Utility Representation

Utility Representation, $U(x)$ - use function to represent preferences so we can use optimization for proofs; function contains information about preferences but function itself ("utils") mean nothing

Formally - function $U(x)$ represents preferences $R$ if $U(x) \geq U(y) \iff x R y$

Ordinal - rankings matter; ordering based on function must be the same as with preferences

Not Cardinal - magnitudes (values) don't matter; e.g., $U(x) = 2U(y)$ means nothing other than $U(x) > U(y)$; also $U(x) - U(y)$ means nothing other than $>0$ or $<0$ for ranking

Infinite Number of Functions - if we find one function that works there will be an infinite number of functions that work because of transformations

Utility Transformation - domain is $R^n$ (e.g., $x = (x_1, x_2,..., x_n)$), range is a number; transformation is another function $V(x) = F(U(x))$ that does something to the utility number (not $x$)

Examples - $V(x) = aU(x) + b$ (linear); $V(x) = [U(x)]^2$; $V(x) = [U(x)]^{1/2}$; $V(x) = \ln U(x)$

Increasing Transformation - example of type of transformation that is valid for utility representations because it preserves the rankings; two definitions (use either):

1) $U(x) > U(y) \Rightarrow F(U(x)) > F(U(y))$

2) $dF/dU > 0$ (but need to make sure $F(U(x))$ exists)

Examples - linear transformation valid if $a > 0$; other three transformations listed above valid only if $U(x) > 0$

Valid Properties - only properties that can be obtained from utility representation are those that hold for all utility representations (i.e., invariant or ordinal); these are properties that rely on the preference relation $R$ (i.e., rankings); also called ordinal property

First Derivative - $\frac{dU}{dx_i} > 0$... look at $V(x) = F(U(x))$: $\frac{dV}{dx_i} = \frac{dF}{dU} \cdot \frac{dU}{dx_i} > 0$ for all increasing transformations $V(x)$ ($dF/dU > 0$) : any property based on first derivative is valid

Second Derivative - $\frac{d^2U}{dx_i dx_j} > 0$... look at $V(x) = F(U(x))$

$$\frac{d^2V}{dx_i dx_j} = \frac{d}{dx_j} \left( \frac{dF(U)}{dU} \frac{dU}{dx_i} \right) = \frac{dU}{dx_i} \left( \frac{d^2F(U)}{dU^2} \frac{dU}{dx_j} + \frac{dF(U)}{dU} \frac{d^2U}{dx_i dx_j} \right);$$

know the second term is positive for all increasing transformations, but don't know anything about the first term; could be $>0$ or $<0$ : any property based on second derivative is not valid

Ratio of Transformations - this is valid; utility transformation doesn't change ratio of first derivatives (i.e., slopes of indifference curves)

$$\frac{dV}{dx_i} / \frac{dV}{dx_j} = \frac{dU}{dx_i} / \frac{dU}{dx_j} = \frac{dU}{dx_i} / \frac{dU}{dx_j}$$

Concavity - if $U$ is concave, that means level curves are increasing at a decreasing rate, but we only care that they are increasing; rate doesn't matter; can have a transformation that keeps level curves increasing, but has them at an increasing rate (i.e., convex); can
check hessians; if $H_U$ is negative definite, don't have to have $H_V$ be negative definite:. concavity is not valid

**Quasiconcavity** - concavity shows the relationship between level curves (i.e., how to label them), but quasiconcavity shows the shape of the level curves (i.e., set of points above the level curve is convex); so QC only depends on preferences so properties based on QC are valid (proved this the long way in HW)

$U(x) \geq U(y) \Rightarrow U(\lambda x + (1 - \lambda)y) \geq U(y)$ i.e., $x \sim y \Rightarrow (\lambda x + (1 - \lambda)y) \sim y$

### Indifference Curves

**Indifference Map** - set of indifference curves

**Indifference Curve** - $IC(x) \equiv \{y : y I x\}$; useful to illustrate arguments and results

- **Definition Using Preferences** - $IC(x) =$ boundary of $R^2(x)$
- **Definition Using Utility Representation** - $IC(x) = \{x : U(x) =$ constant$\}$

**Properties** - to be equivalent to preferences with the five properties must have: an IC goes through every bundle and ICs are non-intersecting, downward sloping, continuous, "thin" lines that are convex to the origin

**Proof** (of non-intersecting):

Assume two ICs intersect as shown in picture ∴ $x I y$ and $x I z$

By transitivity of preferences $y I z$

But by monotonicity of preferences $y P z$ because $y$ has more of both goods ∴ IC cannot intersect

### Ordinary Demands

**Budget Set, $B(P,I)$** - specific class of alternative set $A$: $B(P,I) \equiv \{x : P \cdot x \leq I$ and $x \geq 0\}$

$P_i x_i =$ amount spent on good $i$; $I =$ income; $P \cdot x = P_1 x_1 + P_2 x_2 + \ldots + P_n x_n = \sum_{i=1}^{n} P_i x_i$

**Properties** -

- **Convex** - $x \in B(P,I)$ and $y \in B(P,I) \Rightarrow \lambda x + (1 - \lambda)y \in B(P,I)$
- **Closed** - boundaries are included because of $\leq$ and $\geq$ in definition
- **Bounded** - if $I < \infty$ and $P > 0$ (or demand < supply at $P = 0$)

**Demand Function, $x^o(P,I)$** - $x^o(P,I) = C(B(P,I),R)$; it’s a function if preferences are strictly convex (i.e., $C(B(P,I),R)$ is unique bundle), otherwise it’s called a demand correspondence because there can be multiple bundles ($x$) for each price

**Observable** - we care about demand because it can be observed (unlike preferences)

**Properties** - 5 properties make demand function equivalent to "standard consumer"

1. **Complete** - $x^o(P,I)$ defined for all $P > 0$ and $0 < I < \infty$

   Assuming $I$ is fixed (for now) and $I$ independent of $P$

2. **"Sort of" Continuous** - if preferences ($R$) are strictly convex, then $x^o(P,I)$ is continuous

Small change in budget line from $\Delta P$, w/ strictly convex pref. doesn’t change optimal solution much ∴ $D(P,I)$ is continuous
3. **Adding Up Property** - if preferences (R) are strictly convex, then $P \cdot x^o(P,I) = I$

**Implication** - demands not unrelated; if you know first $n - 1$ demands, the last demand is given by solving formula above for $x^o_n(P,I)$:

$$\sum_{j=1}^{n} P_j x^o_j(P,I) = I \Rightarrow x^o_n(P,I) = \frac{1}{P_n} \left( I - \sum_{j=1}^{n-1} P_j x^o_j(P,I) \right)$$

**Second Implication** - limits on demand: $0 \leq x^o_j(P,I) \leq I/P_j$

4. **Homogeneous of Degree 0 in Prices and Income** - proportional changes in prices and income have no effect on demand; $x^o(iP,tl) = x^o(P,I)$

**Proof:**

$x \in B(P,I) \Rightarrow P \cdot x \leq I; x \in B(t(P,tl)) \Rightarrow tP \cdot x \leq tl$

$P \cdot x \leq I \Rightarrow tP \cdot x \leq tl \Rightarrow B(P,I) = B(t(P,tl))$

5. **Convex** - if preferences (R) are convex, then $x^o(P,I)$ is convex set; if R is strictly convex, then $x^o(P,I)$ is single bundle

**Other Properties** - there are others (e.g., downward sloping); come from comparative statics (hard way) or through duality in optimization

**Comparative Statics** - start with choice set based on preferences: $C(B(P,I),R)$ which can be written as optimization of utility representation: Max $U(x)$ s.t., $P \cdot x \leq I$ and $x \geq 0$

**Lagrangian** - $L = U(x) - \lambda(P, x - I)$

**K-T Conditions** - ignore first order conditions for goods with zero value; also know $\lambda > 0$ because monotonicity of preferences says solution on boundary

$$\frac{\partial L}{\partial x_j} = U_j(x) - \lambda P_j = 0, j = 1,2,...,n\text{ and } -\frac{\partial L}{\partial \lambda} = P \cdot x - I = 0$$

**Economic Interpretations** -

- $MB = MC - \frac{\partial L}{\partial x_j} = \left( \frac{\partial U}{\partial x_j} - \lambda P_j \right) = 0 \Rightarrow P_j \text{ is } $ lost ($/unit)$
- $\lambda$ is value of $ lost (units/$)

- **IC Slope = Budget Line Slope** - $\frac{\partial U}{\partial x_j} - \lambda P_j = 0 = \frac{\partial U}{\partial x_k} - \lambda P_k \Rightarrow \frac{\partial U / \partial x_j}{\partial U / \partial x_k} = \frac{P_j}{P_k}$

- **Marginal Rate of Substitution** - $-1$ times slope of indifference curve; take total differential of $U(x_j, x_k) = 0$ and solve for $dx_j/dx_k$:

$$\frac{\partial U}{\partial x_j} dx_j + \frac{\partial U}{\partial x_k} dx_k = 0 \Rightarrow \frac{dx_k}{dx_j} = -\frac{\partial U / \partial x_j}{\partial U / \partial x_k}$$

- **Slope of Budget Line** - $P_j x_j + P_k x_k + \ldots = I$; take total differential with respect to $x_j$ and $x_k$: to find $dx_k/dx_j$ (slope of budget line with respect to $x_j$ and $x_k$):

$$P_j dx_j + P_k dx_k = 0 \Rightarrow \frac{dx_k}{dx_j} = -\frac{P_j}{P_k}$$

- **List K-T Conditions** - $n + 1$ equations, $n + 1$ unknowns (will come back to these for comparative statics wrt $P_1$)

$$U_1(x) - \lambda P_1 = 0 \ldots$$

$$U_n(x) - \lambda P_n = 0$$

$$P_1 x_1 + P_2 x_2 + \ldots + P_n x_n = I$$
Comparative Static for Income -

Totally differentiate K-T conditions with respect to $I$ (which goes into $x$ and $\lambda$)

\[
\begin{align*}
U_{11} \frac{\partial x_1}{\partial I} + U_{12} \frac{\partial x_2}{\partial I} + \cdots + U_{1n} \frac{\partial x_n}{\partial I} - \frac{\partial \lambda}{\partial I} P_1 &= 0 \\
& \vdots \\
U_{n1} \frac{\partial x_1}{\partial I} + U_{n2} \frac{\partial x_2}{\partial I} + \cdots + U_{nn} \frac{\partial x_n}{\partial I} - \frac{\partial \lambda}{\partial I} P_n &= 0 \\
P_1 \frac{\partial x_1}{\partial I} + P_2 \frac{\partial x_2}{\partial I} + \cdots + P_n \frac{\partial x_n}{\partial I} &= 1
\end{align*}
\]

Note: The last line says it's impossible for all goods to be inferior

Use first order condition $P_j = U_j/\lambda$ and substitute for last term in first $n$ equations and all terms in $(n + 1)^{th}$ equation

\[
\begin{align*}
U_{11} \frac{\partial x_1}{\partial I} + U_{12} \frac{\partial x_2}{\partial I} + \cdots + U_{1n} \frac{\partial x_n}{\partial I} + U_1 \left( -\frac{1}{\lambda} \frac{\partial \lambda}{\partial I} \right) &= 0 \\
& \vdots \\
U_{n1} \frac{\partial x_1}{\partial I} + U_{n2} \frac{\partial x_2}{\partial I} + \cdots + U_{nn} \frac{\partial x_n}{\partial I} + U_n \left( -\frac{1}{\lambda} \frac{\partial \lambda}{\partial I} \right) &= 0 \\
\frac{U_1}{\lambda} \frac{\partial x_1}{\partial I} + \frac{U_2}{\lambda} \frac{\partial x_2}{\partial I} + \cdots + \frac{U_n}{\lambda} \frac{\partial x_n}{\partial I} &= \lambda
\end{align*}
\]

Write with matrices:

\[
\begin{bmatrix}
U_{11} & U_{12} & \cdots & U_{1n} & U_1 \\
U_{21} & U_{22} & \cdots & U_{2n} & U_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
U_{n1} & U_{n2} & \cdots & U_{nn} & U_n \\
U_1 & U_2 & \cdots & U_n & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x_1}{\partial I} \\
\frac{\partial x_2}{\partial I} \\
\vdots \\
\frac{\partial x_n}{\partial I} \\
-\frac{1}{\lambda} \left( \frac{\partial \lambda}{\partial I} \right)
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \lambda \end{bmatrix}
\]

First matrix is a bordered hessian!

Solve for $\frac{\partial x_1}{\partial I}$ using Cramer's Rule:

\[
\frac{\partial x_1}{\partial I} = \frac{\begin{vmatrix}
U_{12} & \cdots & U_{1n} & U_1 \\
U_{22} & \cdots & U_{2n} & U_2 \\
\vdots & \vdots & \ddots & \vdots \\
U_{n2} & \cdots & U_{nn} & U_n \\
\frac{\partial \lambda}{\partial I}
\end{vmatrix}}{\det(BH)}
\]

Technically requirement for quasiconcave is $(-1)^k |BH_k| \geq 0 \quad \forall k \geq 2$, but $= 0$ part is problem here so we usually assume sufficient condition (i.e., $(-1)^k |BH_k| > 0$), in this case $k = n$ and $(-1)^n = (-1)^{n+2}$ :. sign of $\frac{\partial x_1}{\partial I}$ is determined by sign of numerator:
2 Variable Case - sign of $\partial x_1 / \partial I$ is same as sign of

\[
\begin{vmatrix}
0 & U_{12} & \cdots & U_{1n} & U_1 \\
0 & U_{22} & \cdots & U_{2n} & U_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & U_{n2} & \cdots & U_{nn} & U_n \\
\lambda & U_1 & \cdots & U_n & 0
\end{vmatrix} = (-1)^{n+2} \lambda
\]

Could try to interpret this by using substitutes/complements (properties of $U_{12}$) and diminishing marginal utility ($U_{22}$), but second derivative is not invariant property so these conclusions wouldn’t hold for all utility transformations. \(∴\) we can only say that overall sign of \((U_{12}U_2 - U_{12}U_2)\) is ordinal (invariant to utility transformation) and the same as sign of $\partial x_1 / \partial I$

$\partial x_1 / \partial I$ Directly -

Inferior Good - quantity demanded declines as income increases; $\partial x_j / \partial I < 0$

Income Consumption Curve (ICC) - shows how consumption changes with income; if it slopes back towards the vertical axis, $x_1$ is inferior in that range; if it slopes down towards the horizontal axis, $x_2$ good is inferior in that range;

Note: this also means can’t have all goods be inferior.

Indifference Curve for Inferior Good - the ICs pictured here are continuous, downward sloping "thin" lines that don’t intersect and are convex to origin; there are similar ICs that go through every bundle \(∴\) there exist preferences that lead to inferior goods

Real World - narrowly defined goods lead to inferior goods (e.g., relatively less expensive cuts of meat; relatively cheap compact cars); more broadly defined goods will not be inferior (e.g., food, transportation); doesn’t have to be inferior everywhere (e.g., cheap car could go from normal to inferior and back to normal)

Share of Income - share of income spent on good $j$: $s_j = P_j x_j / I$

Necessary Good - amount spent as percent of income declines as income rises; $ds_j / dI < 0$; technically this would include inferior goods, but usually don’t include them (i.e., also require $\partial x_j / \partial I > 0$)

Luxury Good - amount spent as percent of income increases as income increases; $ds_j / dI > 0$

Homothetic Good - amount spent as percent of income stays constant when income changes; has unitary income elasticity

Change in Share - take derivative of $s_j$ ($x_j$ is function of $I$ so use chain rule)
\[
\frac{ds_j}{dl} = \frac{P_j}{I} \frac{dx_j}{dl} - \frac{P_jx_j}{I^2}
\]

Multiply first term by 1 = \((x_j/I)\) \((x_j/I)\)

\[
\frac{ds_j}{dl} = \frac{P_j}{I} \left( \frac{1}{I} \frac{dx_j}{dl} \right) x_j - \frac{P_jx_j}{I^2} = \frac{P_jx_j}{I^2} (\varepsilon_{x,l} - 1)
\]

\(\therefore \) \(\varepsilon_{x,l} > 1\) makes \(ds_j/dl > 0\) (luxury) and \(\varepsilon_{x,l} < 1\) makes \(ds_j/dl < 0\) (necessity)

<table>
<thead>
<tr>
<th>Inferior Good</th>
<th>(\frac{dx_j}{dl})</th>
<th>(\frac{ds_j}{dl})</th>
<th>(\varepsilon_{x,l})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0, 1</td>
<td></td>
</tr>
<tr>
<td>Necessity</td>
<td>&gt; 0</td>
<td>&lt; 0</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>Homothetic</td>
<td>&gt; 0</td>
<td>= 0</td>
<td>1</td>
</tr>
<tr>
<td>Luxury</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>&gt; 1</td>
</tr>
</tbody>
</table>

Limited Income - \(\sum_{j=1}^{n} P_jx_j = I \Rightarrow \sum_{j=1}^{n} P_jx_j / I = 1 \Rightarrow \sum_{j=1}^{n} s_j = 1\)

so sum of \(s_j\)'s equals constant, \(\therefore\) if some \(s_j\) increases (i.e., luxury good with \(ds_j/dl > 0\)), then some other \(s_j\) must decrease (i.e., necessary or inferior good); i.e., if there is a luxury good there has to be a necessary or inferior good

**Comparative Static for Prices**

**Result** - most important result of consumer theory; price change has both income and substitution effects

**Tax Cut Example** - substitution effect \(\Rightarrow\) people work more; income effect \(\Rightarrow\) people work less; don't know which effect is dominant and it's difficult to measure empirically because time horizon (income effects may take longer to observe, but by then tax codes changes again)

Totally differentiate K-T conditions (listed earlier) with respect to \(P_1\) (generalizes to any \(P_j\)); **Note**: since comparative statics looks at optimal solution, \(P_1\) enters into functions for optimal decision variables (i.e., \(x(P, I)\) and \(\lambda(P, I)\))

\[
U_{11} \frac{\partial x_1}{\partial P_1} + U_{12} \frac{\partial x_2}{\partial P_1} + \cdots + U_{1n} \frac{\partial x_n}{\partial P_1} - \lambda \frac{\partial \lambda}{\partial P_1} P_1 = 0
\]

\(U_{j1} \frac{\partial x_1}{\partial P_1} + U_{j2} \frac{\partial x_2}{\partial P_1} + \cdots + U_{jn} \frac{\partial x_n}{\partial P_1} = \frac{\partial \lambda}{\partial P_1} P_j = 0\) \((j = 2, 3, \ldots, n)\)

Use first order condition \(P_j = U_j/\lambda\) and substitute for last term in first \(n\) equations and all terms in \((n + 1)^{th}\) equation; in the first eqn move the \(\lambda\) to the right hand side:

\[
U_{11} \frac{\partial x_1}{\partial P_1} + U_{12} \frac{\partial x_2}{\partial P_1} + \cdots + U_{1n} \frac{\partial x_n}{\partial P_1} - \lambda \frac{\partial \lambda}{\partial P_1} = \lambda
\]

\(U_{j1} \frac{\partial x_1}{\partial P_1} + U_{j2} \frac{\partial x_2}{\partial P_1} + \cdots + U_{jn} \frac{\partial x_n}{\partial P_1} - \frac{\partial \lambda}{\partial P_1} U_j = 0\) \((j = 2, 3, \ldots, n)\)
\[ x_i + \frac{U_1}{\lambda} \frac{\partial x_1}{\partial P_1} + \frac{U_2}{\lambda} \frac{\partial x_2}{\partial P_1} + \cdots + \frac{U_n}{\lambda} \frac{\partial x_n}{\partial P_1} = 0 \]

Multiply last eqn by \( \lambda \) and put first term on other side:
\[
U_1 \frac{\partial x_1}{\partial P_1} + U_2 \frac{\partial x_2}{\partial P_1} + \cdots + U_n \frac{\partial x_n}{\partial P_1} = -\lambda x_i
\]

Write with matrices:
\[
\begin{bmatrix}
U_{11} & U_{12} & \cdots & U_{1n} & U_1 \\
U_{21} & U_{22} & \cdots & U_{2n} & U_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
U_{n1} & U_{n2} & \cdots & U_{nn} & U_n \\
U_1 & U_2 & \cdots & U_n & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x_1}{\partial P_1} \\
\frac{\partial x_2}{\partial P_1} \\
\vdots \\
\frac{\partial x_n}{\partial P_1} \\
\lambda
\end{bmatrix}
= \begin{bmatrix}
\lambda \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

First matrix is a bordered hessian; it will always be a \( \text{BH} \) for constrained optimization... the trick is figuring out what the other terms look like

Solve for \( \frac{\partial x_1}{\partial P_1} \) using Cramer's Rule:
\[
\frac{\partial x_1}{\partial P_1} = \frac{\lambda U_{11} U_{22} \cdots U_{nn} - U_1 U_{12} U_{22} \cdots U_{nn} + \cdots + (-1)^{n+1} U_1 U_{12} \cdots U_{n1}}{|\text{BH}|}
\]

Expand by cofactors using the first column:
\[
\frac{\partial x_1}{\partial P_1} = \lambda U_{11} U_{22} \cdots U_{nn} - U_1 U_{12} U_{22} \cdots U_{nn} + \cdots + (-1)^{n+2} U_1 U_{12} \cdots U_{n1}
\]

The first term is the own substitution effect \( (S_{11}) \) and the second is simply \( x_i \) times \( \frac{\partial x_1}{\partial P_1} \):

\[ \text{Substitution Effect} \quad \text{Income Effect} \]

Slutsky Equation: \[ \frac{\partial x_i}{\partial P_1} = S_{11} - x_i \frac{\partial x_i}{\partial I} \] general: \[ \frac{\partial x_i}{\partial P_j} = S_{ij} - x_j \frac{\partial x_i}{\partial I} \]

Own Substitution Effect \( (S_{ii}) \) - first term from \( \frac{\partial x_i}{\partial P_1} \) above; \( \lambda \) times principal minor of \( \text{BH} \) divided by determinant of \( \text{BH} \); principal minor has 1 row/col removed so it has opposite sign as \( \text{BH} \) and \( S_{ii} < 0 \) (because we assume \( U(x) \) is quasiconcave)
More formally - $S_{11} = \left( \frac{\partial x_1}{\partial P_1} \right)^{U=\text{c}}$

general: $S_{ij} = \left( \frac{\partial x_1}{\partial P_j} \right)^{U=\text{c}}$

i.e., effect of price $j$ on good $i$ while holding utility constant

**Hixian Compensation** - in order to keep utility constant with a changing price, we need an increase in income to offset the higher price; don’t know how much it is unless we know full indifference map

**Cross Substitution Effect ($S_{ij}$)** - effect of price $j$ on good $i$; similar to $S_{11}$ above only it’s a different row and col used to expand the derivative so the term in the numerator is not a principal minor $\therefore S_{ij}$ is uncertain

The sign of $\frac{\partial x_1}{\partial P_1}$ depends on $\frac{\partial x_1}{\partial I}$; for a normal good ($\frac{\partial x_1}{\partial I} > 0$, it is clear that $\frac{\partial x_1}{\partial P_1} < 0$ (i.e., $P_1 \uparrow \Rightarrow x_1 \downarrow$... standard law of demand); for an inferior good, however, $\frac{\partial x_1}{\partial I} < 0$ so the effect of price is indeterminate; usually the substitution effect dominates so the law of demand holds, but there is the possibility that it won’t

**Giffen Good** - inferior good in which income effect dominates so $\frac{\partial x_1}{\partial P_1} > 0$ (i.e., violates law of demand); named after first person to “observe” $P_1 \uparrow$ and $x_1 \uparrow$; situation was in Ireland during potato famine; British law essentially prohibited imports; this limited substitutes so $S_{11}$ would be small; for potatoes in Ireland $x_1$ would be large; potatoes are an inferior good so Giffen’s story is plausible, but "Giffen didn’t say it and it didn’t happen, but it's plausible"; Giffen goods are hard to observe and are more an irritant to theorists by making results ambiguous

**Other Problems** - there are other cases with upwards loping demand curves (i.e., $\frac{\partial x_1}{\partial P_1} > 0$); usually they are things in which demand is based on perceived quality through price so utility isn’t just $U(x)$, but $U(x, P)$ which we chose to ignore in our analysis because we assumed $x$ and $P$ were independent; example: diamonds

**Slutsky Equation Graphically** - when $P_1 \uparrow$, the budget line gets steeper (assuming $x_1$ on the horizontal axis)

**Income Effect** - the new budget line is below the old one so the original bundle is now infeasible; essentially the consumer is now “poorer”

**Substitution Effect** - used to have $U_1/U_2 = P_1/P_2$; since $P_1 \uparrow$, that no longer holds and consumers will substitute away from good 1 to reestablish equilibrium

**Conceptual View** - picture increasing income to offset higher price (e.g., social security cost of living allowance); on the graph, that means move the new budget line up until it’s tangent to the original indifference curve; this would be point C; the move from point C to point B is the income effect; the move from point A to point C is the substitution effect

**Quantity Matters** - initial amount of good being consumed matters to individual because the more he consumes, the more he’ll be impacted by a change in price

**Not Measurable** - this conceptual view isn’t operational because we’d need to know the consumer’s full indifference map; all we know is a specific point, but there could be lots of indifference curves through that point (each having a different income and substitution effect)
**Substitutes** - goods that could be used in place of each other

**Gross** - defined by including the income effect; \( \frac{\partial x_i}{\partial P_j} > 0 \) (i.e., \( P_1 \uparrow \Rightarrow x_2 \uparrow \))

**Net** - doesn't include income effect; also called Hixian substitutes; \( S_{ij} > 0 \)

**Compliments** - goods used together; defined as gross and net just like substitutes except \( \frac{\partial x_i}{\partial P_j} < 0 \) (i.e., \( P_1 \uparrow \Rightarrow x_2 \downarrow \))

**Formal Definitions** - formal definition of substitutes is debated; want something that's ordinal, symmetric (i sub for j ⇒ j sub for i), and unbiased (goods are equally likely to be substitutes or compliments)

**Classical Substitutes** - based on \( U_{ij} \); not good because it's based on second derivative which is not invariant (i.e., not consistent for preferences)

**Hixian Substitutes** - defined using \( S_{ij} \); this is ordinal and symmetric (\( S_{ij} = S_{ji} \)... takes lots of algebra to prove); biased because if there are just two goods, the best you can get is \( S_{ij} = 0 \) (independent) even if the goods are perfect compliments (can never get \( S_{ij} < 0 \))

**Gross Substitutes** - check for symmetry: does \( \frac{\partial x_i}{\partial P_j} = \frac{\partial x_j}{\partial P_i} \)? use Slutsky equation

\[
S_{ij} - x_j \frac{\partial x_i}{\partial I} = S_{ji} - x_i \frac{\partial x_j}{\partial I}
\]

\( S_{ij} \) cancels \( S_{ji} \); cross the \( x_i \) and \( x_j \) and multiply both sides by \( I \) to get

\[
I \frac{\partial x_i}{\partial I} - \frac{\partial x_j}{\partial I} = \frac{\partial x_j}{\partial I} \Rightarrow \epsilon_{x_i,I} = \epsilon_{x_j,I}
\]

\( \therefore \) gross substitute definition is symmetric only if the goods have the same income elasticities (i.e., homothetic preferences); it's possible that income elasticities are so different that \( \frac{\partial x_i}{\partial P_j} \) & \( \frac{\partial x_j}{\partial P_i} \) have different signs

**Indifference Curves** - close to straight lines are substitutes and close to right angles are compliments

**Nothing's Perfect** - no definition really works so pick one based on the context (e.g., whichever works best when trying to "say something interesting" in comparative statics)

**Hixian Terms** (\( S_{ij} \)) - not directly observable; don't know how much to compensate, but can compute them indirectly by rewriting Slutsky eqn:

\[
S_{ij} = \frac{\partial x_i}{\partial P_j} + x_j \frac{\partial x_i}{\partial I}
\]

\( x_i \) is directly observed, the other terms are computed

**Slutsky Matrix** - matrix of all the Hixian Terms; puts restriction on demand; (1) diagonal elements ≤ 0; (2) symmetric; (3) negative semi-definite

\[
\begin{bmatrix}
S_{11} & \cdots & S_{1n} \\
\vdots & \ddots & \vdots \\
S_{n1} & \cdots & S_{nn}
\end{bmatrix}
\]

**Checking Demand Properties** - look at five properties listed earlier and then check the Slutsky matrix... but this is hard stuff
Elasticity ($\varepsilon_{y,x}$) - elasticity of $y$ with respect to $x$; percentage change in $y$ divided by percentage change in $x$

**Divide By What?** - $\Delta y = y_1 - y_0 \ldots$ do we divide by $y_1$ or $y_0$? some say to divide by the average (arc elasticity)

**Point Elasticity** - $\Delta y \to 0$; related to slope, but not the same

$$
\varepsilon_{y,x} = \frac{\% \Delta y}{\% \Delta x} = \frac{\Delta y}{y} \frac{\Delta x}{x} \to \frac{dy}{y} \frac{dx}{x} = \frac{x}{y} \frac{dy}{dx}
$$

**Why Use It?** - independent of units (doesn't change when you change units like slope does); sometimes comes up in comparative statics