Weaken Assumptions of Nash Theorem

Nash's theorem was great, but lots of work came afterward trying to weaken the assumptions (i.e., strengthen the theory)

**Continuity of \( u^i \)** - Dasgupta & Maskin (Review of Economic Study, 1986) looked at several approaches to weaken this assumption

(A) Can make up for losing continuity of \( u^i \) by adding two assumptions:

1. \( u^i \) is USC
2. \( V(s_{-i}) \) is continuous

**Theorem** - if (1) and (2) (plus other Nash assumptions: \( S^i \) compact and convex and \( u^i \) quasiconcave) then there exists a Nash equilibrium in pure strategies (i.e., in \( S^i \))

**Upper Semi Continuity** - property similar to UHC, but relates to functions, not correspondences; basically says that maximum limit point of a sequence converging to \( x_0 \) is in the function; formally: given sequence \( x_n \to x_0 \), \( \limsup_{n \to \infty} f(x_n) \leq f(x_0) \)

**Lim sup** - of sequence \( y_n \) is the smallest value \( y \) such that for every \( \epsilon > 0 \) there exists \( N \) in the sequence such that \( y + \epsilon \geq y_n \) for all \( n > N \); in English, the lim sup is the "least upper bound"

(Note: \( y_n \) is an element of the sequence where \( n = 1,2,\ldots,\infty \))

**Why USC** - if \( u^i \) is USC, it will always have a maximum value (but a function that is LSC may not have a maximum as the center graph above demonstrates)

**Continuous Maximized Value** - \( V(s_{-i}) = \max_{s_i} u^i(s_i, s_{-i}) \) s.t. \( s_i \in S^i \); this was a result of the Berge Maximum Theorem before, but Berge doesn't apply if we remove assumption of continuity of \( u^i \)

**Bad Assumption** - not bad so much as poorly written; the second assumption doesn't address "the primitives" (i.e., \( S^i \) and \( u^i \))

**Useful?** - this is stronger than Nash's theorem because it uses weaker assumptions, but is it really useful? (How often will \( V(s_{-i}) \) be continuous when \( u^i \) is not?)

(B) In a second attempt to eliminate the continuity of \( u^i \) assumption, Dasgupta & Maskin came up with "many more extremely obscure conditions" one of them being the sum of \( u^i \) is USC (weaker than each \( u^i \) being USC)... this isn't the most important condition, just the "most comprehensible"

**Theorem** - given these many obscure conditions, equilibrium exists in mixed strategies

**Existence vs. Equilibrium** - it's nice to have an existence proof to know that equilibrium exists, but it may be easier to find the equilibrium rather than verify the assumptions in this case
Symmetric Equilibrium - another thing Dasgupta & Maskin did was prove a result that was "common knowledge": every symmetric game has symmetric equilibrium in mixed strategies; the only assumption they used was a weak continuity assumption to ensure equilibrium exists

Importance - when it exists, the symmetric equilibrium is usually the most important (most likely) or easiest to find (because it adds structure to the problem)

No Continuity - Luce & Raiffa show examples where $u^i$ is not continuous and there is no equilibrium in pure or mixed strategies

Example - strategy space: $S^i \equiv [0,1]$; Payoffs: $u^i(s) = \begin{cases} 0 & \text{if } s_1 = s_2 = 1 \\ s_i & \text{otherwise} \end{cases}$

No pure strategy - want to be close to 1 as possible without being at 1; always want to play opponent's strategy plus $\varepsilon$

No mixed strategy - only worry about opponent playing 2 (other strategies don't affect your payoff); problem is player has to be indifferent between all strategies played so player can't mix with 1 (or any other strategy)

Quasiconcavity of $u^i$ - (summary of result on p.8)

Problem - if $u^i$ is not quasiconcave, we could have best replies that cross at gaps (i.e., no intersection)

(A) Glicksburg proved if $u^i$ is continuous and $S^i$ is compact, then $\exists$ an equilibrium in mixed strategies (dropped $u^i$ is quasiconcave and $S^i$ is convex)

Proof - (a) can try to apply Nash by showing that all mixed strategies result in convex strategy space and quasiconcave expected payoffs, but that's difficult because there are lots of distributions to try (# of distributions not bounded!)

(b) Second option is to look at a finite approximation which will satisfy Nash (in mixed extension); then get finer approximation; show limit of approximation converges (could solve numerically)

(B) If we drop $u^i$ is quasiconcave assumption from Nash, when will a pure strategy equilibrium exist? Several people tackle this: Novshek, Roberts & Sonnenschein, Vives

Elegant Way - need super modularity and lattices (and a fixed point theorem on lattices)... not pretty

Brute Force - want to find something that we can use that guarantees a pure strategy equilibrium; we'll use the basic Cournot problem for the discussion:

$$\pi^1(q_1, q_2) = F(q_1, q_2)q_1 - C_1(q_1)$$

Assume $F$ & $C$ are continuous $\Rightarrow \pi^1$ is continuous

Assume $0 \leq q_i \leq K$ (arbitrary upper bound) $\Rightarrow$ strategy space is compact and convex (trivial in this case because it's one dimensional)

So far we have enough to apply Glicksburg's theorem and say we have an equilibrium in mixed strategies, but we want to guarantee pure strategy

Single Peaked - Nash Theorem requires $\pi^i$ to be quasiconcave, but we're really interested in it being "single peaked"... any of these work:

Concave - could check this; it implies quasiconcave, but it's easier to verify:

$$\frac{\partial \pi^1}{\partial q_1} = \frac{dF}{dQ} q_1 + F - \frac{dC_1}{dq_1}$$
\[
\frac{\partial^2 \pi_i}{\partial q_i^2} = 2 \frac{dF}{dQ} + \frac{d^2 F}{dQ^2} q_i - \frac{d^2 C_i}{dq_i^2} \quad \text{... need this to be } \leq 0
\]

\[
\frac{dF}{dQ} < 0 \quad \text{... demand slopes down (if we assume no Giffen goods)}
\]

\[- \frac{d^2 C_i}{dq_i^2} < 0 \quad \text{... if we assume decreasing returns}
\]

\[
\frac{d^2 F}{dQ^2} \quad ??\quad \text{... we want this to be } < 0
\]

Note: linear demand and constant MC ensures we have a unique solution because they ensure: \( \frac{d^2 C_i}{dq_i^2} = \frac{d^2 F}{dQ^2} = 0 \)

**Other choice?** - concave would be good, but (a) we can’t guarantee it because the middle term is ambiguous, (b) we don’t need global concavity to have a single peak

**Double Peak** - consider problem where \( \pi^1 \) is double peaked...

problem is we may not have convex valued best reply

Given double peaked \( \pi^1 \) for \( \overline{q}_2 \) and \( \overline{q}_2 \), there exists some \( \hat{q}_2 \) with \( \overline{q}_2 < \hat{q}_2 < \overline{q}_2 \) where the peaks are at the same height; at this point the best reply is not convex

Note: if there is a mixed strategy equilibrium it will occur where player 2 picks \( \hat{q}_2 \); player 1 mixes between his two best replies to this strategy in order to make \( \hat{q}_2 \) player 2’s best reply to the mixed strategy

Back to pure strategy:

**Monotonicity Theorem** - In a 2 player game where each player chooses from a 1 dimensional strategy space that is compact and convex, if the best replies are weakly monotonic in the same direction (i.e., both are nondecreasing or both are nonincreasing) then \( \exists \) Nash equilibrium in pure strategies

**Generalizing** - case for nondecreasing generalizes, but nonincreasing only works for two players

**Intuition** - consider nonincreasing best replies:

\( R^1(0) < R^2(s_1) \)

\( R^1(s_2_{\max}) > R^2(s_1) \)

\( R^1(s_2) \) can’t start below and end above without crossing \( R^1(s_2) \) so there’s at least one intersection (i.e. pure strategy Nash equilibrium)

Nonincreasing best replies case is similar (can’t start below and end above without crossing)

**Nondecreasing Best Replies**

Same situation except \( R^1(s_2) \) starts above \( R^2(s_1) \) and then ends up below. Jumps in \( R^1(s_2) \) have to be horizontal to the right

1) Any \( s_1 \) in this area is an equilibrium, so consider \( R^1(0) \); further left, that means \( R^1(s_2) \) is below \( R^2(s_1) \)

2) If we have nonincreasing best reply, \( R^1(s_{2\max}) \) can’t be here. Note that means \( R^1(s_{2\max}) \) is above \( R^2(s_1) \)

3) For nonincreasing best reply, any jumps in \( R^1(s_2) \) must be horizontal to the left so \( R^1(s_2) \) remains below \( R^2(s_1) \)
Mapping \( S^1 \to S^1 \) - another way to think about this is to map from \( S^1 \) back to \( S^1 \) by using player 1’s best reply to players 2’s best reply to \( s_1 \in S^1 \):

\[ s_1' \in \bigcup_{s_1 \in R^1(s_1)} R^1(s_2) \] ... also written \( s_1' \in R^1(R^2(s_1)) \)

If \( R^1 \) and \( R^2 \) are nondecreasing then \( R^1(R^2(s_1)) \) is nondecreasing

If \( R^1 \) and \( R^2 \) are nonincreasing then \( R^1(R^2(s_1)) \) is nonincreasing

Mapping \( s_1' \in R^1(R^2(s_1)) \) (nondecreasing)

1) Any point that crosses the \( 45^\circ \) line is an equilibrium:

\( s_1' = s_1 \) means \( s_1 \) and \( s_2 \) are best replies to each other

2) Pick \( s_1' \) above the \( 45^\circ \) line (otherwise it starts on it we know there’s an equilibrium)

3) Since \( R^1(s_2) \) is nondecreasing, \( s_1' \) at \( s_1 \) can’t be below \( s_1' \) at \( s_1 = 0 \); it also has to be below \( 45^\circ \) line (otherwise we know there’s an equilibrium)

4) Jumps in \( s_1' \) are vertical and upward because \( R^1(s_2) \) is nondecreasing

**Generalizing** - case for nondecreasing generalizes, but nonincreasing only works for two players

**Proof:** define three best replies: \( x \in f(y,z) \), \( y \in g(x,z) \), \( z \in h(x,y) \)

Fix \( z \) and think about it as a two player game with best replies: \( \hat{f}(y) = f(y,z) \)

and \( \hat{g}(x) = g(x,z) \)

To ensure the best replies lead to equilibria in the three player game, substitute the best replies to the fixed \( z \):

\( x(y) = f(y(z), z) \) and \( y(z) = g(x(z), z) \)

Now we want to find out how these best replies change wrt \( z \) to make sure they’re monotonic in the same direction (hence the theorem for two players holds)

 Totally differentiate wrt \( z \):

\[
\begin{align*}
\frac{dx}{dz} &= \frac{\partial f}{\partial y} \frac{dy}{dz} + \frac{\partial f}{\partial z} \\
\frac{dy}{dz} &= \frac{\partial g}{\partial x} \frac{dx}{dz} + \frac{\partial g}{\partial z} \\
\frac{dz}{dz} &= \frac{dx}{dx} + \frac{dy}{dy}
\end{align*}
\]

or in matrix form:

\[
\begin{bmatrix}
1 & -\frac{\partial f}{\partial y} \\
-\frac{\partial g}{\partial x} & 1
\end{bmatrix}
\begin{bmatrix}
\frac{dx}{dz} \\
\frac{dy}{dz}
\end{bmatrix}
= \begin{bmatrix}
\frac{df}{dz} \\
\frac{dg}{dz}
\end{bmatrix}
\]

Solve with Cramer’s Rule:

\[
\begin{align*}
\frac{dx}{dz} &= \frac{\begin{vmatrix}
\frac{df}{dz} & \frac{dg}{dz} \\
\frac{\partial f}{\partial y} & \frac{\partial g}{\partial y}
\end{vmatrix}}{\begin{vmatrix}
1 & -\frac{\partial f}{\partial y} \\
-\frac{\partial g}{\partial x} & 1
\end{vmatrix}} = \frac{\frac{\partial f}{\partial z} + \frac{\partial g}{\partial y}}{\frac{\partial g}{\partial z} + \frac{\partial g}{\partial x}} \\
\frac{dy}{dz} &= \frac{\begin{vmatrix}
-\frac{\partial f}{\partial y} & 1 \\
\frac{\partial g}{\partial y} & \frac{\partial g}{\partial x}
\end{vmatrix}}{\begin{vmatrix}
1 & -\frac{\partial f}{\partial y} \\
-\frac{\partial g}{\partial x} & 1
\end{vmatrix}} = \frac{-\frac{\partial f}{\partial y} \frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}}{\frac{\partial g}{\partial z} + \frac{\partial g}{\partial x}}
\end{align*}
\]
**Case 1:** \( f \) and \( g \) are nondecreasing: 
\[
\frac{\partial f}{\partial y} \geq 0, \quad \frac{\partial f}{\partial z} \geq 0, \quad \frac{\partial g}{\partial x} \geq 0, \quad \frac{\partial g}{\partial z} \geq 0
\]

Consider \( dx/dz \) first:
Numerator is positive (adding positive \# to product of positive \#s)
Denominator is ambiguous \((1 - \text{positive \#})\)

**Stability Argument** - game wouldn’t’ be stable if responses are greater than actions they’re in response to \((\text{i.e.,} \frac{\partial g}{\partial x} < 1 \text{ and } \frac{\partial f}{\partial y} < 1)\)... that means denominator is positive

The same argument holds for \( dy/dz \)

That shows the best replies for the two player simplification of the three player game (holding third player’s strategy constant) are nondecreasing

Now we need to make sure there’s a strategy for the third player that is a best reply to the other players' best replies to his strategy (so an equilibrium exists); look at third player's best reply by mapping from \( z \) to itself: pick \( z, \) find \( x(z) \) & \( y(z), \) then \( z'=h(x(z), y(x)) \)

Totally differentiate this wrt \( z \):
\[
\frac{dz'}{dz} = \frac{\partial h}{\partial x} \frac{dx}{dz} + \frac{\partial h}{\partial y} \frac{dy}{dz} \geq 0
\]
All the terms are nondecreasing so \( dz'/dz \) is nondecreasing; we argued on top of p.4 that this means there’s an equilibrium

**Case 2:** \( f \) and \( g \) are nonincreasing: 
\[
\frac{\partial f}{\partial y} \leq 0, \quad \frac{\partial f}{\partial z} \leq 0, \quad \frac{\partial g}{\partial x} \leq 0, \quad \frac{\partial g}{\partial z} \leq 0
\]

Consider \( dx/dz \) first:
Denominator is positive based on stability argument above
Numerator is ambiguous \((-\text{negative + positive})\)

The same argument holds for \( dy/dz \)

\[
\frac{dx}{dz} + \frac{dy}{dz} = \frac{\frac{\partial f}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial z} + \frac{\partial g}{\partial z} + \frac{\partial f}{\partial z} \frac{\partial g}{\partial x}}{1 - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}} = \frac{\frac{\partial f}{\partial z} \left(1 + \frac{\partial g}{\partial x}\right) + \frac{\partial g}{\partial z} \left(1 + \frac{\partial f}{\partial y}\right)}{1 - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}} \leq 0
\]

From stability argument terms in parentheses are positive so numerator is sum of two negative numbers; denominator is positive from stability argument .. sum of these two ambiguous terms is nonincreasing (but that’s not enough to guarantee a crossing between the best replies)

Even though we’re not sure the first two player’s best replies to the third player cross, we can still look at the third player’s best reply and differentiate like we did before:
\[
\frac{dz'}{dz} = \frac{\partial h}{\partial x} \frac{dx}{dz} + \frac{\partial h}{\partial y} \frac{dy}{dz}
\]
This is two negative terms multiplied by ambiguous terms so this is also ambiguous... that means for the nonincreasing case, we can’t guarantee an equilibrium exists with three players

**Problem with Proof** - not "rigorous" because we assumed differentiability

**Problem with Theorem** - addresses best replies not the primitives (strategies and payoffs)... look at an example for solution to this problem:
Consider payoffs $u'(x, y)$ & $u^*(x, y)$ for strategies: $0 \leq x \leq 1$ & $0 \leq y \leq 1$

Derive best replies (assuming an interior solution and everything is differentiable)

\[
\frac{\partial u'(x, y)}{\partial x} = 0 \quad \text{solve for best reply } x(y)
\]

We can derive $dx/dy$ from comparative statics; totally differentiate FOC wrt $y$:

\[
\frac{\partial^2 u'}{\partial x^2} \frac{dx}{dy} + \frac{\partial^2 u'}{\partial x \partial y} = 0 \Rightarrow \frac{dx}{dy} = -\frac{\partial^2 u'}{\partial x \partial y} / \frac{\partial^2 u'}{\partial x^2}
\]

We do not need to impose globally concavity, but if this is a best reply (i.e., max $u'$), we know it's locally concave thus denominator is negative and we know

Sign of $\frac{dx}{dy}$ is same as sign of $\frac{\partial^2 u'}{\partial x \partial y}$

\[
\therefore \frac{\partial^2 u'}{\partial x \partial y} \geq 0 \Rightarrow \frac{dx}{dy} \geq 0 \quad \text{(best reply slopes upward... nondecreasing)}
\]

We get similar result for $u^*$:

Sign of $\frac{dy}{dx}$ is same as sign of $\frac{\partial^2 u^*}{\partial x \partial y}$

**Summary** - if $\frac{\partial^2 u^*}{\partial x^2} < 0$ & $\frac{\partial^2 u^*}{\partial y^2} < 0$ locally and $\frac{\partial^2 u'}{\partial x \partial y}$ & $\frac{\partial^2 u^*}{\partial x \partial y}$ have the same sign globally, then there exists a pure strategy Nash equilibrium... didn’t address the jumps, but this will hold for those too

**Jumps** - assume 2 peaks, but this argument generalizes; consider the picture shown here; at $x_1$, the left peak is the best reply; at $x_2$ the right peak is the best reply (somewhere in between the peaks are the same height so there is a jump in the best reply)

- if $x_2 > x_1$, the jump will be up
- if $x_2 < x_1$, the jump will be down

Define $G(x) = u^* - u'^* (x, y') - u^* (x, y')$

$G(x) > 0 \Rightarrow y''(x)$ is best reply (i.e., the best reply)

$G(x) < 0 \Rightarrow y'(x)$ is global max

$G(x) = 0 \Rightarrow$ multiple optima; jump occurs in $y$’s best reply

**Use Continuity** - $u'$ (payoffs) are continuous so if $G(\hat{x}) = 0$, we know $G(\hat{x} + \epsilon)$ and $G(\hat{x} - \epsilon)$ are near zero (for small $\epsilon$); 3 cases:

(a) $G(\hat{x} - \epsilon) > 0 > G(\hat{x} + \epsilon)$ (shown in picture); for $x < \hat{x}$ (near $\hat{x}$), $G(x) > 0$ ($y''(x)$ is best reply); for $x > \hat{x}$ (near $\hat{x}$), $G(x) < 0$ ($y'(x)$ is best reply)... that means as $x \uparrow$ jump in $y$’s best reply is **down**

(b) $G(\hat{x} - \epsilon) < 0 < G(\hat{x} + \epsilon)$ ... jump is **up**

(c) $(G(\hat{x} - \epsilon) < 0$ & $G(\hat{x} + \epsilon) < 0)$ or $(G(\hat{x} - \epsilon) > 0$ & $G(\hat{x} + \epsilon) > 0)$ (i.e., both have same sign)
Using Derivatives - the small changes above can be interpreted as the derivative of $G$ wrt $x$ evaluated at $\hat{x}$:

(a) $\frac{dG(\hat{x})}{dx} < 0 \Rightarrow$ jump down

(b) $\frac{dG(\hat{x})}{dx} > 0 \Rightarrow$ jump up

(c) $\frac{dG(\hat{x})}{dx} = 0 \Rightarrow$ inflection point; need to check higher derivatives

Taking Derivative - $G(x) = u^2(x, y''(x)) - u^2(x, y'(x))$ ... note that $u^2(x, y''(x))$ & $u^2(x, y'(x))$ are maximized value functions so we can use the envelope theorem (don't need to worry about how changes in $x$ change $y'(x)$ or $y''(x)$ and how those changes affect $u^2$; I'll write it out anyway and show why we don't worry about these):

$$\frac{dG(x)}{dx} = \frac{\partial u^2(x, y''(x))}{\partial x} + \frac{\partial u^2(x, y'(x))}{\partial y} \frac{dy''(x)}{dx} - \frac{\partial u^2(x, y''(x))}{\partial x} - \frac{\partial u^2(x, y'(x))}{\partial y} \frac{dy'(x)}{dx}$$

Note: to get case (c) above, these derivatives must be equal; that means $\partial u^2 / \partial x$ is not dependent on $y$; another way to say that is $\partial^2 u^2 / \partial x \partial y = 0$

Calculus - fundamental theorem of calculus: $H(x, a) - H(x, b) = \int_{a}^{b} \frac{\partial H(x, y)}{\partial y} dy$

$$\frac{dG(\hat{x})}{dx} = \frac{\partial u^2(\hat{x}, y''(x))}{\partial x} - \frac{\partial u^2(\hat{x}, y'(x))}{\partial x} = \int_{y'(\hat{x})}^{y''(\hat{x})} \frac{\partial^2 u^2(x, y)}{\partial x \partial y} dy$$

$\therefore \frac{\partial^2 u^2}{\partial x \partial y} > 0 \forall \ y$ is sufficient condition for $\frac{dG(\hat{x})}{dx} > 0$ (i.e., jump up)

(really only need condition to hold for $\forall \ y \in [y'(\hat{x}), y''(\hat{x})]$)

Similarly $\frac{\partial^2 u^2}{\partial x \partial y} < 0 \forall \ y$ is sufficient condition $\frac{dG(\hat{x})}{dx} < 0$ (i.e., jump down)

Combine Continuous and Jumps - sufficient condition for jump up at discontinuous portion of best reply (i.e., $\hat{x}$ above) is same sufficient condition for continuous portion of best reply to be nondecreasing; condition for jump down is same as condition for nonincreasing
Quasiconcavity Summary - alternatives for dropping quasiconcavity of payoffs ($u^i$) assumption in Nash Theorem and still have pure strategy equilibrium:

1. $\frac{\partial^2 u^i}{\partial x_i^2} < 0 \quad \forall \ i$ (concavity $\Rightarrow$ quasiconcavity)... not really weaker assumption, but usually easier to verify

2. $\frac{\partial^2 u^i}{\partial x_i \partial x_j} \leq 0 \quad \forall \ i, j \quad \text{or} \quad \frac{\partial^2 u^i}{\partial x_i \partial x_j} \geq 0 \quad \forall \ i, j$

   This is the monotonicity theorem (both best replies are weakly monotonic in same direction);
   Checking this condition is "the same level of complexity" as the first condition when there are only two players (remember it only generalizes for the $\geq 0$ case)

Cournot Example - apply rules to Cournot game; payoffs are $\pi^i$; strategies are $q_i$;

$\pi^i = F(q_i + q_j)q_i - C^i(q_i)$ (define $Q = q_i + q_j$)

$\frac{\partial \pi^i}{\partial q_i} = \frac{dF}{dQ} q_i + F(Q) - \frac{\partial C^i}{\partial q_i}$

Test (1): $\frac{\partial^2 \pi^i}{\partial q_i^2} = \frac{d^2 F}{dQ^2} q_i + \frac{dF}{dQ} + \frac{dF}{dQ} - \frac{\partial^2 C^i}{\partial q_i^2}$ (also have to check for $\frac{\partial^2 \pi^i}{\partial q_j^2}$)

Test (2): $\frac{\partial^2 \pi^i}{\partial q_i \partial q_j} = \frac{d^2 F}{dQ^2} q_i + \frac{dF}{dQ}$ (also have to check for $\frac{\partial^2 \pi^i}{\partial q_i \partial q_j}$)

These are separate conditions; only have to check one; the second is usually easier to check for reasonably symmetric game (i.e., same cost functions)