text{ as big as possible... can't actually get to } S, \text{ but}
\]
\[ \lim_{b_1 \to S} E[u^1] = (1-q)[p(2S + R) + (1-p)R] = (1-q)(2pS + R) \]

10 of 13
Compare this to playing $b_1 = S$ : (suspect $b_1 = S$ is better so write with $>$ and try to verify it)
$$p(2S + R) > (1-q)(2pS + R)$$
$$p2S + pR > 2pS + R - q2pS - qR$$
$$2pqS > (1-p-q)R ... this is true (under out assumptions about $p$ & $q$)
because the left term is positive and the right term is negative

(c) $b_1 = 0$ - player 1 wins if player 2 gets it wrong, regardless of what player 1 answers
$$E[u^1] = (1-q)(S + R)$$

Compare this to playing $0 < b_1 < S$ : (yeah $b_1 = S$ is better than $0 < b_1 < S$ , but this trick allows us to rule out $b_1 = 0$ because $0 < b_1 < S$ is better)
$$(1-q)(S + R) < (1-q)(2pS + R) ... this is the case because $p \geq 1/2$$

$$b_1 = b_2 = S \text{ is an equilibrium}$$

Consider $b_2 = 0$ - look at three options: $b_1 = S$, $0 < b_1 < S$, $b_1 = 0$

(a) $b_1 = S$ - same as above:
$$E[u^1] = p(2S + R)$$

(b) $0 < b_1 < S$ - only win if player 1 gets it right
$$E[u^1] = p(S + b_1 + R)$$

Player 1 wants to maximize this wrt $b_1$ so look at FOC:
$$\frac{\partial E[u^1]}{\partial b_1} = p > 0 \implies \text{make } b_1 \text{ as big as possible... can't actually get to } S \text{, but }$$
$$\lim_{b_1 \to S} E[u^1] = p(2S + R) ... this is the same as the first case

(c) $b_1 = 0$ - player 1 wins regardless of how he or player 2 answer the question
$$E[u^1] = S + R$$

Figure out when this would be better than playing $b_1 = S$ :
$$S + R \geq p(2S + R) \implies p \leq \frac{S + R}{2S + R}$$

$$b_1 = b_2 = 0 \text{ is an equilibrium} \text{ if } p \leq \frac{S + R}{2S + R} \& q \leq \frac{S + R}{2S + R}$$

Mixing Between $S$ and $0$ - the first two equilibria we found for a coordination game: both players want to do the same thing, but they don’t know what the opponent is doing; since coordination game as a mixed strategy equilibrium this problem probably has one too

Lottery - think of $b_1$ being a lottery: $b_1 = S$ with probability $\alpha_i$ and $b_1 = 0$ with probability $1 - \alpha_i$; where $\alpha_i$ is chosen to make player 2 indifferent

(skip derivation): $\alpha_i = \frac{S + R - p(2S + R)}{q(S + R)}$
\[ \alpha_i \leq 1 \Rightarrow S + R - p(2S + R) \leq q(S + R) \Rightarrow (1 - q)(S + R) \leq p(2S + R) \ldots \text{this always holds for } p \geq 1/2 \text{ & } q \geq 1/2 \]

\[ \alpha_i \geq 0 \Rightarrow S + R - p(2S + R) \geq 0 \ldots \text{this is the same restriction for } b_1 = b_2 = 0 \text{ to be an equilibrium} \]

Symmetry - we can get the same thing for player 2 since \( S_1 = S_2 = S \)

\[ : b_1 = (S, 0, \alpha_1) \text{ & } b_2 = (S, 0, \alpha_2) \text{ is an equilibrium} \]

where \[ \alpha_1 = \frac{S + R - p(2S + R)}{q(S + R)} \text{ & } \alpha_2 = \frac{S + R - q(2S + R)}{p(S + R)} \]

if \( p \leq \frac{S + R}{2S + R} \text{ & } q \leq \frac{S + R}{2S + R} \)

Consider \( 0 < b_2 < S \) - we can look at this as \( b_2 = S - \epsilon \); look at five options: \( b_1 = S , S - \epsilon < b_1 < S , b_1 = S - \epsilon , 0 < b_1 < S - \epsilon , b_1 = 0 \)

(a) \( b_i = S \) - same as before

\[ E[u^1] = p(2S + R) \]

(b) \( S - \epsilon < b_i < S \) - player 1 is betting more than player 2, so the only way he can win is to get the question right (regardless of what player 2 does):

\[ E[u^1] = p(S + b_i + R) < p(2S + R) \Rightarrow \text{ playing } b_i = S \text{ is better} \]

(c) \( b_i = S - \epsilon \) - player 1 is betting the same as player 2 so he wins if both answer correctly or if both answer incorrectly or if player 1 is right and player 2 is wrong; in other words, player 1 always wins if he gets it right or if both get it wrong:

\[ E[u^1] = p(2S - \epsilon + R) + (1 - p)(1 - q)(\epsilon + R) \]

Figure out when (if) this is better than \( b_i = S \):

\[ p(2S - \epsilon + R) + (1 - p)(1 - q)(\epsilon + R) \geq p(2S + R) \]

Solve for \( \epsilon \): \[ \epsilon \leq \frac{(1 - p)(1 - q)R}{p - (1 - p)(1 - q)} \]

(d) \( 0 < b_1 < S - \epsilon \) - player 1 is better less than player 2 so he only wins if player 2 is wrong (but his payoff depends on his answer):

\[ E[u^1] = (1 - q)\left[p(S + b_i + R) + (1 - p)(S - b_i + R)\right] \]

Player 1 wants to maximize this wrt \( b_i \) so look at FOC:

\[ \frac{\partial E[u^1]}{\partial b_i} = (1 - q)(2p - 1) > 0 \Rightarrow \text{ make } b_i \text{ as big as possible} \]

\[ \lim_{b_i \to S - \epsilon} E[u^1] = (1 - q)\left[p(2S - \epsilon + R) + (1 - p)(1 - q)(\epsilon + R)\right] \ldots \text{ this less than case (c) so we can drop this as a potential best reply} \]

(e) \( b_i = 0 \) - player 1 wins if player 2 gets it wrong, regardless of what player 1 answers

\[ E[u^1] = (1 - q)(S + R) < p(2S + R) \Rightarrow \text{ playing } b_i = S \text{ is better} \]
\[ \therefore b_1 = b_2 = S - \varepsilon \text{ is an equilibrium if } \varepsilon \leq \min \left[ \frac{(1-p)(1-q)R}{p-(1-p)(1-q)}, S \right] \]

**Lots of Equilibria** - in other words, there are lots of pure strategy (and even more mixed strategy) equilibria

**Problem** - players don't know what opponents are doing; it's worse than the typical coordination problem with only two equilibria

**Rational?** - with \( p = q = 1/2, b_1 = b_2 = 0 \) is Pareto optimal, but both players betting zero is almost never observed

**Extension** - Slutsky also studied \( S_1 > S_2 \ldots \) must have \( 3S_2 > 2S_1 \) (Slutsky forgot why) and must have \( 2S_2 > S_1 \) (otherwise, it's a runaway game... player 1 can always win)

**Continuum of Equilibria** - \( b_1 = 2S_2 + 1 - S_1 \) and \( b_2 = S_2 - 1 \)

**Example** - \( S_1 = 10,000 \) and \( S_2 = 8,000 \); equilibria are:

(6001,4001), (6002,4002), ...., (9999,7999)

**Real World** - (6001,7999) is most common outcome, both players are playing strategies from Nash equilibria, but the result is not an equilibrium

**Consistent?** - real world doesn't play Nash equilibria, but also don't see players use best replies to observed behavior (i.e., not adapting)

**Excuses** - (a) players are mimicking rather than computing; (b) this analysis is "beyond the computational capacity of players", (c) players could be minimizing probability of losing so the $ amount is not significant

**The Point** - what should we get from all this?

**Methodology** - how to go about finding mixed strategy (and sometimes pure) equilibria

**Skepticism** - problematic to say game theory is good predictive tool; this is a game with simple rules and large payoffs (in some cases equivalent to a year’s salary), yet the players aren't using these techniques; why would large companies use it?

**Len’s Answer** - most people have to be paid to be this geeky; large companies are looking at payoffs that are large enough to employ analysts to do the number crunching