6.1. Determine which strategies are dominated in the following normal-form games.

(a) B dominates A; L dominates R
(b) L dominates R; with R gone, U dominates M and D; with M and D gone, L dominates C
(c) 0.5W & 0.5X dominates Z; with Z gone, U dominates M and D; with M and D gone, X dominates W and Y
(d) No strategies are dominant (although (U,L) and (D,R) are Nash equilibria)

6.3. Consider a version of the Cournot duopoly game, where firms 1 and 2 simultaneously and independently select quantities to produce in a market. The quantity selected by firm $i$ is denoted $q_i$ and must be greater than or equal to zero, for $i = 1, 2$. The market price is given by $p = 100 - 2q_1 - 2q_2$. Suppose that each firm produces at a cost of 20 per unit. Further, assume that each firm’s payoff is defined as its profit. Is it ever a best response for player 1 to choose $q_1 = 25$? Suppose that player 1 has the belief that player 2 is equally likely to select each of the quantities 6, 11, and 13. What is player 1’s best response?

(a) Profit: $\pi^1 = pq_1 - 20q_1 = (100 - 2q_1 - 2q_2)q_1 - 20q_1 = 100q_1 - 2q_1^2 - 2q_1q_2 - 20q_1 = 80q_1 - 2q_1^2 - 2q_1q_2$

Best Reply for Firm 1: solve $\frac{\partial \pi^1}{\partial q_1} = 0$ for $q_1$

$\frac{\partial \pi^1}{\partial q_1} = 80 - 4q_1 - 2q_2 = 0$

$R^1(q_2) = 20 - \frac{1}{2}q_2$

$R^1(25) = 20 - \frac{1}{2}25 = 15$ which is infeasible. ♦. it is never a best response for player 1 to choose $q_1 = 25$

(b) $E(q_2) = (6 + 11 + 13)/3 = 10$

$R^1(E(q_2)) = 20 - \frac{1}{2}10 = 15$

This is the same as $E(R^1(q_2)) = [(20 - \frac{1}{2}(6)) + (20 - \frac{1}{2}(11)) + (20 - \frac{1}{2}(13))]/3 = 15$
7.3. In the game pictured here, is it ever rational for player 1 to select strategy C? Why?

No. Using a mixed strategy of A with probability 0.5 and B with probability 0.5, player 1's expected payoff is 1.5 so it dominates strategy C.

7.5. Find the set of rationalizable strategies for the following game.

Note that each player has more than one dominated strategy. Discuss why, in the iterative process of deleting dominated strategies, the order in which dominated strategies are deleted does not matter.

c dominates b and d; with b and d gone x dominates y and w dominates z; with y and z gone, c dominates a; with a gone x dominates w; \( \therefore \) the set of rationalizable strategies only has strategy x for player 1 and strategy c for player 2: \( \{(c,x)\} \)

By definition a dominated strategy is one that a rational player would not use. Strategies are dominated by non-dominated strategies. Saying that the order of deleting strategies matters suggests that a dominated strategy (say B dominated by A) is used to remove another dominated strategy (say C). But that first dominated strategy (B) is dominated by A which also dominates C. Therefore, we could just as easily delete C based on domination by A or domination by B. Strategy B will then be deleted because it's dominated by A. The order doesn't matter.
9.1. Find the Nash equilibria of the games in the figures in Exercise 3(a-d) at the end of Chapter 4. Remember to convert these games into the normal form first.

(a) Nash equilibria: \{ (B, CF), (B, DF) \}; Note: that means player 1 plays strategy B and player 2 plays strategy F.

(b) Nash equilibra: \{ (IU, I), (O, O) \}; Note: (IU, I) is strict, but (O, O) isn't. We could look at OU and OI for player 1, but these is redundant and not necessary.

(c) Nash equilibria: \{ (DE, AC), (DE, BC), (U, BD) \}; Note: that means player 1 plays strategy D and player 2 plays strategy C, or player 1 plays strategy U and player two plays strategy B. We could look at UE and UF for player 1, but these are redundant and not necessary.
(c) Nash equilibria: none. We could add strategies for player 1 (UXW, UXZ, UYW, UXZ, DXW, DXZ, DYW, DYZ), but that doesn't fundamentally change the problem; it just copies the four rows here because X & Y are irrelevant when player 1 plays D and W & Z are irrelevant when player 1 plays U.

9.5. Find the Nash equilibrium of the following normal-form game: \( S_1 = [0, 1], S_2 = [0, 1], u_1(s_1, s_2) = 3s_1 - 2s_1s_2 - 2s_1^2 \), and \( u_2(s_1, s_2) = s_2 + 2s_1s_2 - 2s_2^2 \). (The solution is interior so you can use calculus.)

Best Reply Functions:
\[
\frac{\partial u_1(s_1, s_2)}{\partial s_1} = 3 - 2s_2 - 4s_1 = 0 \quad \Rightarrow \quad s_1 = \frac{3}{4} - \frac{1}{2}s_2
\]
\[
\frac{\partial u_2(s_1, s_2)}{\partial s_2} = 1 + 2s_1 - 4s_2 = 0 \quad \Rightarrow \quad s_2 = \frac{1}{4} + \frac{1}{2}s_1
\]

Nash equilibrium is when each player is playing best reply to opponent's strategy (i.e., intersection of best reply functions):
\[
s_1 = \frac{3}{4} - \frac{1}{2}(\frac{1}{4} + \frac{1}{2}s_1) = \frac{3}{4} - \frac{1}{8} - \frac{1}{4}s_1 \quad \Rightarrow \quad s_1 = \frac{4}{5}(\frac{5}{8}) = \frac{4}{8} = \frac{1}{2}
\]
\[
s_1 = \frac{1}{4} + \frac{1}{2}(\frac{1}{2}) = \frac{1}{2}
\]

10.1. Consider a more general Cournot model than the one presented in this chapter. Suppose there are \( n \) firms. The firms simultaneously and independently select quantities to bring to the market. Firm \( i \)'s quantity is denoted \( q_i \), which is constrained to be greater than or equal to zero. All of the units of the good are sold, but the prevailing market price depends on the total quantity in the industry, which is \( Q = \sum_{i=1}^n q_i \). Suppose the price is given by \( p = a - bQ \) and suppose each firm produces with marginal cost \( c \). There is no fixed cost for the firms. Assume \( a > c > 0 \) and \( b > 0 \). Note that firm \( i \)'s profit is given by \( u_i = p(Q)q_i - cq_i = (a - bQ)q_i - cq_i \). Defining \( Q_i \), as the sum of the quantities produced by
all firms except firm $i$, we have $u_i = (a - bq_i - bQ_i)q_i - cq_i$. Each firm maximizes its own profit.

(a) Represent this game in the normal form by describing the strategy spaces and payoff functions.

(b) Find firm $i$'s best response function as a function of $Q_i$. Graph this function.

(c) Compute the Nash equilibrium of this game. Report the equilibrium quantities, price, and total output. Hint: Summing the best-response functions over the different players will help.

(d) Show that, for the Cournot duopoly game ($n = 2$), the set of rationalizable strategies coincides with the Nash equilibrium.

(a) Players: $I = \{1, 2, \ldots, n\}$

Strategy Spaces: $S^i = [0, \infty)$ ($\forall i \in I$) because player $i$ can’t produce a negative quantity. The upper bound could be tightened to a more realistic quantity in two ways. First, realize that it doesn’t make sense for anyone to produce to the point that prices go negative. Therefore, solving $a - bQ = 0$ yields an upper bound of $a/b$. A tighter bound comes from realizing that a firm producing by itself will behave like a monopoly. That means $MR = MC = c$ yields an upper bound of $(a - c)/2b$.

Payoff Functions: $u_i(q_1, q_2, \ldots, q_n) = (a - bq_i - bQ_i)q_i - cq_i$, where $q_i \in S^i \forall i \in I$

(b) Best Response for firm $i$: $R_i(Q_{-i}) = a - 2bq_i - bQ_i - c = 0 \Rightarrow \frac{a-c}{2b} - Q_{-i} \Rightarrow q_i = \frac{a-c}{2b}$

(c) Nash equilibrium means players play best responses to opponents' best responses.

$q_i^\ast = \frac{a-c}{2b} - \frac{Q_{-i}^\ast}{2}$

$Q^\ast = \sum_{i=1}^{n} q_i^\ast = \sum_{i=1}^{n} \left( \frac{a-c}{2b} - \frac{Q_{-i}^\ast}{2} \right) = \frac{n(a-c)}{2b} - \frac{1}{2} \sum_{i=1}^{n} Q_{-i}^\ast$

Tricky part... sub $Q_{-i} = \sum_{j \neq i} q_j$ and realize $\sum_{i=1}^{n} \sum_{j \neq i} q_j = (n-1)\sum_{i=1}^{n} q_i$

$\sum_{i=1}^{n} q_i^\ast = \frac{n(a-c)}{2b} - \frac{n-1}{2} \sum_{i=1}^{n} q_i^\ast$

Move sums to same side

$\left(1 + \frac{n-1}{2}\right) \sum_{i=1}^{n} q_i^\ast = \frac{n(a-c)}{2b}$
Solve for $Q^*$

$$Q^* = \sum_{i=1}^{n} q_i^* = \frac{n(a-c)}{2b} \left( \frac{2}{n+1} \right) = \frac{n(a-c)b}{b(n+1)}$$

Substitute $Q^*$ to solve for $p^*$

$$p^* = a - bQ^* = a - b \left( \frac{n(a-c)}{b(n+1)} \right) = \frac{an + a - na - nc}{n+1} = \frac{a + nc}{n+1}$$

Substitute $Q^*$ to solve for $q_i^*$

$$q_i^* = \frac{a-c}{2b} - \frac{Q_{-i}^*}{2}$$

Trick is to substitute $Q_{-i}^* = Q^* - q_i^*$

$$q_i^* = \frac{a-c - (Q^* - q_i^*)}{2} = \frac{a-c}{2b} - \frac{1}{2} \left( \frac{n(a-c)}{b(n+1)} \right) + q_i^*$$

Put all $q_i^*$ on same side

$$\frac{1}{2} q_i^* = \frac{na + a - nc - c}{2b(n+1)} - \frac{na - nc}{2b(n+1)} = \frac{a-c}{2b(n+1)}$$

Solve for $q_i^*$

$$q_i^* = \left( \frac{a-c}{2b(n+1)} \right)^2 = \frac{a-c}{b(n+1)}$$

(d) $n = 2$

$$Q^* = \frac{2(a-c)}{3b} ; \quad p^* = \frac{a+2c}{3} ; \quad q_i^* = \frac{a-c}{3b} = \frac{Q^*}{2}$$

From textbook problem (p.96) $a = 1000, b = 1, \text{ and } c = 100$

$$Q^* = \frac{2(1000-100)}{3(1)} = 600; \quad p^* = \frac{1000 + 2(100)}{3} = 400; \quad q_i^* = \frac{1000 - 100}{3(1)} = 300$$

Which is the same solution given on p.96.

**Documentation.**

I reviewed my work with Josh Kneifel. He caught an error in 6.1.a (my drawing figure didn’t match my verbiage). We also worked out problem 10.1 together. Ozde Oztekin also caught an error in 6.1b and d (I didn’t copy the problem correctly).