Consumer Theory - Expenditure Function & Compensated Demand

**Expenditure Function** - $E(P, u) \equiv \min P \cdot x \text{ st } U(x) \geq u \text{ and } x \geq 0$; optimized value function of the dual to the utility maximization problem (i.e., trying to minimize what consumer would have to spend at given prices in order to achieve a specific value of utility)

**Isoexpenditure Lines** - level curves for expenditure; show various combinations of consumption that cost same amount of income

**First Order Condition** - same as original problem (Max $U$): slopes of indifference curve and budget line must be equal (unless on corner solution)

**Utility Consumption Curve** - connects all solution points for various levels of utility; will be identical to the income consumption curve described under ordinary demands

**Duality** - solving the Max $U$ and Min $P \cdot x$ leads to same result with slight technical difference:

- **Ordinary Demand** - solution to Max $U$ problem; function of $P$ and $I$... $x^O(P, I)$
- **Compensated Demand** - solution to Min $P \cdot x$ problem; uses $\Delta I$ to "compensate" for $\Delta P_j$ in order to maintain same level of utility; function of $P$ and $u$... $x^C(P, u)$

**Demands are Related** - these are important identities; first one plugs expenditure function (an income value) into the ordinary demand function which yields the same result as the compensated demand; it's kind of simply and silly: start with a target utility $u$; $E(P, u)$ tells how much income $I$ is required to attain that level of utility; by plugging this income into ordinary demand, we're saying, given $I$, maximize utility; because we set the problem up this way, the answer should be $u$; the point of doing this is to make it easier to generate properties of ordinary demand (like the Slutsky equation)

$$x^C(P, E(P, u)) \equiv x^O(P, u) \quad \text{and} \quad x^C(P, V(P, I)) \equiv x^O(P, I)$$

**Expenditure and Comp Demand Related** - $x^C_i(P, u) = \frac{\partial E(P, u)}{\partial P_i}$ and $E(P, u) \equiv P \cdot x^C(P, u)$

**Expenditure and Indirect Utility** - $E(P, u)$ and $V(P, I)$ usually aren't known so they're not good for empirical work, but they're good for theory (e.g., generate Slutsky equation); they're inverses of each other: start with $E(P, u) = u$, plug in $V(P, I)$ for $u$ and then solve for $V$ (don't need to solve two optimization problems): $E(P, V(P, I)) \equiv I$ and $V(P, E(P, u)) \equiv I$

**"Easy" Properties** - generating Slutsky Equation

Recall from Roy's Formula that $\partial V / \partial P_i = -\lambda x_i^O$ and $\partial V / \partial I = \lambda$; these will come in handy in a bit

Totally differentiate the identity for $x^C$ wrt $I$; note the chain rule for the second argument, but realize our assumption that $P$ and $I$ are independent allows us to ignore the first term; also note that $V(P, I) = u$ so rather than write $\partial V$ in the first term, we use $\partial u$ (this will be more clear later because we're dealing with $x^C(P, u)$)
\[ \frac{\partial x^c}{\partial u} \cdot \frac{\partial V}{\partial l} = \frac{\partial x^0}{\partial l} \Rightarrow \frac{\partial x^c}{\partial u} = \frac{\partial x^0}{\partial l} \]

Take the identity for \( x^c \) and totally differentiate wrt \( P_j \); note that we need to use the chain rule on the left side because there are two arguments

\[ \frac{\partial x^c}{\partial P_j} + \frac{\partial x^0}{\partial P_j} \cdot \frac{\partial V}{\partial l} = \frac{\partial x^c}{\partial P_j} \Rightarrow \frac{\partial x^c}{\partial P_j} = \frac{\partial x^0}{\partial P_j} \]

That's the Slutsky equation; Note that \( x^c(P,u) \) has constant utility so when price changes, I has to change as well... that is

\[ \frac{\partial x^c}{\partial P_j} = \frac{\partial x^0}{\partial P_j} \( U(x) = u \) = S_{ij} \]

Try it the other way - Min \( P \cdot x \) st \( U(x) \geq u \) and \( x \geq 0 \Rightarrow \) lagrangian \( L = P \cdot x^c - \beta U(x) - u \); at optimal solution \( E(P,u) = L \); we can use the Envelope Theorem to find \( \partial E/\partial P_j = x^c_j \); note also that at the optimal solution \( x^c_j = x^0_j \) (functions are different, but values at optimum are the same)

Now take the identity for \( x^0 \) and totally differentiate wrt \( P_j \); note the same trick of using \( \partial l \) in place of \( \partial u \) (before we did \( \partial u \) place of \( \partial V \))

\[ \frac{\partial x^0}{\partial P_j} + \frac{\partial x^c}{\partial P_j} \cdot \frac{\partial E}{\partial l} = \frac{\partial x^0}{\partial P_j} \Rightarrow \frac{\partial x^0}{\partial P_j} = \frac{\partial x^c}{\partial P_j} \frac{\partial E}{\partial l} = \frac{\partial x^c}{\partial P_j} \]

(Slutsky Equation)

**Properties of Expenditure Function**

1. **Complete** - \( E(P,u) \) defined for all \( P > 0 \) and \( u \)
2. **Continuous** - \( E(P,u) \) continuous in \( P \) and \( u \) (even if compensating demands aren't) \( E(P,I) = P \cdot x^c(P,u); x^c(P,u) \) may not be a function, but those places still have the same income to achieve the same \( u \) so \( E \) is continuous
3. **Homogeneous of Degree 1 in Prices** - \( E(tP',u) = tE(P,u) \)
4. **Monotonic** - increasing in \( u \) and non-decreasing in \( P_j \)

\[ \frac{\partial E}{\partial u} = -\beta > 0; \quad \frac{\partial E}{\partial P_j} = x_j \geq 0 \]

5. **Concave in \( P \)**

   **Proof:**

   Want to show: \( E(tP'+(1-t)P'',u) \geq tE(P',u) + (1-t)E(P'',u) \) (definition of concave)

   \[ E(tP'+(1-t)P'',u) = (tP'+(1-t)P'') \cdot x^c(tP'+(1-t)P'',u) \] (definition of \( E \))

   \[ = tP' \cdot x^c(tP'+(1-t)P'',u) + (1-t)P'' \cdot x^c(tP'+(1-t)P'',u) \] (linear in \( P' \) & \( P'' \))

   \[ E(P',u) = P' \cdot x^c(P'+(1-t)P'',u) \text{ and } E(P'',u) = P'' \cdot x^c(tP'+(1-t)P'',u) \text{ because } E \text{ is the minimum of these two optimization problems (Min } P' \cdot x \text{ and Min } P'' \cdot x) \text{; the compensated demands } (x^c(tP'+(1-t)P'',u) \text{ are feasible in these problems)} \]

   \[ E(tP'+(1-t)P'',u) = (tP'+(1-t)P'') \cdot x^c(tP'+(1-t)P'',u) \]

6. **Differentiable** - if \( U \) is strictly quasiconcave (or preferences strictly convex)
Properties of Compensated Demand

1. **Complete** - \( x^c(P, u) \) defined for all \( P > 0 \) and \( u \)

2. "Sort of" Continuous - \( x^c(P, u) \) continuous in \( P \) and \( u \) (like ordinary demand, compensated demand may not be a function so there may be multiple optimal solutions (many \( x^c \)) but it will always be a convex set)

3. **Homogeneous of Degree 0 in Prices** - \( x^c(tP, u) = x^c(P, u) \) (tangency point doesn't change)

4. **Nonnegativity** - \( x^c(P, u) \geq 0 \); note that adding up property from ordinary demand is not useful here: \( U(x^c(P, u)) = u \) ... can't do much with that

5. **Negative Semidefinite** - substitution matrix (all derivatives of \( x^c(P, u) \) wrt \( P_j \)) is symmetric and negative semidefinite

Proof:

Substitution Matrix = Slutsky Matrix (latter is measurable)

\[
\begin{bmatrix}
\frac{\partial x_1^c}{\partial P_1} & \cdots & \frac{\partial x_1^c}{\partial P_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial x_n^c}{\partial P_1} & \cdots & \frac{\partial x_n^c}{\partial P_n}
\end{bmatrix}
= \begin{bmatrix}
S_{11} & \cdots & S_{1n} \\
\vdots & \ddots & \vdots \\
S_{n1} & \cdots & S_{nn}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial x_1^o}{\partial P_1} + x_1^o \frac{\partial x_1^o}{\partial I} & \cdots & \frac{\partial x_n^o}{\partial P_1} + x_n^o \frac{\partial x_n^o}{\partial I} \\
\vdots & \ddots & \vdots \\
\frac{\partial x_1^o}{\partial P_n} + x_1^o \frac{\partial x_1^o}{\partial I} & \cdots & \frac{\partial x_n^o}{\partial P_n} + x_n^o \frac{\partial x_n^o}{\partial I}
\end{bmatrix}
\]

Earlier we used the Envelope Theorem on \( E(P, u) \) to find \( \frac{\partial E}{\partial P_j} = x^c_j \)

Take total derivative wrt \( P_k \):

\[
\frac{\partial^2 E}{\partial P_j \partial P_k} = \frac{\partial x_j^c}{\partial P_k}
\]

\( \therefore \) substitution matrix is hessian of \( E(P, u) \) which we saw earlier was convex so it has to be negative semidefinite

Also, by Young's Theorem, the hessian is symmetric

**Results** -

a. \( \frac{\partial x_i^c}{\partial P_i} \leq 0 \); own effects are negative (we also proved this with comparative statics)

b. \( \frac{\partial x_i^c}{\partial P_j} = \frac{\partial x_j^c}{\partial P_i} \); symmetric (cross effects are the same)

b. \( 2 \times 2 \) matrix:

\[
\frac{\partial x_1^c}{\partial P_1} \cdot \frac{\partial x_2^c}{\partial P_2} - \left( \frac{\partial x_1^c}{\partial P_2} \right)^2 \geq 0 \); own effects outweigh the cross effect

**Finishing Ordinary Demand**

6. Slutsky Matrix is symmetric and negative semidefinite

**Cobb-Douglas** - specific type of utility function: \( U(x_1, x_2) = x_1^\alpha x_2^\beta \)

**Fraction of Income** -

\[
\frac{P_1 x_1}{I} = \frac{\alpha}{\alpha + \beta} \quad \text{and} \quad \frac{P_2 x_2}{I} = \frac{\beta}{\alpha + \beta}; \text{fraction of income spent on good } i
\]

is same regardless of level of utility (not the same between goods unless \( \alpha = \beta \))
Example

$U = x_1 x_2$ (same one from indirect utility notes)

Max $x_1 x_2$ s.t. $P_1 x_1 + P_2 x_2 \leq I$ or Min $P_1 x_1 + P_2 x_2$ s.t. $x_1 x_2 \geq u$ (solve min problem)

**Lagrangian** - $L = P_1 x_1 + P_2 x_2 - \beta (x_1 x_2 - u)$

Not worried about corner solutions if $u > 0$ so we can ignore them

**K-T Conditions** - $\frac{\partial L}{\partial x_1} = P_1 - \beta x_2 = 0$, $\frac{\partial L}{\partial x_2} = P_2 - \beta x_1 = 0$, and $-\frac{\partial L}{\partial \beta} = x_1 x_2 - u = 0$

**Solution** - From first two equations: $2 \Rightarrow P_1 x_1 = P_2 x_2 \Rightarrow x_2 = \frac{P_1 x_1}{P_2}$

**Notes**: spend the same amount on both goods; $\beta$ is marginal expenditure of utils (how much you have to spend to improve $u$); $\beta$ is inverse of shadow price form maximization problem

Substitute $x_2$ into last equation: $x_1 = \frac{P_2}{P_1} u = P_1^{1/2} P_2^{1/2} u^{1/2}$

$x_2 = P_1 P_2^{-1} (P_1^{1/2} P_2^{1/2} u^{1/2}) = P_1^{1/2} P_2^{-1/2} u^{1/2}$

**Ordinary Demands** - from indirect utility notes $x_1^0 = \frac{I}{2 P_1}$ & $x_2^0 = \frac{I}{2 P_2}$

**Gross Independent** - $\frac{\partial x_1^0}{\partial P_j} = 0$

**Hixian Substitutes** - $\frac{\partial x_1^c}{\partial P_j} = \frac{\partial x_2^c}{\partial P_j} = \frac{1}{2} P_1^{-1/2} P_2^{-1/2} u^{1/2} > 0$

**Check Compensated Demand Properties** -

1. Defined $\forall P > 0$ and $u... yes (if $u > 0$)
2. "Sort of" continuous... yes
3. Homogeneous... $x_1(tP, u) = (t P_1)^{1/2} (t P_2)^{1/2} u^{1/2} = P_1^{-1/2} P_2^{1/2} u^{1/2} = x_1(P, u)... yes$
4. Nonnegativity... yes
5. Substitution matrix is negative semidefinite... testing this is same as testing if $E(tP, u)$ is concave which we'll do in a minute.

**Expenditure Function** -

$E(tP, u) = P \cdot x^c = P_1 x_1^c + P_2 x_2^c = P_1 (P_1^{-1/2} P_2^{1/2} u^{1/2}) + P_2 (P_1^{1/2} P_2^{-1/2} u^{1/2}) = 2 P_1^{1/2} P_2^{1/2} u^{1/2}$

**Check Expenditure Function Properties** -

1. Defined $\forall P > 0$ and $u... yes (if $u > 0$)
2. Continuous in $P$ and $u... yes because sqrt is continuous function
3. Homogeneous... $E(tP, u) = 2(tP_1)^{1/2} (tP_2)^{1/2} u^{1/2} = t 2 P_1^{1/2} P_2^{1/2} u^{1/2} = tE(P, u)... yes$
4. Increasing in $u... \frac{\partial E(tP, u)}{\partial u} = P_1^{1/2} P_2^{-1/2} u^{-1/2} > 0... yes$
5. Non-decreasing in $P_j... \frac{\partial E(tP, u)}{\partial P_j} = P_1^{-1/2} P_2^{1/2} u^{1/2} \geq 0... yes$
6. Concave in $P... show hessian is negative semidefinite... yes (see below)

To make it easier, use $\frac{\partial^2 E}{\partial P_j \partial P_k} = \frac{\partial x_j^c}{\partial P_k}$

$E_{11} = \frac{\partial x_1^c}{\partial P_1} = -\frac{1}{2} P_1^{-3/2} P_2^{1/2} u^{1/2} < 0$

$E_{12} = \frac{\partial x_1^c}{\partial P_2} = -\frac{1}{2} P_1^{-1/2} P_2^{-3/2} u^{1/2}$

$E_{22} = \frac{\partial x_2^c}{\partial P_2} = -\frac{1}{2} P_1^{1/2} P_2^{-3/2} u^{1/2} < 0$
\[ |H_x| = \begin{vmatrix} -\frac{1}{2} P_1^{-3/2} P_2^{1/2} u^{1/2} & -\frac{1}{2} P_1^{-1/2} P_2^{-1/2} u^{1/2} \\ -\frac{1}{2} P_1^{-1/2} P_2^{-1/2} u^{1/2} & -\frac{1}{2} P_1^{1/2} P_2^{-3/2} u^{1/2} \end{vmatrix} = \frac{1}{4} P_1^{-1} P_2^{-1} u = -\frac{1}{4} P_1^{-1} P_2^{-1} u = 0 \] (this happens because of homogeneity); this is OK because negative semidefinite requires this to be \( \geq 0 \); also need diagonal elements to be \( \leq 0 \)...

### Solving for \( V \) and \( x^0 \)

\[ E(P_1, P_2, V(P, I)) = I = 2 P_1^{1/2} P_2^{1/2} [V(P, I)]^{1/2} \Rightarrow V(P, I) = \frac{I^2}{4 P_1 P_2} \] ... same as we had before

\[ x_i^c(P_1, P_2, V(P, I)) = x_i^0 \]

\[ x_1^0 = P_1^{-1/2} P_2^{1/2} \left( \frac{I^2}{4 P_1 P_2} \right)^{1/2} = \frac{I}{2 P_1} \quad \text{and} \quad x_2^0 = P_1^{1/2} P_2^{-1/2} \left( \frac{I^2}{4 P_1 P_2} \right)^{1/2} = \frac{I}{2 P_2} \] ... same as we had before

### Back to \( U \)
- can now get from any representation of a standard consumer to another, except back to utility representation; we can related compensated demands to utility by applying the envelope theorem to the expenditure function:

\[ \frac{\partial E(P, u)}{\partial P_i} = x_i^c, \quad i = 1 \ldots n \]

This gives us \( n \) equations, but have homogeneity of degree 0 for compensated demand so if we know \( n - 1 \) prices, we know the last price; if we divide all prices by \( P_n \), we get:

\[ \frac{\partial E(P_1, P_1, P_1, \ldots, P_{n-1}, P_{n-1})}{\partial P_i} = x_i^c \]

This is now \( n \) equations and \( n - 1 \) unknowns (price ratios); the last equation relates \( u \) to \( x_i^c \)

### Example
- using same function we've been working with:

\[ x_1^c(P_1, P_2, u) = P_1^{-1/2} P_2^{1/2} u^{1/2} \quad \& \quad x_2^c(P_1, P_2, u) = P_1^{1/2} P_2^{-1/2} u^{1/2} \]

\[ x_1^c(P_1 / P_2, 1, u) = (P_1 / P_2)^{-1/2} (1) u^{1/2} \quad \& \quad x_2^c(P_1 / P_2, 1, u) = (P_1 / P_2)^{1/2} (1) u^{1/2} \]

From the \( x_1^c \) equation, \( (P_1 / P_2)^{1/2} = u^{1/2} / x_1^c \); now substitute that into the \( x_2^c \) equation and solve for \( u \)

\[ x_2^c = \left( u^{1/2} / x_1^c \right) u^{1/2} \Rightarrow u = x_1 x_2 \]

### Back and Forth
- can get to just about anything form just about anything else (e.g., find \( U(x) \) from \( E(P, u) \) or find \( x^0(P, I) \) from \( V(P, I) \)... all the relationships are shown in the summary page

### Integrability Problem
- if the Slutsky matrix is symmetric, you can (in theory) get back to \( U(x) \) from \( x^0 \) by system of partial differential equations

### Other Topics
- to be covered next: revealed preference, non-fixed income (income as function of price), aggregation across individuals, additional assumptions (e.g., additivity: \( U(x) + H(y) \)... used to separate commodities), consumer surplus