Refinements

Refinements - change set of equilibria to find "better" set of equilibria by eliminating some that are less plausible

Strategic Form

Eliminate Weakly Dominated Strategies -

Purpose - throwing out strictly dominated strategies (even iteratively) never eliminates a Nash equilibrium (because Nash equilibrium never has player using a strictly dominated strategy); goal is to eliminate weakly dominated strategies to get a better defined equilibrium

Example - two people meeting at the airport; any place they meet is a Nash equilibrium, but there are better places to meet (e.g., ticket counter vs. 5th hangar)

Example - in this game both (6, 3) and (6, 0) are Nash equilibria (as well as any mixed strategy for player 1), but they aren't equally compelling because row 1 weakly dominates row 2 : we could argue (6, 3) is the "best" Nash equilibrium

Why Best - another argument for (6, 3) is that player 1 is only indifferent between A and B if player 2 plays X with probability 1; if player 2 plays Y (presumably by mistake), then row 1 is a better choice because it has a better payoff than row 2 in the case of player 2's mistake... this is trembling hand perfect argument

Order Matters - order of eliminating weakly dominated strategies matters; could cause problems if it changes the remaining set of equilibria (if so, we need to look closer at the structure of the game to see the order of moves, etc.); this is why some theorists argue against iterated weakly dominated strategies

How Likely - some may argue that probability of having two payoffs being exactly equal is unlikely... until you consider extensive form where only difference between two strategies is irrelevant (i.e., have same payoff); e.g., strategy leading down equilibrium paths, 1 says X and 1 says Y

Pareto Superior - sum of payoffs being greater implies a better equilibrium

Example - only 2 pure strategy Nash equilibria; which is better? most would argue (10, 10), but notice that A is weakly dominated by B so some theorists would say (3, 4) is more reasonable

Trembling Hand Perfect - consider possibility that players make random errors; look at limit of probabilities of errors to get subset of original Nash equilibria

Theorem - finite, 2 player game, trembling hand perfect and eliminating weakly dominated strategies are the same

Nonrandom Errors - depend on strategy you're looking at (e.g., verbal directions "B" sounds

With error E(A) < E(B) so player 1 should pick strategy B
Random Errors - random noise with same probability distribution for all non-equilibrium strategies

Incorporate Into Game - create new payoff matrix based on probability 1 - \( \varepsilon \) of hitting strategy A and probability \( \varepsilon \) of making an error; enter expected values in each box (e.g., for (A,X) payoff would be \((1 - \varepsilon)(1 - \delta)(10) + ...\) (messy expression); each set of strategies will produce an expected payoff for each player: \( E(\varepsilon, \delta) \); now goal is to find \( \lim_{\varepsilon, \delta \to 0} E(\varepsilon, \delta) \)

If errors may be significant, we may want something other than this limit, but need more info to model them (e.g., if one strategy is particularly bad so player is extra cautious, we could set up probability distribution over errors to make that strategy less likely)

Equilibrium Correspondence - closed set that relates \( E(\varepsilon, \delta) \) to different values of \( \varepsilon \) and \( \delta \)

Example - football coaches consider errors when calling plays (e.g., if you punt it may be blocked [your error] or it could be fumbled [opponent’s error]); even if perfectly executed play may be best call, when accounting for errors, it may not be the best play to pick

Example - three player game with no weakly dominated strategies; assume (A,X,Left), (B,X,Left), and (B,Y,Right) are Nash equilibria and all players are aiming at (A,X,Left)

<table>
<thead>
<tr>
<th>Player 1</th>
<th>1 - ( \gamma )</th>
<th>( \gamma )</th>
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</thead>
<tbody>
<tr>
<td>A</td>
<td>1, 1, 1</td>
<td>0, 0, 0</td>
</tr>
<tr>
<td>B</td>
<td>1, 1, 1</td>
<td>0, 0, ?</td>
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Player 1 has A weakly dominating B in Left, but B weakly dominating A in Right

Look at player 1’s expected value of A and B to determine which strategy is trembling hand perfect

- \( EV^A = 1(1 - \varepsilon)(1 - \delta) + 1(\varepsilon)(1 - \delta) + 0(1 - \varepsilon)(\delta) + 0(\varepsilon)(\delta) = 1 - \varepsilon - \delta + \varepsilon\delta + \varepsilon - \varepsilon\delta = 1 - \delta \)
- \( EV^B = 1(1 - \varepsilon)(1 - \delta) + 0(\varepsilon)(1 - \delta) + 0(1 - \varepsilon)(\delta) + 1(\varepsilon)(\delta) = 1 - \varepsilon - \delta + 2\varepsilon\delta \)

\( EV^A - EV^B = \varepsilon - 2\varepsilon\delta = \varepsilon(1 - 2\delta) \)

For small \( \delta, EV^A - EV^B > 0 \) . A is better than B even though it doesn’t dominate (not even weakly dominate); the only time player 1 is better using B is if both opponents have errors (less likely than one of the two making an error)

Conclusion - since player 1 will play A, we can remove the other two Nash equilibria for trembling hand perfect (only (A,X,Left) remains); THP throws out weakly dominated strategies and more

Own Errors -

- Aim at A: \( EV^A(1 - \gamma) + EV^B(\gamma) \)
- Aim at B: \( EV^B(1 - \gamma) + EV^A(\gamma) \)

Difference: \( (1 - 2\gamma)(EV^A - EV^B) \); we know second term is > 0 so as long as \( \gamma < 1/2 \), player 1’s choice is independent of \( \gamma \)

Realistic? - not really; football teams look at probability of their own errors (interceptions, fumble, etc.)
Extensive Form

Tree Rules - a little more in depth on the details behind extensive form

- **Successor** - nodes that can be reached from a given node by following arrows
- **Immediate Successor** - node that is at the end of any arrow leading away from a given node
- **Predecessor** - analogous to successors except we trace backward through the tree; also applies to *immediate predecessor*
- **Path** - sequence of nodes that (1) starts with the initial node, (2) ends with a terminal node, and (3) has the property that successive nodes in the sequence are immediate successors of each other

**Rule 1** - every node is a successor of the initial node, and the initial node is the only one with this property

**Rule 2** - each node except the initial node has exactly one immediate predecessor; the initial node has no predecessors

**Rule 3** - multiple branches extending from the same node have different action labels

**Rule 4** - each information set contains decision nodes for only one of the players; if this weren't the case, at some point in the game, the players won't know who is to make a decision

**Rule 5** - all nodes in a given information set must have the same number of immediate successors and they must have the same set of action labels on the branches leading to these successors; if this weren't the case, the player would be able to distinguish between the nodes

**Equilibrium Path** - what we observe if we watch the game

**Refinements** - mostly target nodes that aren't on the equilibrium path because they don't affect expected payoffs in game

**Perfect Information** - players know everything that happened before (i.e., aware of previous decisions by other players); equivalent to saying all information sets have only 1 node or saying it's a sequential game; beliefs are automatic (1 or 0 because player knows opponents' actions)

**Imperfect Information** - information set has 2 or more nodes; in these cases, player must have belief about opponent's actions (i.e., assign a probability distribution)

**Refinement** - in general the probability distribution assigned doesn't matter if the information set is not on the equilibrium path, but refinement says people shouldn't believe someone will do something that's not rational

**Sequential Rationality** - players ought to demonstrate rationality whenever they are called on to make decisions; optimal strategy for a player should maximize his or her expected payoff, conditional on every information set at which this player has the move

**Common Knowledge** - if sequential rationality is common knowledge, then each player will "look ahead" to consider what players will do in the future in response to his move at a particular information set

**Conditionally Dominated** - strategy $s_i$ for player $i$ is conditionally dominated if, contingent on reaching some information set of player $i$, there is another strategy $\sigma_i$ that strictly dominates it

**Backward Induction** - simple version of finding conditionally dominated strategies; process of analyzing game from back to front; at each information set, remove non-optimal actions so set essentially becomes a terminal node; works best with perfect information because can't deal with information sets with multiple nodes; essentially same as eliminating weakly dominated strategies
Result - every finite game with perfect information has a pure-strategy Nash equilibrium; backward induction identifies an equilibrium

Good - generally yields unique equilibrium (most other refinements don't)

Bad - only works with finite time horizon

Example - strategies for player 3: (L,L,L), (R,L,L), etc. get identical payoffs with (*,L,L) so only difference comes from L or R in first component .: if one of these is better than the other, it weakly dominates (only 1 payoff is different)

Example - using backward induction, we eliminate strategy F for player 1 and strategy D for player 2; move to next iteration; now eliminate strategy A for player 2; eliminate strategy R for player 1; Nash equilibrium therefore is (LE, BC), that is, strategies L and E for player 1 and B and C for player 2

Subgame Perfect - if game comes to initial node of a subgame, the game will progress as if it were the subgame so we should find Nash equilibria to all subgames (don't believe people will do irrational things); closely related to throwing out weakly dominated strategies

Subgame - can't break up an information set and has to start from a single node;

Note: in game of perfect information, every node initiates a subgame

Example - in this game, it is not credible for player 2 to say he will retaliate; look at subgame starting with player 2's decision, Nash equilibrium is (Attack, No); Note: (No, Retaliate) is not a plausible equilibrium because it ignores the real-time dimension of the game

Multiple Nodes - still assume optimizing; first look for dominated strategies, then take limit of beliefs (sequence of believes as errors → 0; vs. trembling hand perfect which is sequence of equilibria as errors → 0)

Example - 11.3. This exercise explores how, in a mixed-strategy equilibrium, players must put positive probability only on best responses. Consider the game in the following figure.
Compute the pure and mixed Nash equilibria for this game and note how they depend on $x$. In particular, what is the difference between $x > 1$ and $x < 1$?

**Note:** $p_1, p_2, p_3 \& q_1, q_2, q_3$ are probabilities each player assigns to a strategy when playing a mixed strategy; $p_3 = 1 - p_2 - p_2$ and $q_3 = 1 - q_1 - q_2$ (only use $p_3$ and $q_3$ to make notation look simpler)

$(U, L)$ is only pure strategy Nash equilibrium (as long as $x > 0$)

**Mixed strategies** - have to look at all combinations

**Fully Mixed** - using all three strategies implies $EV_U = EV_C = EV_D$ and $EV_L = EV_M = EV_R$

$EV_U = xq_1 + xq_2 + xq_3 = x$

$EV_C = 2q_2$

$EV_D = 2q_3 = 2(1 - q_1 - q_2)$

$x = 2q_2 \Rightarrow q_2 = x/2$

$2(1 - q_1 - q_2) = x \Rightarrow (1 - q_1 - x/2) = x/2 \Rightarrow q_1 = 1 - x$

Note: this strategy requires $x < 1$ (in order to have $q_1 < 1$); can have $x = 1$, but that results in degenerate solution (one of the three strategies will have probability 0 so it'll be same as paired strategy case)

$EV_L = xp_1 + xp_2 + xp_3 = x$

$EV_M = 2p_3 = 2(1 - p_1 - p_2)$

$EV_R = 2p_2$... these are basically the same as player 1's

**Fully mixed strategy:** $p_1 = q_1 = 1 - x$; $p_2 = p_3 = q_2 = q_3 = x/2$

**Paired Strategies** - using strategies two at a time has 3 cases for each player; looking at all combinations means $3 \times 3 = 9$ possibilities; first check for dominated strategies so we do least amount of work possible.

$(U,C)$ and $(L,M)$ (i.e., $p_3 = q_3 = 0$) $\Rightarrow$ L dominates M (i.e. not a Nash equilibrium)

$(U,C)$ and $(L,R)$ (i.e., $p_3 = q_3 = 0$) $\Rightarrow$ U dominates C

$(U,D)$ and $(M,R)$ (i.e., $p_3 = q_3 = 0$) $\Rightarrow$ R weakly dominates M

$(U,D)$ and $(L,M)$ (i.e., $p_2 = q_3 = 0$) $\Rightarrow$ U dominates D

$(U,D)$ and $(L,R)$ (i.e., $p_2 = q_2 = 0$) $\Rightarrow$ L dominates R

$(U,D)$ and $(M,R)$ (i.e., $p_2 = q_3 = 0$) $\Rightarrow$ M weakly dominates R

$(C,D)$ and $(L,M)$ (i.e., $p_3 = q_3 = 0$) $\Rightarrow$ L dominates M

$(C,D)$ and $(L,R)$ (i.e., $p_1 = q_2 = 0$) $\Rightarrow$ L dominates R

$(C,D)$ and $(M,R)$ (i.e., $p_3 = q_3 = 0$) $\Rightarrow$ nothing dominates... can have a mixed strategy

$EV_C = 2q_3 = EV_D = 2q_3 = 2(1 - q_1 - q_2) \Rightarrow q_2 = q_2 = 1/2$

$EV_M = 2p_3 = 2(1 - p_1 - p_2) = EV_R = 2p_2 \Rightarrow p_2 = p_3 = 1/2$

This only looked at first rule ($EV$ of all strategies being used are equal); now need to check second rule ($EV$ of strategies being used $\geq EV$ of strategies not used); $EV_C = EV_D \geq EV_U \Rightarrow 2q_2 \geq x \Rightarrow x \leq 1$ (same result when looking at $EV_M = EV_R \geq EV_L$)

**Solution** -

$x > 1$ 1 equilibrium: $(U,L)$ (pure strategy only)

$x = 1$ 2 equilibria: $(U,L)$ (pure strategy) and $p_1 = q_2 = 0$ and $p_2 = p_3 = q_2 = q_3 = 1/2$

$x < 1$ 3 equilibria: previous 2 plus $p_1 = q_1 = 1 - x$; $p_2 = p_3 = q_2 = q_3 = x/2$

**Theorem** - generally will have odd number of equilibria; case of even number is generally degenerate... like polynomial: