1. A firm has two identical plants and a fixed amount of labor. The firm wishes to maximize output from both plants.

Production: \( Y_i \leq f(l_i) \), \( l_i \geq 0 \), \( f(0) = 0 \), and \( f'(l_i) > 0 \)

Labor: \( l_1 + l_2 \leq \bar{L} \)

A. Set this up as a maximization problem.

B. Using Kuhn-Tucker conditions, characterize the way the firm will split its output between the two firms in each of the following cases:

(1) \( f''(l_i) \leq 0 \)
(2) \( f''(l_i) > 0 \)
(3) There exists an \( \bar{l} \) such that for

- \( l_i < \bar{l} \) \( f''(l_i) > 0 \)
- \( l_i = \bar{l} \) \( f''(l_i) = 0 \)
- \( l_i > \bar{l} \) \( f''(l_i) < 0 \)

(Part 3 is tricky and depends upon the relation between \( \bar{L} \) and \( l^* \) which is the point where \( f'(l^*) = \frac{f(l^*)}{l^*} \). Multiple solutions are possible.)

(A) Although \( Y_i \leq f(l_i) \), it’s clear that we would not want to produce less than capacity since the objective is to maximize output so we only need to consider \( Y_i = f(l_i) \).

\[
\begin{aligned}
\max_{l_1, l_2} & \quad f(l_1) + f(l_2) \\
\text{s.t.} & \quad l_1 + l_2 - \bar{L} \leq 0 : \lambda \\
& \quad l_1 \geq 0, \quad l_2 \geq 0
\end{aligned}
\]

(B) Lagrangian: \( L = f(l_1) + f(l_2) - \lambda(l_1 + l_2 - \bar{L}) \)

K-T Conditions:

(1a) \( l_1 > 0 \Rightarrow \frac{\partial L}{\partial l_1} = f'(l_1) = \lambda = 0 \)
(1b) \( l_1 = 0 \Rightarrow \frac{\partial L}{\partial l_1} = f'(l_1) = \lambda \leq 0 \)
(2a) \( l_2 > 0 \Rightarrow \frac{\partial L}{\partial l_2} = f'(l_2) = \lambda = 0 \)
(2b) \( l_2 = 0 \Rightarrow \frac{\partial L}{\partial l_2} = f'(l_2) = \lambda \leq 0 \)
(3a) \( \lambda > 0 \Rightarrow -\frac{\partial L}{\partial \lambda} = l_1 + l_2 - \bar{L} = 0 \)
(3b) \( \lambda = 0 \Rightarrow -\frac{\partial L}{\partial \lambda} = l_1 + l_2 - \bar{L} \leq 0 \)
In order to use the K-T conditions, we must have $L > 0$ to avoid having an empty constraint set. Given that labor is available, we can rule out having both $l_1$ and $l_2$ equal to 0 because an obvious feasible solution better than $l_1 = l_2 = 0$ would be $l_1 = L$ and $l_2 = 0$. Finally, assuming we can have fractional amounts of labor, the optimal solution would not waste labor, so the constraint will be binding (i.e., $\lambda > 0$). This reasoning limits the different cases to three: (1a, 2a, 3a), (1a, 2b, 3a), and (1b, 2a, 3a). Which case is generates the optimal solution depends on the properties of $f(l_i)$.

(1) $f'(l_i) > 0$ and $f''(l_i) \leq 0$ means we have an upward sloping concave function. It's not strictly concave so it can be linear (or have linear portions). The general graph of $f(l_i)$ will look like the ones shown below on the top. The objective function which adds $f(l_i)$ in two dimensions will also be concave. The graphs below show a general example for strictly concave and linear versions.

First consider the case where the objective is strictly concave. This objective function will also be quasiconcave because any concave function in also quasiconcave. The constraint is linear which is both quasiconcave and quasiconvex. Given a quasiconcave objective and quasiconvex constraint, a solution that satisfies the K-T conditions will be the optimal (i.e., global max). Therefore, for the strictly concave scenario, we only need to find one solution that satisfies the K-T conditions.

**Case 1:** (1a, 2a, 3a)

\[
\begin{align*}
f'(l_1) - \lambda &= 0 \\
f'(l_2) - \lambda &= 0 \\
l_1 + l_2 - L &= 0
\end{align*}
\]

From the first two conditions we have $f'(l_1) = f'(l_2)$. Since the function is increasing with decreasing slopes (i.e., no inflection points), we can say $f'(l_1) = f'(l_2) \Rightarrow l_1 = l_2$. Substitute that into the third equation and the solution is

$$l_1 = l_2 = \frac{L}{2}$$
Now check the inequalities. Since $\bar{L} > 0$, both $l_1$ and $l_2$ are $> 0$. For the third condition $\lambda = f'(l_1) = f'(l_2) > 0$ (property of $f'(l_i)$). Therefore, the K-T conditions hold and this is the optimal solution. We don’t have to check the other two cases.

This solution makes sense based on the strictly concave graph given earlier. The first derivative of the production function tells us how much more output will be generated at a plant from an additional unit of labor; that is $f'(l_i) = MPL$, the marginal product of labor. If this value is decreasing (i.e., $f''(l_i) < 0$), each additional unit of labor at a plant will generate less output; that’s called diminishing MPL. Therefore, it makes sense to alternate adding labor to each plant to keep the marginal product of labor equal between the two plants.

Note that the solution just described also holds for a linear production function. In fact, looking at the other conditions

**Case 2:** (1a, 2b, 3a)

\[
\begin{align*}
  f'(l_1) - \lambda &= 0 \\
  l_2 &= 0 \\
  l_1 + l_2 - \bar{L} &= 0 
\end{align*}
\]

From the first condition, $\lambda = f'(l_1) > 0$ (which satisfies the inequality of the third condition). Substituting the second condition into the third, $l_1 = \bar{L} > 0$ (which satisfies the inequality of the first condition). The solution is

\[
\begin{align*}
  l_1 &= \bar{L} \quad \text{and} \quad l_2 = 0 
\end{align*}
\]

The only inequality left to check is the second condition: $f'(l_2) - \lambda \leq 0$. Substitute $\lambda = f'(l_1)$ and this becomes $f'(l_2) - f'(l_1) \leq 0$, which is true because in the linear case $f'(l_2) = f'(l_1)$. (Note that the solution in this case would not satisfy the K-T conditions if the objective where strictly concave.)

**Case 3:** (1b, 2a, 3a)

\[
\begin{align*}
  l_1 &= 0 \\
  f'(l_2) - \lambda &= 0 \\
  l_1 + l_2 - \bar{L} &= 0 
\end{align*}
\]

This mirrors Case 2. The solution is

\[
\begin{align*}
  l_1 &= 0 \quad \text{and} \quad l_2 = \bar{L} 
\end{align*}
\]

As with Case 2, the solution satisfies the K-T conditions if the objective function is linear (but not if it is strictly concave).
There are actually an infinite number of solutions if the objective function is linear since any combination of \( l_1 \) and \( l_2 \) that adds up to \( L \) would maximize production. That is:

\[
\{ (l_1, l_2) : l_1 + l_2 = L \}
\]

This makes intuitive sense because a linear production function has constant returns to scale. In this case, the **marginal product of labor is the same at each plant regardless of how much labor is used.**

(2) \( f'(l_i) > 0 \) and \( f''(l_i) > 0 \) means we have an upward sloping strictly convex function as pictured below on the left. Since it's a monotonic function, \( f(l_i) \) is still quasiconcave, but when you add two of them together, the resulting function (i.e., the objective) is quasiconvex. This does not satisfy the requirements for having a global maximum.

This problem has the same lagrangian and K-T conditions as part B1, but we need to verify the inequalities.

**Case 1:** (1a, 2a, 3a)

\[
\begin{align*}
  f'(l_1) - \lambda &= 0 \\
  f'(l_2) - \lambda &= 0 \\
  l_1 + l_2 - L &= 0 
\end{align*}
\]

The solution to this was found in part B1: \( l_1 = l_2 = L/2 \)

Now check the inequalities. Since \( L > 0 \), both \( l_1 \) and \( l_2 \) are \( > 0 \). For the third condition \( \lambda = f'(l_1) = f'(l_2) > 0 \) (property of \( f'(l_i) \)). Therefore, the K-T conditions hold, however, this is **not an optimal solution** to this problem. Since \( f''(l_i) > 0 \), the solution could be improved by increasing \( l_1 \). The trade off involves decreasing \( l_2 \) by the same amount to keep the solution feasible. The cost to the objective function is less than the gain because \( f'(l_1) > f'(l_2) \) (this follows from the fact that \( l_1 > l_2 \) and \( f''(l_i) > 0 \)). This reasoning eventually takes us to a corner solution as Cases 2 and 3 show.

**Case 2:** (1a, 2b, 3a)

\[
  f'(l_1) - \lambda = 0
\]
\[ l_2 = 0 \\
l_1 + l_2 - \overline{L} = 0 \]

The solution to this was found in part B1. \[ l_1 = \overline{L} \text{ and } l_2 = 0 \]

Now check the inequalities. Since \( \overline{L} > 0 \), \( l_1 > 0 \). The second condition is \( f'(l_2) - \lambda \leq 0 \). Substitute \( \lambda = f'(l_1) \) and this becomes \( f'(l_2) - f'(l_1) \leq 0 \), which is true because \( l_1 > l_2 \) and \( f''(l_1) > 0 \). The third condition follows from the first: \( \lambda = f'(l_1) > 0 \). Therefore, the K-T conditions hold for this solution. This is an optimal solution. Intuitively, \( f''(l_1) > 0 \) means the marginal product of labor is increasing. Therefore, it makes sense to put all the labor into a single plant.

**Case 3: (1b, 2a, 3a)**

\[ l_1 = 0 \]

\[ f'(l_2) - \lambda = 0 \]

\[ l_1 + l_2 - \overline{L} = 0 \]

The solution to this was found in part B1. \[ l_1 = 0 \text{ and } l_2 = \overline{L} \]

This case mirrors Case 2 so it is also an optimal solution.

(3) This production function transitions from a strictly convex function (increasing MPL) to a strictly concave function (diminishing MPL) as pictured here. The graph on top plots a general function and the lower one looks at MPL and APL (average product of labor; \( f(l_i)/l_i \)). (Note, that the graphs don’t necessarily show the same function due to limited artistic ability.) There are several solutions for the problem depending on the labor capacity (\(\overline{L} \)) and the point at which MPL = APL (\( l^* \)).

Before going on we should consider why APL is important for this problem. Note that total output from a plant is equal to the APL at that plant times the labor used at that plant. This is one case where the math is actually more clear than the verbiage:

\[ f(l_i) = \text{APL} \cdot l_i = \left( \frac{f(l_i)}{l_i} \right) l_i \]

Therefore, when deciding between two possible combinations of output, we can simply look at the APL for each plant times the respective amounts of labor used at those plants.

Another thing to consider are the possible solutions to this problem. From the K-T conditions covered earlier there were three possible solutions: \((\overline{L}/2, \overline{L}/2), (\overline{L}, 0), \) and \((0, \overline{L})\). Note, however, that we came to the first solution by taking advantage
of there being no inflection points, something that is no longer true. The original condition was $f'(l_1) = f'(l_2)$. The solution of this form must satisfy the constraint so we can rewrite it as $f'(l_1) = f'(L - l_1)$. Note that we can always swap production from plant 1 and plant 2 so there are actually five different solutions, but ignoring the symmetry, there are only three basic solutions. Here is the total amount produced in each possible solution:

<table>
<thead>
<tr>
<th>Solution Point</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(L,0)$</td>
<td>$(L,0)$</td>
</tr>
<tr>
<td>$(L/2,L/2)$</td>
<td>$(L/2,L/2)$</td>
</tr>
<tr>
<td>$(x,L-x)$</td>
<td>$(x,L-x)$</td>
</tr>
</tbody>
</table>

Output at $(L/2,L/2)$ comes from $f(L/2)/L + f(L/2)/L = f(L/2)/L$.

Not that to compare the case of one plant producing everything or two plants equally sharing production, all we need to do is compare the APL in each case. The case were two plants split production unevenly cannot be answered without specific information about the production function.

**Case 1:** $L \geq 2l^*$

$L \geq 2l^* \Rightarrow L/2 \geq l^*$

That means $L$ and $L/2$ are to the right of the maximum point of APL. Therefore, APL at $L/2$ exceeds APL at $L$ and the solution with both plants producing the same amount is best. This solution is also better than the third option because APL at $x$ and APL at $(L-x)$ will both be less than APL at $L/2$. (Stare at the graph long enough and this should be clear.) Solution:

$l_1 = l_2 = L/2$

**Case 2:** $L \leq l^*$

$L \leq l^* \Rightarrow L/2 \leq l^*$

That means $L$ and $L/2$ are to the left of the maximum point of APL. Therefore, APL at $L$ exceeds APL at $L/2$ and the solution with a single plant producing is best. The third solution is not feasible in this case because it is not possible to have $f'(x) = f'(L - x)$ when capacity $(L)$ is below the maximum APL. Solution:
Case 3: \( l^* < \overline{L} < 2l^* \)

\( l^* < \overline{L} < 2l^* \Rightarrow \overline{L}/2 < l^* \)

That means \( \overline{L} \) and \( \overline{L}/2 \) are on different sides of the maximum point of APL. It is not possible to tell if APL at \( \overline{L} \) exceeds APL at \( \overline{L}/2 \) or vice versa. What is evident (based on \( \overline{L} \) and \( \overline{L}/2 \) being on opposite sides of \( l^* \)) is that there exists an \( \overline{L} \) such that APL at \( \overline{L} \) equals APL at \( \overline{L}/2 \) (see graphs below). At this point, the firm should be indifferent between splitting production equally (\( \overline{L}/2 \)) between the plants or consolidating it all (\( \overline{L} \)) in a single plant. If capacity goes beyond this level, it would be better to split production. If \( \overline{L} \) is lower, the better option is to split production. The only problem is the pesky \((x, \overline{L} - x)\) solution. This cannot be compared without more information about \( f'(i) \). Even if splitting production this way is better, the basic finding of this example still holds:

Either produce all at one firm or split production so that MPL is equal between the firms.

\[
\begin{align*}
Q & \quad \text{MPL} & \quad \text{APL} \\
\frac{L}{2} & \quad l^* & \quad \overline{L} \\
\end{align*}
\]

\[
\begin{align*}
Q & \quad \text{MPL} & \quad \text{APL} \\
\frac{L}{2} & \quad l^* & \quad \overline{L} \\
\end{align*}
\]

2. max \( x^{1/3}y^{2/3} \)

\[
\begin{align*}
\text{s.t.} & \quad 8x + 4y - 12 \leq 0 \\
& \quad x^2 + 2y^2 - 6 \leq 0 \\
& \quad x \geq 0, \quad y \geq 0
\end{align*}
\]

The first constraint is linear so it’s fine (quasiconvex). The second constraint is the inside of an oval which is also quasiconvex, but just to be sure, look at the determinant of the bordered hessian:

\[
\begin{vmatrix}
0 & 2x & 4y \\
2x & 2 & 0 \\
4y & 0 & 4
\end{vmatrix} = -32y^2 - 32x^2 < 0 \quad \therefore \text{quasiconvex}
\]

There is no need to check the objective function because it is a Cobb-Douglas function which is quasiconcave. The graph above shows that the feasible region

\[
\begin{align*}
x & = \frac{2}{3} & \lambda_1 = 0.0716 \\
y & = \frac{5}{3} & \lambda_2 = 0.0307 \\
x^{1/3}y^{2/3} & = 1.228
\end{align*}
\]

(work shown below)
is not empty, therefore there will only be one solution to the K-T conditions and it will be the global maximum.

Lagrangian: \( L = x^{1/3} y^{2/3} - \lambda_1 (8x + 4y - 12) - \lambda_2 (x^2 + 2y^2 - 6) \)

K-T Conditions:

1. \( x > 0 \Rightarrow \frac{\partial L}{\partial x} = 1/3x^{2/3}y^{2/3} - 8\lambda_1 - 2\lambda_2 x = 0 \)
2. \( x = 0 \Rightarrow \frac{\partial L}{\partial x} = 1/3x^{2/3}y^{2/3} - 8\lambda_1 - 2\lambda_2 x \leq 0 \)
3. \( y > 0 \Rightarrow \frac{\partial L}{\partial y} = 2/3x^{1/3}y^{1/3} - 4\lambda_1 - 4\lambda_2 y = 0 \)
4. \( y = 0 \Rightarrow \frac{\partial L}{\partial y} = 2/3x^{1/3}y^{1/3} - 4\lambda_1 - 4\lambda_2 y \leq 0 \)

As with the sample problem in class, it’s clear that we do not need to worry about the cases where \( x = 0 \) or \( y = 0 \). Also, it’s evident that at least one of the constraints must hold so we’re limited to only checking three cases: (1a, 2a, 3a, 4a), (1a, 2a, 3a, 4b), and (1a, 2a, 3b, 4a).

Case 1: (1a, 2a, 3a, 4a)

\[
\begin{align*}
\frac{1}{3}x^{2/3}y^{2/3} - 8\lambda_1 - 2\lambda_2 x &= 0 \\
2/3x^{1/3}y^{1/3} - 4\lambda_1 - 4\lambda_2 y &= 0 \\
8x + 4y - 12 &= 0 \\
x^2 + 2y^2 &= 6 = 0
\end{align*}
\]

From the third condition: \( y = 3 - 2x \)

Substitute that into the fourth condition:

\[
\begin{align*}
x^2 + 2(3 - 2x)^2 &= 6 = 0 \\
x^2 + 2(9 - 12x + 4x^2) - 6 &= 0 \\
9x^2 - 24x + 12 &= 0 \\
3x^2 - 8x + 4 &= 0 \\
(3x - 2)(x - 2) &= 0 \\
x &= 2/3 \text{ or } 2
\end{align*}
\]

Put that back into the third condition: \( y = 3 - 2x = 3 - 2(2/3) = 5/3 \) or -1

Can’t have a negative value for \( y \) so the solution is: \( x = 2/3, y = 5/3 \)

Cancel out \( \lambda_1 \) by subtracting 2 times the second condition from the first:

\[
\begin{align*}
\frac{1}{3}x^{2/3}y^{2/3} - 8\lambda_1 - 2\lambda_2 x &= 0 \\
\frac{-4}{3}x^{1/3}y^{1/3} + 8\lambda_1 + 8\lambda_2 y &= 0 \\
\frac{1}{3}x^{2/3}y^{2/3} - \frac{-4}{3}x^{1/3}y^{1/3} - 2\lambda_2 x + 8\lambda_2 y &= 0
\end{align*}
\]
Plug in $x$ and $y$ and solve for $\lambda_2$

$$\frac{1}{3}(2/3)^{2/3}(5/3)^{2/3} - 4/3(2/3)^{1/3}(5/3)^{1/3} - 2\lambda_2(2/3) + 8\lambda_2(5/3) = 0$$

$$\lambda_2 = 0.0307 \text{ (calculator)}$$

Plug that into either of the other formulas to get $\lambda_1 = 0.0716 \text{ (calculator)}$

All four inequalities are satisfied (i.e., all values are $> 0$), so this solution satisfies the K-T conditions and as stated earlier is the optimal solution for this problem.

**Documentation**

Prof Slutsky clarified the notation in problem 1 in class. Specifically, he pointed out that the last two conditions for the production function were properties of the function, not constraints in the problem. He also flat out told us the objective was to maximize the sum of the production from each plant. On part 3 of problem 1, Prof Slutsky told me my interpretation of MPL and APL was correct and I was on the right track by looking at setting the first derivatives equal. He gave more specific guidance on how to approach the problem during his office hours.