Single Play Nash Equilibrium

**General n-Person Game** - defined (in strategic form) by
- **Set of players** \( T = \{1, 2, \ldots, n\} \)
- **Strategy space** for each player \( S^i \)
- **Payoffs** for each player \( u^i(s_1, s_2, \ldots, s_n), \) all \( s_i \in S^i \)

**Externalities** - enter through payoffs not strategies; can do it with strategies, but using payoffs is more convenient because we can take strategy space as fixed without imposing a strong assumption (makes existence proof easier)

**Pseudo Game** - (Friedman) opponents' strategy choices limit players strategy space; not necessary to this with strategy space because it can be done through actions; think of strategies as "will try to..." and actions as "will/can do..."

**Sequentiality** - sequence of play is built into the strategies

**Other Players' Strategies** - for player \( i \), the strategies played by the other players are denoted by \( s_{-i} \)

**Nash Equilibrium (NE)** - is a set of strategies \((s_1^*, s_2^*, \ldots, s_n^*)\) with each \( s_i^* \in S^i \) such that for every individual \( u^i(s_1^*, s_2^*, \ldots, s_n^*) \geq u^i(s_1, s_2^*, \ldots, s_n) \) \( \forall s_i \in S^i \)

(i.e., each player is playing his best reply to the opponents' best replies)

**Existence** - don't have enough to determine if NE exists; can come up with games (description of players, strategy space and payoffs) that don't have NE

**No NE Game** - \( n \) players (players); each player announces a number (strategy space); player with largest numbers wins \( x \) dollars and others get nothing (payoffs); this is a game, but there is no NE because players can always find a larger number; problem is that strategy space is not bounded (or closed)

**Assumptions for NE** - set of sufficient conditions to get NE; NE could exist without these but general existence proof is very difficult without them; since players are maximizing their expected payoffs it makes sense that these assumptions (and the existence proof) mirror consumer optimization
- \( 1 \) \( S^i \) **Compact** (Closed & Bounded) - if not, could have possibility of no best reply
- \( 2 \) \( S^i \) **Convex Set** - combined with \#3 ensures best replies are convex sets
- \( 3 \) \( u^i \) **Quasiconcave** - combined with \#2 ensures best replies are convex sets
- \( 4 \) \( u^i \) **Jointly Continuous** - guarantees best reply changes with respect to opponent's strategy in well behaved way

**Nash Theorem** - any game in strategic form satisfying assumptions \( 1 \)-(4) above has a Nash equilibrium in \( S^i \).

**Finite Games** - not included because \( S^i \) not a convex set; the mixed extension of a finite game will satisfy the assumptions and have a NE

**Outline of Proof**: this is very long and will require lots of set up; here's the short version:
- Write optimization problem
- Find best replies
- Get properties of best replies
- Use Berge Maximum Theorem
- Use fixed point theorem
Proof:

**Optimization Problem** - this is faced by each player in the game:
\[
\max_{s_i} u'(s_i, s_{-i}) \quad \text{s.t. } s_i \in S^i
\]

**Best Reply Correspondence** - solution to maximization problem: \(s_i(s_{-i})\)

By assumptions (1) and (4) (\(S^i\) is compact and \(u^i\) jointly continuous) and fact that player \(i\)’s strategy is independent of other player’s strategies (i.e., \(S^i\) is constant, which means it’s also continuous), the Berge Maximum Theorem can be applied (see below for gory details)
\[
M(s_{-i}) = \max \left\{ u^i(s_i, s_{-i}) \mid s_i \in S^i \right\}
\]
\[
\psi(s_{-i}) = \left\{ s_i \in S^i : u^i(s_i, s_{-i}) = M(s_{-i}) \right\}
\]

Assumptions (2) and (3) \((S^i)\) is convex and \(u^i\) quasiconcave mean \(\psi(s_{-i})\) is also convex valued (which will allow us to apply a fixed point theorem later)

Pick \(x, y \in \psi^i(s_{-i})\)
\[
\lambda x + (1 - \lambda) y \in S^i \quad \text{because } S^i \text{ is compact (2)}
\]
\[
u^i(\lambda x + (1 - \lambda) y, s_{-i}) \geq u^i(x, s_{-i}) \quad \text{because } u^i \text{ is quasiconcave (3)}
\]

But can't have \(u^i(\lambda x + (1 - \lambda) y, s_{-i}) > u^i(x, s_{-i}) \) because \(x \in \psi^i(s_{-i})\)
\[
u^i(\lambda x + (1 - \lambda) y, s_{-i}) = u^i(x, s_{-i})
\]

Which means \(\lambda x + (1 - \lambda) y \in \psi^i(s_{-i})\)

Define \(S = \prod_i S^i = S^1 \times S^2 \times \ldots \) (Cartesian product)

Subtle change in notation: \(S^i\) is player \(i\)’s strategy space; \(S\) (without superscript) is Cartesian product of all the players’ strategy spaces

\(S\) is nonempty, compact, and convex (because it is a Cartesian product of a finite number of nonempty, compact, and convex sets)

Define \(\psi : S \to 2^S\) (mapping from the Cartesian product of all the players’ strategy spaces into the set of subsets of the Cartesian product)
\[
\psi(s) = \prod_i \psi^i(s_{-i})
\]

More notation: \(\psi^i(s_{-i})\) is set containing all of player \(i\)’s best replies to \(s_{-i}\) (strategies played by other players); \(\psi(s)\) maps any strategy \(s\) (1 strategy for each player) into the Cartesian product of each player’s best reply to the other players’ strategies; \(s\) can be written as \((s_1, s_2, \ldots, s_n) = (s_i, s_{-i}) = (s_{-j}, s_{-j})\)

\(\psi(s)\) is nonempty, compact, convex valued, and UHC (because it is a Cartesian product of nonempty, compact, convex valued, and UHC sets... this is something else Berge proved, but we won’t)

**Kakutani Fixed Point Theorem** - consider a mapping \(\psi : S \to 2^S\) where \(S\) is a compact and convex set and \(\psi\) is nonempty, compact, convex valued and UHC, then \(\exists s^*\) with \(s^* \in \psi(s^*)\)

All the assumptions are satisfied so \(\exists s^* \in \psi(s^*)\)
\[
\therefore s^* \in \psi^i(s^*) \quad \text{... i.e., } u^i(s^*_i, s^*_{-i}) \geq u^i(s_i, s_{-i}) \quad \forall s_i \in S^i
\]
That means there exists a set of strategies $s^*$ where each player's strategy is a best reply to the other players' (i.e., a Nash equilibrium)

Note: applying Kakutani only works because of the way we set up the mapping $\psi$; we can find other mappings that satisfy Kakutani but the fixed point is not a Nash equilibrium (final exam question from ECO 7120)

(Berge Maximum Theorem - if $F: S \times T \to R$ is a continuous, numerical function and

$\Gamma: S \to 2^T$ is a compact valued, continuous correspondence such that $\forall s \in S$, $\Gamma(s)$ is nonempty, then the numerical function $M$ defined by $M(s) = \max \{ F(s,t) \forall t \in \Gamma(s) \}$ is continuous in $s$ and the correspondence $\psi: S \to 2^T$ defined by

$\psi(s) = \{ t \in \Gamma(s) : F(s,t) = M(s) \}$ is nonnull and compact valued at each $s \in S$ and is upper hemi continuous in $S$

English - if the objective function ($F$) is continuous and the constraint set ($\Gamma(s)$) is nonempty, compact and continuous (i.e., both UHC and LHC), then the maximized value function ($M(s)$) is continuous and the maximizer ($\psi(s)$) exists, is compact and UHC

Translation -

$F: S \times T \to R$ ... $F$ is a mapping from the domain $S \times T$ (all combinations of elements of the sets $S$ and $T$) into the real numbers

$\Gamma: S \to 2^T$ ... $\Gamma$ is a mapping from the set $S$ into the power set of $T$ (the set of subsets of $T$); this makes $\Gamma$ a correspondence, not a function

Consumer Analogy -

$S =$ parameters (price, income)

$T =$ choice variables (quantity)

$F(s,t) =$ objective function (utility)

$\Gamma(s) =$ constraint correspondence (budget set)

$M(s) =$ indirect utility

$\psi(s) =$ demand correspondence

Used in Game Theory -

$s = s_{-i} =$ parameters (other player's strategies)

$t = s_i =$ choice variables (player's strategy)

$F(s,t) = u^i(s, s_{-i}) =$ objective function (utility)

$\Gamma(s) =$ $S'$ constraint correspondence (player's strategy space; constant by construction)

Proof: (Theorem's assumptions highlighted)

(1) $\psi(s)$ is nonnull

Weierstrass's Theorem - a continuous function whose domain is a closed and bounded interval must have both a global maximum and a global minimum in this domain

Since $\Gamma(s)$ is nonempty and compact and $F(s)$ is continuous, Weierstrass's Theorem applies; that means $F(s)$ attains a maximum on $\Gamma(s)$ so there always exists an optimizer (i.e., $\psi(s)$ is nonnull)

(2) $\psi(s)$ is compact valued

$\psi(s) \subset \Gamma(s)$

Since $\psi(s)$ is a subset of $\Gamma(s)$ which is bounded, $\psi(s)$ is also bounded
Consider any sequence \( t^n \in \psi(s^0) \) with \( t^n \to \bar{t} \), where \( \bar{t} \in \Gamma(s^0) \) (i.e., a sequence of maximizers for given parameter \( s^0 \) that converges to any feasible value \( \bar{t} \)).

That means either (a) \( F(s^0, \bar{t}) < M(s^0) \) (i.e., \( \bar{t} \) is not in the set of maximizers) or (b) \( F(s^0, \bar{t}) = M(s^0) \); can’t have \( F(s^0, \bar{t}) > M(s^0) \) because \( M(s^0) \) is the maximized value at \( F(s^0, t) \).

Consider case (a) from continuity of \( F(s, t) \), after some \( k \)th term in the sequence \( F(s^0, t^j) < M(s^0) \) for all \( j > k \) (i.e., the sequence eventually has to go below \( M(s^0) \) to converge on \( F(s^0, \bar{t}) \), but that contradicts \( t^j \in \psi(s^0) \) so case (b) must hold.

Case (b) implies \( \bar{t} \in \psi(s^0) \) which means \( \psi(s) \) is closed.

A closed and bounded set is compact (definition).

(3) \( \psi(s) \) is UHC.

Need to show \( \exists \) convergent subsequences; these subsequences converge to \( \bar{t} \) which is a maximizer (i.e., \( \in \psi(s^0) \)).

(i) Consider any \( s^0 \) and any sequence \( s^n \to s^0 \) and any sequence \( t^n \in \psi(s^n) \).

We know \( t^n \in \Gamma(s^n) \) because optimizer has to be feasible.

\( \Gamma(s) \) is \textbf{UHC} (because it’s continuous).

By definition of UHC, \( t^n \) has a subsequence that converges to a value in \( \Gamma(s^0) \);

Label the sequence \( t^k \) and say it converges to \( \bar{t} \in \Gamma(s^0) \).

(ii) Consider any \( \hat{t} \in \Gamma(s^0) \) and consider the sequence \( s^k \) that corresponds to the convergent sequence \( t^k \) from the previous step (\( \therefore s^k \to s^0 \)).

\( \Gamma(s) \) is \textbf{LHC} (because it’s continuous).

By definition of LHC, \( \exists \) a sequence \( \hat{t}^k \in \Gamma(s^k) \) with \( \hat{t}^k \to \hat{t} \).

(iii) Since \( t^k \) is a subsequence of \( t^n \in \psi(s^n) \), we have \( t^k \in \psi(s^0) \).

By definition of maximized value function \( M(s) \), \( M(s^k) = F(s^k, t^k) \).

Previous step said \( \hat{t}^k \in \Gamma(s^k) \) (i.e., \( \hat{t}^k \) is sequence of feasible values).

\( \therefore F(s^k, t^k) \geq F(s^k, \hat{t}^k) \).

(iv) Since \( F(s, t) \) is \textbf{continuous}, we can take the limit of both sides and the inequality still holds (recall: \( s^k \to s^0 \), \( t^k \to \bar{t} \), and \( \hat{t}^k \to \hat{t} \)).

\( F(s^0, \bar{t}) \geq F(s^0, \hat{t}) \forall \hat{t} \in \Gamma(s^0) \) (the \( \forall \) comes from (ii) were we picked any \( \hat{t} \)).

(v) This means \( \bar{t} \) maximizes \( F \) for \( s^0 \) (i.e., \( \bar{t} \in \psi(s^0) \)) so \( \psi(s) \) is UHC.

(4) \( M(s) \) is continuous (i.e., \( \forall \) sequences \( s^k \to s^0 \), \( M(s^k) \to M(s^0) \)).

\( M(s^k) = F(s^k, t^k) \) for \( t^k \in \psi(s^k) \).

We already showed \( \psi(s) \) is UHC so convergent subsequence exists with \( F(s^k, \hat{t}^k) \to F(s^0, \bar{t}) \).

Since \( \bar{t} \in \psi(s^0) \), we know \( F(s^0, \bar{t}) \geq F(s^0, t) \forall t \in \Gamma(s^0) \).

That means \( M(s^0) = F(s^0, \bar{t}) \) so \( M(s^k) \to M(s^0) \).
(End of proof of Berge Maximum Theorem)

These notes are combined with notes from ECO 7120 (new notes in blue)

**General Maximization Problem** - \( \max_x F(x, \alpha) \) s.t. \( x \in G(\alpha) \), where \( x \) is a vector of decision variables and \( \alpha \) is a vector of parameters (e.g., prices and endowments); \( G(\alpha) \) is the "constraint set" which identifies all possible value that \( x \) can take on

**New Notation** - \( \max_t F(s, t) \) s.t. \( t \in \Gamma(s) \),

**Maximized Value Function** - \( V(\alpha) \equiv \max_x F(x, \alpha) \); value of the function at its maximum (e.g., indirect utility function)

**New Notation** - \( M(s) = \max_t F(s, t) \)

**Maximizer** - \( x(\alpha) \) such that \( F(x(\alpha), \alpha) \geq F(y, \alpha) \) \( \forall y \in G(\alpha) \); the value of the decision variables that maximizes the function (e.g., demand function/correspondence)

**New Notation** - \( \tilde{t} \in \psi(s) \) s.t. \( F(s, \tilde{t}) \geq F(s, t) \) \( \forall t \in \Gamma(s) \) or \( \psi(s) \equiv \{ t \in \Gamma(s) : F(s, t) = M(s) \} \)

**Function** - each \( \alpha \) is allowed to map to at most one value of \( x(\alpha) \); if function has upper hemi continuity (or LHC), it is continuous

**Correspondence** - each \( \alpha \) is allowed to map to one or more values of \( x(\alpha) \); if correspondence has upper & lower hemi continuity, it is continuous

**Berge Maximum Theorem** - if \( F(x, \alpha) \) is continuous in \( x \) and \( \alpha \) and \( G(\alpha) \) is compact (closed and bounded) for each \( \alpha \) and continuous in \( \alpha \), then the maximized value function \( (V(\alpha)) \) is continuous and the maximizer \( (x(\alpha)) \) is upper hemi continuous

**Note1:** if \( G(\alpha) \) is not bounded for a given set of parameters, then the problem may not have a solution (e.g., zero price could result in infinite demand for good so there's no way to maximize utility)

**Note2:** If a parameter does not enter a function (as they don't in utility maximization we're studying), the function is continuous in that parameter

**Continuous** - Function \( F(x) \) is continuous if for any sequence \( x^n \to x^0 \),

(i) \( F(x^n) \to y \) (sequence in range determined by applying function to sequence \( x^n \) converges to some value \( y \))

(ii) \( F(x^0) \) exists

(iii) \( F(x^0) = y \) (most non-continuous functions violate this part of the definition)

**Upper Hemi Continuity** - this is the "sort of" continuous we talked about in micro; Consider sequence of points \( \alpha^n \) that converges to \( \alpha^0 \) (blue dots in graphs); upper hemi continuity says that any series determined by \( x(\alpha^n) \) (red dots) converges to a point in \( x(\alpha^0) \)

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Are UH Continuous                  Are Not UH Continuous

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Formally - given the convergent sequence $\alpha^n \to \alpha^0$, then any sequence $y^n \in x(\alpha^n)$, with $y^n \to \bar{y}$ has $\bar{y} \in x(\alpha^0)$

Another Way - if sequence of points in the correspondence converges to $(\alpha^0, \bar{y})$, then $(\alpha^0, \bar{y})$ must be in the correspondence

Convergence - only look at convergent sequences; some sequences will jump back and forth and the limit doesn't exist; for these sequences, we can use sub-sequence that will converge

ECO 7405 Def'n - consider any sequence $s^n \to s^0$ and any sequence $t^n \in \psi(s^n)$ (aside: since $\psi(s)$ is not single valued (i.e., a correspondence) there could be many sequences), $\psi(s)$ is UHC at $s^0$ if there exists a convergent subsequence $\tilde{t}^n \to \tilde{t}$ and if for any convergent subsequence $\bar{t} \in \psi(s^0)$

Lower Hemi Continuity - works backwards from UHC; take any point $\bar{y}$ in the correspondence at $\alpha^0$; for any sequence of points $\alpha^n$ that converges to $\alpha^0$, there exists a sequence in the correspondence that converges to $\bar{y}$

Difference - LHC is a very subtle difference (for me anyway) from UHC; basically, UHC says we look at a sequence in the correspondence to see if it converges to a point in the correspondence; LHC says we look at a point in the correspondence and then see if we can find a sequence in the correspondence that converges to that point... clear as mud?

Are LH Continuous Are Not LH Continuous

Formally - take any $\bar{y} \in x(\alpha^0)$; for any convergent sequence $\alpha^n \to \alpha^0 \exists y^n \in x(\alpha^n)$ such that $y^n \to \bar{y}$

ECO 7405 Def'n - assume $t^0 \in \psi(s^0)$; consider any sequence $s^n \to s^0$; $\psi(s)$ is LHC if there exists a sequence $t^n \in \psi(s^n)$ such that $t^n \to t^0$

Strict Convexity Assumption - if preferences are strictly convex (i.e., $G(\alpha)$ is a strictly convex set; $F(x, \alpha)$ is strictly quasiconcave), then there will be a unique optimizer so $x(\alpha)$ is a function (not a correspondence)

Convexity Assumption - if $\Gamma(s)$ is convex valued and $F(s, t)$ is quasiconcave, then $\psi(s)$ is convex valued

Standard Correspondence - if goods are perfect substitutes, the demand correspondence is not continuous (because it's not a function); it is UHC, but not LHC (to show not LHC, pick a point in the middle of the flat section... no sequence of prices will generate a sequence of demands that converge to that point)... don't forget these axes are reverse here since $p_1$ is the independent variable
Finite Strategy Spaces - Nash theorem doesn't apply to games with finite number of players, each with finite number of strategies ($S$ not a convex set)

Mixed Extension - if we allow players to use probability distributions over their strategies, the Nash assumptions will be satisfied

Mixed Strategy Space - $\Pi^i \equiv \left\{ \pi^i : \pi^i_j \geq 0 \forall j \text{ and } \sum_j \pi^i_j = 1 \right\}$

- $\Pi^i$ is set of all mixed strategies for player $i$
- $\pi^i$ is a vector of probabilities, one for each of player $i$'s pure strategies
- $\pi^i_j$ is the probability player $i$ puts on strategy $j$

$\Pi^i$ is compact - bounded by 0 and 1; closed (why?)

$\Pi^i$ is convex - linear

Payoffs - $a^1_{ijk}$ = payoff to player 1 when 1 plays $i$, 2 plays $j$ and 3 plays $k$

Probability - $Pr[a^1_{ijk}] = \pi^i_1 \pi^j_2 \pi^k_3$

Expected Utility - $u^1(\pi^1, \pi^2, \pi^3) = \sum_i \sum_j \sum_k a^1_{ijk} \pi^i_1 \pi^j_2 \pi^k_3$

- $u^1(\pi^1, \pi^2, \pi^3)$ is continuous - multiplication and addition are continuous functions
- $u^1(\pi^1, \pi^2, \pi^3)$ is quasiconcave - fix $\pi^2$ and $\pi^3$ and $u^1$ is linear in $\pi^1$ (linear means concave means quasiconcave)
- $\therefore$ mixed extension satisfies the four assumptions of the Nash Theorem

Nash's Contribution - the proof of Nash's theorem "isn't that hard", but he won the Nobel Prize for more than the theorem; Nash corrected and consolidated various other notions of equilibrium by formally specifying the structure of the game

Poorly Defined - prior to Nash, the concept of equilibrium wasn't well defined

Cournot Equilibrium - 2 firms making simultaneous decisions on output
Stakelberg - 2 firms making sequential decision on output (leader-follower)
Fellner - 1940s wrote "Competition Among The Few"; tried to combine all notions of equilibrium between different oligopoly models: "conjectural variation models"

Example - $\max_{q_i} u^1(q_i, q_j)$

- Cournot... $\frac{\partial u^1}{\partial q_i} = \frac{\partial \pi^i}{\partial q_j} = 0$
- Stakelberg... $\frac{\partial \pi^i}{\partial q_i} + \frac{\partial \pi^i}{\partial q_j} \frac{dq_j}{dq_i} = 0$ (leader) and $\frac{\partial \pi^i}{\partial q_j} = 0$ (follower)

- Conjectural Variation... $\frac{dq_j}{dq_i}$, sometimes written $\frac{dq_j^i}{dq_i}$; what player $i$ believes player $j$ will do in response to player $i$'s change in quantity

- General Model... $\frac{\partial u^1}{\partial q_i} + \frac{\partial u^1}{\partial q_j} \frac{dq_j}{dq_i} = 0$ and $\frac{\partial u^1}{\partial q_j} + \frac{\partial u^1}{\partial q_i} \frac{dq_i}{dq_j} = 0$

- General Model - was thought to open a lot of possibilities because it could derive the other models (e.g., $dq_j / dq_i = dq_i / dq_j = 0$ is Cournot)

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Bresnahan - said conjectures may not be consistent with actual replies so he proposed consistent conjectural variation; a whole series of "standard" conjectures for various types of models

Game Theorists - eventually said conjectural variations didn't make sense; there are only two options in a single-stage, two-player game: simultaneous moves or leader-follower; a player can't change what he's doing if he can't observe the rival's move and you can't have both move after the other so general model for conjectural variations isn't right

Nash - corrected conjectural variations by formally specifying structure of game; Cournot and Stakelberg equilibria are both Nash equilibria to different games (different structure)