Optimization Parameters

Exogenous Parameters ($\alpha$) - optimization problem is composed of decision (choice) variables and exogenous parameters (e.g., price, budget, preferences, etc.); parameters can be in objective function, in constraints, or in both; use vector $\alpha$ to denote parameters; can use single vector by specifying first derivatives (e.g., $\partial f/\partial \alpha_k = 0$ means parameter $\alpha_k$ is not in the objective function)

New Questions -
1. How does parameter change value of solution (objective function)... envelope theorem
2. How does parameter change value of choice variables

Optimal Solutions - value of choice values that maximize $f$ for different values of parameters means we can write $x$ as a function of $\alpha$; use obscure macro notation: $x(\alpha)$ (e.g., demand is a function of price); find this relationship by plugging in new parameters methodically or by using comparative statics

Envelope Theorem

Objective Value - define value of objective function at maximum point for various values of $\alpha$: $V(\alpha) \equiv f(x(\alpha), \alpha)$; interested in properties of $V(\alpha)$, usually only look at first derivative; need more information on specific function for second derivative to be useful

General Lagrangian - $L(x, \lambda, \alpha) = f(x, \alpha) - \sum_{i=1}^{m} \lambda_i g^i(x, \alpha)$

Optimal Lagrangian - $L(x(\alpha), \lambda(\alpha), \alpha) \equiv V(\alpha)$

Change in Objective Function - take derivative of $f(x(\alpha), \alpha)$:

$$\frac{\partial V}{\partial \alpha_k} = \frac{\partial f}{\partial \alpha_k} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial \alpha_k} + \frac{\partial f}{\partial \alpha_k}$$

This is how choice variable changes when $\alpha_k$ changes; we find this with comparative statics, but won't need it for this

Envelope Theorem - don't need to worry about how parameters change choice variables, only the change in the objective (or constraint); that means to find $\partial V/\partial \alpha_k$, we only need to worry about places where $\alpha_k$ enters directly, not $x(\alpha)$ or $\lambda(\alpha)$

Proof:

$$\frac{\partial V(\alpha)}{\partial \alpha_k} = \frac{\partial L(x, \lambda, \alpha)}{\partial \alpha_k} = \frac{\partial}{\partial \alpha_k} \left[ f(x(\alpha), \alpha) - \sum_{i=1}^{m} \lambda_i g^i(x, \alpha) \right]$$

Rearrange to group $\partial x_j/\partial \alpha_k$ terms:

$$\left( \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \frac{\partial g^i}{\partial x_j} \frac{\partial x_j}{\partial \alpha_k} \right) \frac{\partial x_j}{\partial \alpha_k} + \frac{\partial f}{\partial \alpha_k} - \sum_{i=1}^{m} \lambda_i \frac{\partial g^i}{\partial \alpha_k}$$
From K-T conditions for $x_j$:
1. $x_j > 0$ and $\frac{\partial f}{\partial x_j} = 0$
2. $x_j = 0$ and $\frac{\partial f}{\partial x_j} = 0$
3. $x_j = 0$ and $\frac{\partial f}{\partial x_j} < 0$

From first two cases, the first term in the derivative above drops out.

Third case - corner solution; would like to make $x_j$ smaller, but you can't because of a constraint; $\frac{\partial f}{\partial x_j}$ is continuous (by assumption) so small change in $\alpha_k$ ("in the neighborhood") doesn't change the inequality; \therefore since $\frac{\partial f}{\partial x_j} < 0$, $x_j = 0$ (K-T condition) for small change in $\alpha_k$ (i.e., $\partial x_j/\partial \alpha_k = 0$)... so first term in the derivative above drops out.

Use K-T conditions for $\lambda_i$ and same reasoning means third term in derivative above drops out, now:

$$\frac{\partial V(\alpha)}{\partial \alpha_k} = \frac{\partial f}{\partial \alpha_k} - \sum_{i=1}^{m} \lambda_i \frac{\partial g^i}{\partial \alpha_k}$$

.. i.e., only need to look at terms of $L$ where $\alpha_k$ enters directly.

** will use this a lot later in the semester.

**Example** - (proof that $\lambda \geq 0$)

Max $f(x)$

\begin{align*}
\text{s.t.} \quad & g^i(x) \leq K_i, \ i = 1,2,...,m \\
& x \geq 0
\end{align*}

Parameter ($K_i$) - in constraint only and only one parameter per constraint; used to shift constraint (see graph); can rewrite constraint: $g^i(x) - K_i \leq 0$

**Lagrangian** - $L = f(x) - \sum_{i=1}^{m} \lambda_i \left[ g^i(x) - K_i \right]$

**Use Envelope Theorem** - only look at terms with $K_i$: $\frac{\partial V}{\partial K_i} = \lambda_i$

**Conclusion** - since increasing $K_i$ increases feasible set, objective can't decline (i.e., $V(K_i) \geq V(0) \Rightarrow \partial V/\partial K_i \geq 0$) \therefore $\lambda_i \geq 0$

**Comparative Statics**

**Comparative Statics** - looks at $\partial x_j/\partial \alpha_k$ (how particular parameter changes value of choice variable); can get general relationship (e.g., + or -) without knowing explicit function.

**General Approach** - brute force, but makes no assumptions; checklist:

1. Write out lagrangian
2. Take K-T conditions
3. Discard conditions where $\partial L/\partial x_j < 0$ and $x_j = 0$ (same with $\partial L/\partial \lambda_i < 0$ and $\lambda_i = 0$); these aren't "interesting" because comparative static is done: $\partial x_j/\partial \alpha_k = 0$; **note:** still need to worry about $\partial L/\partial x_j = 0$ and $x_j = 0$

**System of Equations** - $\partial L/\partial x_j = 0, j = 1,2,... \hat{n}$ (deleted some)

$\partial L/\partial \lambda_i = 0, i = 1,2,... \hat{m}$

4. Implicitly solve the system and substitute solution values into K-T conditions; this converts them from equalities (=) to identities (≡) is $\alpha_k$ (or $\alpha$ if you want to do them all... makes notation more complicated, but still works)
5. Totally differentiate system of \( \dot{\mathbf{n}} + \mathbf{m} \) identities with respect to \( \alpha_k \); will be linear in \( \partial \mathbf{x} / \partial \alpha_k \).
6. Write system in matrix form: \( \mathbf{A} \mathbf{z} = \mathbf{b} \)
   - \( \mathbf{A} \) - "endogenous effects induced by \( R \)"; in unconstrained problem \( \mathbf{A} \) is hessian of choice variables; for constrained problem it's bordered hessian
   - \( \mathbf{z} \) - vector of comparative statics: \( (\partial \mathbf{x}_i / \partial \alpha_k, \cdots, \partial \mathbf{x}_n / \partial \alpha_k) \)
   - \( \mathbf{b} \) - vector of how parameters affect marginal benefit of each choice variable: \( (\partial^2 f / \partial x_i \partial \alpha_k) \)
7. Solve using Cramer's Rule - to solve for \( i \)^th term in \( \mathbf{z} \), calculate determinant of matrix created by replacing \( i \)th column in \( \mathbf{A} \) with \( \mathbf{b} \) and divide by determinant of \( \mathbf{A} \) (see example)
8. Say something interesting about \( \mathbf{z} \)
   - Sign - usually best thing you can determine is sign; use all info you can from assuming sufficient first and second order conditions (usually enough for denominator and part of numerator in Cramer's Rule); usually need additional information on objective and/or constraints to get useful results

**Example** - \( x_1 \) & \( x_2 \) = amount of each crop to plant; \( R \) = amount of rain; unconstrained optimization problem: Max \( f(x_1, x_2, R) \)
   1. Don't need lagrangian (unconstrained)
   2. \( \partial f(x_1, x_2, R) / \partial x_1 = f_1(x_1, x_2, R) = 0 \) and \( \partial f(x_1, x_2, R) / \partial x_2 = f_2(x_1, x_2, R) = 0 \)
   3. Don't need
   4. \( x_1(R) \) and \( x_2(R) \)... nasty macro-style notation there; that's \( x_1 \) as a function of \( R \)
   5. Totally differentiate:
      \[
      f_{11} \frac{dx_1}{dR} + f_{12} \frac{dx_2}{dR} + f_{13} = 0 \quad \text{and} \quad f_{21} \frac{dx_1}{dR} + f_{22} \frac{dx_2}{dR} + f_{23} = 0
      \]
   6. Matrix notation: \( \mathbf{A} \mathbf{z} = \mathbf{b} \)
      \[
      \begin{bmatrix}
      f_{11} & f_{12} \\
      f_{21} & f_{22}
      \end{bmatrix} \begin{bmatrix}
      dx_1/dR \\
      dx_2/dR
      \end{bmatrix} = \begin{bmatrix}
      -f_{13} \\
      -f_{23}
      \end{bmatrix}
      \]
   7. Cramer's Rule -
      \[
      \frac{dx_1}{dR} = \frac{-f_{13} f_{22} - f_{12} f_{23}}{f_{11} f_{22} - f_{12} f_{21}} \quad \text{and} \quad \frac{dx_2}{dR} = \frac{f_{11} f_{23} - f_{13} f_{21}}{f_{11} f_{22} - f_{12} f_{21}}
      \]
   8. Something Interesting -
      \( |\mathbf{A}| > 0 \) (\( \mathbf{A} \) is negative definite by 2nd order condition)
      \( f_{22} < 0 \), which cancels the "-" in the first term
      Sign of \( dx_1/dR \) is same as sign of \( f_{13} + f_{23}f_{12} \)... can't say more without explicit function
      Special Case: \( f(x_1, x_2, R) = [M(x_1) + H(x_2)]g(R) \)
      \( f_{12} = 0 \)
      Now sign of \( dx_1/dR \) is same as sign of \( f_{13} = dM/dx_1 \cdot dg/dR \)
      From 1st order condition: \( f_1 = dM/dx_1 \cdot g(R) = 0 \), \( \therefore dM/dx_1 = 0 \) (or \( g(R) = 0 \) which makes problem worthless)
      \( f_{13} = dM/dx_1 \cdot dg/dR = 0 \) (not a very good example)

**Multiple Parameters** - \( \mathbf{A} \) will be same for all parameters; can repeat process for each parameter or do it simultaneously: Max \( f(x_1, x_2, R, T) \)

\[
\begin{bmatrix}
  f_{11} & f_{12} \\
  f_{21} & f_{22}
\end{bmatrix} \begin{bmatrix}
  dx_1 \\
  dx_2
\end{bmatrix} = \begin{bmatrix}
  -f_{13} dR - f_{14} dT \\
  -f_{23} dR - f_{24} dT
\end{bmatrix}
\]
Correspondence Principle - comparative statics only reveal interesting things about stable points; if system doesn't get to point in equilibrium that point is irrelevant to comparative statics; proposed by Samuelson

Optimization Statics - the above deals with comparative statics for optimization; first and second order conditions aren't disputed so results are consistent among economists; to do it for other types of analysis, change steps 2 & 3

Equilibrium Statics - Steps 2 & 3 above simply become "write out equilibrium conditions"; more ambiguity in assumptions (e.g., $P$ adjusts vs. $Q$ adjusts for supply and demand equilibrium) so results may vary