1. In order to be able to work at home, Professor Slutsky needs to provide his cat Flash with cat toys that he will leave him alone. There are two types of cat toys that Professor Slutsky is trying to choose between: Hartz Stuffin’ Mice which cost $2 per toy, and Pet Essentials Furry Mice which cost $3 per toy. Depending on the production quality and the friskiness of Flash, Professor Slutsky has ascertained that the lifetime, $x$, in weeks of cat toys has an exponential distribution with probability density function

$$ f(x; \theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x > 0 $$

(1)

Professor Slutsky knows that cat toys come in either one of two qualities: high quality with $\theta = 2$; and low quality with $\theta = 1$. Unfortunately, due to poor eye sight, Professor Slutsky cannot tell by inspection alone whether or not a cat toy is of high or low quality. Having purchased Stuffin’ Mice for years, he knows that they are of low quality, but wants to know whether or not it is worth spending the extra dollar for the Furry Mice.

(a) What is the hypothesis that Professor Slutsky wishes to test?
(b) Professor Slutsky, vaguely remembering his statistics course, decides to buy three Furry Mice and accept the null hypothesis, $H_0$, if the three mice each last for more than 1.5 weeks. Assuming that the lifetime of each toy is independent of the lifetime of the other toys, calculate the following probabilities:

- (i) $\Pr[\text{Accepting } H_0 \mid H_0 \text{ true}]$
- (ii) $\Pr[\text{Rejecting } H_0 \mid H_0 \text{ true}]$
- (iii) $\Pr[\text{Accepting } H_0 \mid H_1 \text{ true}]$
- (iv) $\Pr[\text{Rejecting } H_0 \mid H_1 \text{ true}]$

c) Which of (i), (ii), (iii), and (iv) is the probability of making a type I error? Which is the probability of making a type II error?

d) What is the power of Professor Slutsky’s test?

e) Having studied statistics more recently than Professor Slutsky, you suggest that he test the null hypothesis by noting whether or not the average lifetime of the Furry Mice is greater than or less than 1.5 weeks. Having ascertained that the probability distribution for the sum of the lifetimes ($\bar{x} = x_1 + x_2 + x_3$) of the three mice is given by

$$ f(\bar{x}; \theta) = \frac{\bar{x}^2}{2\theta^3} e^{-\frac{\bar{x}}{\theta}}, \quad \bar{x} > 0 $$

(2)

For your stated $H_0$, calculate

- (i) $\Pr[\text{Accepting } H_0 \mid H_0 \text{ true}]$
- (ii) $\Pr[\text{Rejecting } H_0 \mid H_0 \text{ true}]$
- (iii) $\Pr[\text{Accepting } H_0 \mid H_1 \text{ true}]$
- (iv) $\Pr[\text{Rejecting } H_0 \mid H_1 \text{ true}]$

(f) Which, if either, is the better test? Explain you conclusion.

g) Suppose that the three Furry Mice last for 1.6, 0.8 and 2.4 weeks. How confident is Professor Slutsky that it is worth spending the extra dollar on these mice? What would you suggest Professor Slutsky do in order to resolve his doubts?
(a) Wants to know if the new toy (*Furry Mice*) will be worth the money; he'd be spending 50 percent more ($3 vs. $2), so the *Furry Mice* has to last at least 50 percent longer than the *Stuffin' Mice*. That's $\theta = 1.5$, but there are only two possibilities: 1 or 2. Therefore,

\[ H_0: \theta = 2 \]
\[ H_1: \theta = 1 \]

(b) (i) \( \Pr[\text{Accepting } H_0 \mid H_0 \text{ true}] \)...
\[ \Pr[\text{Exp}(2) \geq 1.5] = (1 - \Pr[\text{Exp}(2) < 1.5])^3 = (1 - 0.4724)^3 = 0.1054 \]

(ii) \( \Pr[\text{Rejecting } H_0 \mid H_0 \text{ true}] \)...
\[ 1 - \Pr[\text{Accepting } H_0 \mid H_0 \text{ true}] = 1 - 0.1054 = 0.8946 \]

(iii) \( \Pr[\text{Accepting } H_0 \mid H_1 \text{ true}] \)...
\[ \Pr[\text{Exp}(1) \geq 1.5] = (1 - \Pr[\text{Exp}(1) < 1.5])^3 = (1 - 0.2231)^3 = 0.0111 \]

(iv) \( \Pr[\text{Rejecting } H_0 \mid H_1 \text{ true}] \)...
\[ 1 - \Pr[\text{Accepting } H_0 \mid H_0 \text{ true}] = 1 - 0.0111 = 0.9889 \]

(c) Type I error = \( \Pr[\text{Rejecting } H_0 \mid H_0 \text{ true}] \) (ii)
Type II error = \( \Pr[\text{Accepting } H_0 \mid H_1 \text{ true}] \) (iii)

(d) Power = \( 1 - \beta \) (Type II error) = 1 - 0.0111 = 0.9889

(e) \( F(\bar{x} ; \theta) = \int_0^{\bar{x}} \frac{x^2}{2\theta^2} e^{-x \theta} dx = \left[ -\frac{x^2}{2\theta^2} e^{-x \theta} - \left(-\frac{x}{\theta} \right) \left(-\frac{x}{\theta} \right) e^{-x \theta} - e^{-x \theta} \right]_0^{\bar{x}} = \frac{-\bar{x}^2}{2\theta^2} e^{-\bar{x} \theta} - \frac{\bar{x}}{\theta} e^{-\bar{x} \theta} - e^{-\bar{x} \theta} + 1 \)

(i) \( \Pr[\text{Accepting } H_0 \mid H_0 \text{ true}] \)...
\[ 1 - \left( -\frac{(4.5)^2}{2(2)^2} e^{-\frac{4.5}{2}} - \frac{4.5}{2} - e^{-\frac{4.5}{2}} + 1 \right) = 0.6093 \]

(ii) \( \Pr[\text{Rejecting } H_0 \mid H_0 \text{ true}] \)...
\[ 1 - \Pr[\text{Accepting } H_0 \mid H_0 \text{ true}] = 1 - 0.6093 = 0.3907 \]

(iii) \( \Pr[\text{Accepting } H_0 \mid H_1 \text{ true}] \)...
\[ 1 - \left( -\frac{(4.5)^2}{2(1)^2} e^{-\frac{4.5}{1}} - \frac{4.5}{1} - e^{-\frac{4.5}{1}} + 1 \right) = 0.1736 \]

(iv) \( \Pr[\text{Rejecting } H_0 \mid H_1 \text{ true}] \)...
\[ 1 - \Pr[\text{Accepting } H_0 \mid H_0 \text{ true}] = 1 - 0.1736 = 0.8264 \]

(f) Slutsky Test - \( \alpha = 0.8946 \); power = 0.9889; New Test - \( \alpha = 0.3907 \); power = 0.8264; significance is nice because we're sure to not reject that the toy is of high quality when it actually is, but in this case power is probably more important to Prof. Slutsky. Power gives the probability of rejecting the hypothesis that the more expensive toy is of high quality when it really isn't. Since Prof. Slutsky is worried about spending too much money, power is more important to him and he should use his test. Could increase rejection region of second test to 8.3 (vs. 4.5; this was found by trial and error in Excel). This gives same power (0.9889), and \( \alpha = 0.783 \) which is better.

(g) \( x_1 = 1.6, x_2 = 0.8, x_3 = 2.4 \Rightarrow \bar{x} = (1.6 + 0.8 + 2.4) = 4.8 \)
\[ 1 - \Pr[\text{f(}\bar{x};\theta) \geq 4.8] = 1 - \left( 1 - \Pr[\text{f(}\bar{x};\theta) \leq 4.8] \right) = F(4.8;1) = \left( -\frac{(4.8)^2}{2(1)^2} e^{-\frac{4.8}{1}} - \frac{2(4.8)}{2(1)} e^{-\frac{4.8}{1}} - e^{-\frac{4.8}{1}} + 1 \right) = 0.8574 \]
Buy more toys to improve the confidence.
2. Mendelian theory indicates that the shape and color of a certain variety of pea ought to be divided into four groups: round and yellow, round and green, angular and yellow, and angular and green, according to the ratios 9/3/3/1. For $n = 556$ peas, the following were observed.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Color</th>
<th>Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>Round</td>
<td>Yellow</td>
<td>315</td>
</tr>
<tr>
<td>Round</td>
<td>Green</td>
<td>108</td>
</tr>
<tr>
<td>Angular</td>
<td>Yellow</td>
<td>101</td>
</tr>
<tr>
<td>Angular</td>
<td>Green</td>
<td>32</td>
</tr>
</tbody>
</table>

Test whether the data agrees with Mendelian theory.

$H_0$: $p_1 = \frac{9}{16}, p_2 = \frac{3}{16}, p_3 = \frac{3}{16}$ (implies $p_4 = 1 - p_1 - p_2 - p_3 = \frac{1}{16}$)

$H_a$: $p_1 \neq \frac{9}{16}, p_2 \neq \frac{3}{16}, p_3 \neq \frac{3}{16}$

Multinomial Distribution

$f(x; \theta) = p_1^{x_1} p_2^{x_2} p_3^{x_3} (1 - p_1 - p_2 - p_3)^{x_4}, x_1 + x_2 + x_3 + x_4,$

where $x = [x_1 \ x_2 \ x_3 \ x_4]^T$ and $\theta = [p_1 \ p_2 \ p_3]^T$

$L(x; \theta) = \prod_{i=1}^{N} f(x_i; \theta) = \prod_{i=1}^{N} p_1^{x_{1i}} p_2^{x_{2i}} p_3^{x_{3i}} (1 - p_1 - p_2 - p_3)^{x_{4i}} = p_1^{n_1} p_2^{n_2} p_3^{n_3} (1 - p_1 - p_2 - p_3)^{N-n_1-n_2-n_3},$ where $n_j = \sum_{i=1}^{N} x_{ji}, j = 1, 2, 3$

**Note:** $E(n_j) = \sum_{i=1}^{N} \frac{n_j}{N} = \sum_{i=1}^{N} p_j = N p_j$

**Note:** for simplicity substitute $n_4 = N - n_1 - n_2 - n_3$

$L(x; \theta) = p_1^{n_1} p_2^{n_2} p_3^{n_3} (1 - p_1 - p_2 - p_3)^{n_4}$

$\ln L(x; \theta) = n_1 \ln(p_1) + n_2 \ln(p_2) + n_3 \ln(p_3) + n_4 \ln(1 - p_1 - p_2 - p_3)$

\[
\text{Score } S(x, \theta) = \begin{bmatrix}
\frac{\partial \ln L(x; \theta)}{\partial p_1} \\
\frac{\partial \ln L(x; \theta)}{\partial p_2} \\
\frac{\partial \ln L(x; \theta)}{\partial p_3}
\end{bmatrix} = \begin{bmatrix}
\frac{n_1}{p_1} - n_4 \frac{1}{1 - p_1 - p_2 - p_3} \\
\frac{n_2}{p_2} - n_4 \frac{1}{1 - p_1 - p_2 - p_3} \\
\frac{n_3}{p_3} - n_4 \frac{1}{1 - p_1 - p_2 - p_3}
\end{bmatrix}
\]

To find MLE, solve $S(x, \theta) = 0$ for $\theta$; lots of complicated algebra will show that

$\hat{p}_j = \frac{n_j}{N}, j = 1, 2, 3$  \hspace{1cm} (4)

$I(\theta) = \begin{bmatrix}
I_{11} & I_{12} & I_{13} \\
I_{21} & I_{22} & I_{23} \\
I_{31} & I_{32} & I_{33}
\end{bmatrix}, \text{ where } I_{i,j} = -E \left[ \frac{\partial^2 \ln L(x; \theta)}{\partial \theta_i \partial \theta_j} \right]$

(I'm only doing this because of very liberal use of cut and paste from the midterm.)
\[I_{11} = -E \left[ \frac{\partial}{\partial p_1} \left( n_1 \frac{1}{p_1} - n_4 \frac{1}{1 - p_1 - p_2 - p_3} \right) \right] = -E \left[ -n_1 \frac{1}{p_1^2} - n_4 \frac{1}{(1 - p_1 - p_2 - p_3)^2} \right] = \]

\[\frac{1}{p_1^2} E(n_1) + \frac{1}{(1 - p_1 - p_2 - p_3)^2} E(n_4) = \]

\[\frac{1}{p_1^2} Np_1 + \frac{1}{(1 - p_1 - p_2 - p_3)^2} N(1 - p_1 - p_2 - p_3) = \frac{N}{p_1} + \frac{N}{(1 - p_1 - p_2 - p_3)^2} \]

\[I_{12} = -E \left[ \frac{\partial}{\partial p_2} \left( n_1 \frac{1}{p_1} - n_4 \frac{1}{1 - p_1 - p_2 - p_3} \right) \right] = -E \left[ -n_4 \frac{1}{(1 - p_1 - p_2 - p_3)^2} \right] = \]

\[\frac{1}{(1 - p_1 - p_2 - p_3)^2} E(n_4) = \frac{1}{(1 - p_1 - p_2 - p_3)^2} N(1 - p_1 - p_2 - p_3) = \]

\[
\frac{N}{(1 - p_1 - p_2 - p_3)^2} \]

(same as \(I_{12}\))

\[I_{13} = -E \left[ \frac{\partial}{\partial p_3} \left( n_1 \frac{1}{p_1} - n_4 \frac{1}{1 - p_1 - p_2 - p_3} \right) \right] = \frac{N}{(1 - p_1 - p_2 - p_3)^2} \]

\[I_{22} = -E \left[ \frac{\partial}{\partial p_2} \left( n_2 \frac{1}{p_2} - n_4 \frac{1}{1 - p_1 - p_2 - p_3} \right) \right] = -E \left[ -n_2 \frac{1}{p_2^2} - n_4 \frac{1}{(1 - p_1 - p_2 - p_3)^2} \right] = \]

\[\frac{1}{p_2^2} E(n_2) + \frac{1}{(1 - p_1 - p_2 - p_3)^2} E(n_4) = \]

\[\frac{1}{p_2^2} Np_2 + \frac{1}{(1 - p_1 - p_2 - p_3)^2} N(1 - p_1 - p_2 - p_3) = \frac{N}{p_2} + \frac{N}{(1 - p_1 - p_2 - p_3)^2} \]

\[I_{23} = -E \left[ \frac{\partial}{\partial p_3} \left( n_2 \frac{1}{p_2} - n_4 \frac{1}{1 - p_1 - p_2 - p_3} \right) \right] = -E \left[ -n_4 \frac{1}{(1 - p_1 - p_2 - p_3)^2} \right] = \]

\[\frac{1}{(1 - p_1 - p_2 - p_3)^2} E(n_4) = \frac{1}{(1 - p_1 - p_2 - p_3)^2} N(1 - p_1 - p_2 - p_3) = \]

\[
\frac{N}{(1 - p_1 - p_2 - p_3)^2} \]

\[I_{33} = -E \left[ \frac{\partial}{\partial p_3} \left( n_3 \frac{1}{p_3} - n_4 \frac{1}{1 - p_1 - p_2 - p_3} \right) \right] = -E \left[ -n_3 \frac{1}{p_3^2} - n_4 \frac{1}{(1 - p_1 - p_2 - p_3)^2} \right] = \]

\[\frac{1}{p_3^2} E(n_3) + \frac{1}{(1 - p_1 - p_2 - p_3)^2} E(n_4) = \]

\[\frac{1}{p_3^2} Np_3 + \frac{1}{(1 - p_1 - p_2 - p_3)^2} N(1 - p_1 - p_2 - p_3) = \frac{N}{p_3} + \frac{N}{(1 - p_1 - p_2 - p_3)^2} \]

(Wasn't that fun?)
\[
I(\theta) = \begin{bmatrix}
\frac{N}{p_1} \frac{N}{(1 - p_1 - p_2 - p_3)} & \frac{N}{(1 - p_1 - p_2 - p_3)} & \frac{N}{(1 - p_1 - p_2 - p_3)} \\
\frac{N}{(1 - p_1 - p_2 - p_3)} & \frac{N}{p_2} \frac{N}{(1 - p_1 - p_2 - p_3)} & \frac{N}{(1 - p_1 - p_2 - p_3)} \\
\frac{N}{(1 - p_1 - p_2 - p_3)} & \frac{N}{(1 - p_1 - p_2 - p_3)} & \frac{N}{p_3} \frac{N}{(1 - p_1 - p_2 - p_3)}
\end{bmatrix}
\]

\textbf{Critical Value} - estimating 3 parameters \((p_1, p_2, p_3)\); 95\% \(\chi^2\) = 7.814 \(\therefore\) reject H\(_0\) if test statistic exceeds 7.814.

\textbf{Summary of Data} - just to make things easier to find

\[
\begin{align*}
\theta_0 &= \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 9/16 \\ 3/16 \\ 3/16 \end{bmatrix} = \begin{bmatrix} 0.5625 \\ 0.1875 \\ 0.1875 \end{bmatrix} \\
\hat{\theta} &= \begin{bmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \end{bmatrix} = \begin{bmatrix} 315/556 \\ 108/556 \\ 101/556 \end{bmatrix} = \begin{bmatrix} 0.5665 \\ 0.1942 \\ 0.1817 \end{bmatrix}
\end{align*}
\]

\[
S(x; \theta_0) = \begin{bmatrix}
\frac{315}{1} & -32 & \frac{1}{0.5625} \\
\frac{108}{1} & -32 & \frac{1}{0.1875} \\
\frac{101}{1} & -32 & \frac{1}{0.1875}
\end{bmatrix} = \begin{bmatrix} 48 \\ 64 \\ 26.6667 \end{bmatrix}
\]

\[
I(\theta_0) = \begin{bmatrix}
\begin{bmatrix} 556 \\ 556 \\ 556 \end{bmatrix} + \begin{bmatrix} 556 \\ 556 \\ 556 \end{bmatrix} & \begin{bmatrix} 0.0625 \\ 0.0625 \\ 0.0625 \end{bmatrix} & \begin{bmatrix} 0.0625 \\ 0.0625 \\ 0.0625 \end{bmatrix} \\
\begin{bmatrix} 556 \\ 556 \\ 556 \end{bmatrix} & \begin{bmatrix} 0.1875 \\ 0.1875 \\ 0.1875 \end{bmatrix} & \begin{bmatrix} 0.1875 \\ 0.1875 \\ 0.1875 \end{bmatrix} \\
\begin{bmatrix} 0.0625 \\ 0.0625 \\ 0.0625 \end{bmatrix} & \begin{bmatrix} 0.1875 \\ 0.1875 \\ 0.1875 \end{bmatrix} & \begin{bmatrix} 0.1875 \\ 0.1875 \\ 0.1875 \end{bmatrix}
\end{bmatrix}
= \begin{bmatrix} 9884.44 \\ 8896 \\ 8896 \\
8896 & 11861.33 & 8896 \\
8896 & 8896 & 11861.33
\end{bmatrix}
\]

\[
I^{-1}(\theta_0) = \begin{bmatrix} 0.000443 & -0.00019 & -0.00019 \\
-0.00019 & 0.000274 & -0.000063 \\
-0.00019 & -0.000063 & 0.000274
\end{bmatrix}
\]
\[ \mathbf{I}(\hat{\theta}) = \begin{bmatrix} 556 & 556 & 556 \\ 0.5665 & 0.0576 & 0.0576 \end{bmatrix} + \begin{bmatrix} 556 & 556 & 556 \\ 0.0576 & 0.1942 & 0.0576 \end{bmatrix} = \begin{bmatrix} 10641.88 & 9660.5 & 9660.5 \\ 9660.5 & 12522.87 & 9660.5 \\ 9660.5 & 9660.5 & 12721.25 \end{bmatrix} \]

**Likelihood Ratio Test**

\[ \xi_{LR} = -2 \left( \ln L(x; \theta_0) - \ln L(x; \hat{\theta}) \right) = -2 \left( 315 \ln(0.5625) + 108 \ln(0.1875) + 101 \ln(0.1875) + 32 \ln(0.0625) - 315 \ln(0.5665) - 108 \ln(0.1942) - 101 \ln(0.1817) - 32 \ln(0.0576) \right) = 0.475 < 7.814 \therefore \text{don't reject } H_0. \] The data seems to agree with Mendelian theory.

**Lagrange Multiplier Test**

\[ \xi_{LM} = S^T (x; \theta_0) I^{-1}(\theta_0) S(x; \theta_0) = \begin{bmatrix} 0.000443 & -0.00019 & -0.00041 \end{bmatrix} \begin{bmatrix} 48 \\ 64 \\ 26.667 \end{bmatrix} = 0.470 < 7.814 : \text{don't reject } H_0. \] The data seems to agree with Mendelian theory.

**Wald Test**

\[ (\hat{\theta} - \theta_0) = \begin{bmatrix} 0.5665 \\ 0.1942 \\ 0.1817 \end{bmatrix} - \begin{bmatrix} 0.5625 \\ 0.1875 \\ 0.1875 \end{bmatrix} = \begin{bmatrix} 0.004 \\ 0.0067 \\ -0.0058 \end{bmatrix} \]

\[ \xi_W = (\hat{\theta} - \theta_0)^T \mathbf{I}(\hat{\theta})(\hat{\theta} - \theta_0) = \begin{bmatrix} 0.004 \\ 0.0067 \\ -0.0058 \end{bmatrix}^T \begin{bmatrix} 0.000443 & -0.00019 & -0.00041 \\ -0.00019 & 0.000274 & -0.000063 \\ -0.00041 & -0.000063 & 0.000274 \end{bmatrix} \begin{bmatrix} 0.004 \\ 0.0067 \\ -0.0058 \end{bmatrix} = 0.487 < 7.815 : \text{don't reject } H_0. \] The data seems to agree with Mendelian theory.

**Documentation.**

Prof Werner gave me the formula for \( F(\bar{x}; \theta) \) in 1e. I worked all of problem 1 with Josh Kneifel and Prof Werner... lovely arguments on the wording of the question and violent agreement about our calculations.