Duopoly Examples

Painful micro-esque detail on finding best response functions

Cournot Duopoly

Assumptions - decision made simultaneously; firms make identical product and compete by setting quantity (Q); take products to auction where price is determined based on total output by both firms; assume risk neutral so we can use expected profit & not worry about utility

Real World - makes sense when firms make Q decision well before output is realized (e.g., agriculture)

Cost - C(Qi); function of individual firm’s output

Price - P(Q1 + Q2); function of total output (that’s where the interaction comes in)

Profit - \( \pi'(Q_1, Q_2) = P(Q_1 + Q_2) \cdot Q_1 - C'(Q_1) \)

Strategy - select \( Q_1 \in [0, Q] \) to maximize expected profit; we can always set Q large enough to avoid corner solutions

Best Reply - given belief that firm 2 will produce \( Q_2 \), firm 1 will max profit by producing \( Q_1 = R'(Q_2) \), best reply function (firm 1’s optimal choice); find by solving:

Max \( \pi'(Q_1, Q_2) = P(Q_1 + Q_2) \cdot Q_1 - C'(Q_1) \)

set 1st derivative = 0 and solve for \( Q_1 \)

Interaction - to find how \( Q_1 \) varies with \( Q_2 \), take total derivative of 1st order cond wrt \( Q_2 \):

\[
\begin{align*}
\frac{dQ_1}{dQ_2} &= \left( -\frac{\partial^2 P}{\partial Q^2} Q_1 + \frac{\partial P}{\partial Q} Q_1 + \frac{\partial P}{\partial Q} \right) \\
\frac{dQ_2}{dQ_1} &= \left( -\frac{\partial^2 P}{\partial Q^2} Q_2 + \frac{\partial P}{\partial Q} \right)
\end{align*}
\]

Another way of looking at this is to work directly with \( \pi'(Q_1, Q_2) \)
1st Order: \( \frac{\partial \pi^1}{\partial Q_1} = 0 \)

Total derivative wrt \( Q_2 \): \( \frac{\partial^2 \pi^1}{\partial Q_2} \cdot dQ_1 + \frac{\partial^2 \pi^1}{\partial Q_1 \partial Q_2} = 0 \)

Assume \( \frac{\partial^2 \pi^1}{\partial Q_1^2} < 0 \) (sufficient 2nd order condition for maximization problem)

\[ \therefore \frac{dQ_1}{dQ_2} \text{ has same sign as } \frac{\partial^2 \pi^1}{\partial Q_1 \partial Q_2} = \frac{\partial^2 P}{\partial Q^2} \cdot \frac{Q_1}{\partial Q} + \frac{\partial P}{\partial Q} ; \text{ 1st term usually } > 0; \text{ 2nd is } < 0 \]

**Linear Demand** - to get further results we assume linear demand so

\[ \frac{\partial^2 P}{\partial Q^2} = 0 \]; that means \( \frac{dQ_1}{dQ_2} = \frac{\partial P}{\partial Q} < 0 \) (i.e., \( Q_2 \uparrow \Rightarrow Q_1 \downarrow \))

**Note:** this is with linear assumption; in theory it’s possible to have \( \frac{dQ_1}{dQ_2} > 0 \) if demand is convex enough (i.e., \( \frac{\partial^2 P}{\partial Q^2} > 0 \))

**Constant Marginal Cost** - also assuming no fixed cost and identical, constant marginal cost \( (c) \) for both firms (i.e., \( C^1(Q_1) = cQ_1 \))

**Monopoly Output** - what a firm would produced if the other firms produces zero

**Competitive Output** - the only way a firm can drive the opponent to produce zero output is to product at the competitive output level \( (P = MC) \)

**Symmetry** - since we assume each firm has identical marginal cost, graph of best response functions is symmetric; had we assumed otherwise, one of the two best response would be steeper (or flatter) than the other

**Equilibrium** - evident that equilibrium should be at the intersection of the two best response functions (neither player has an incentive to change his strategy... that’s a Nash equilibrium)

**Dominant Solvable** - iteratively eliminating dominated strategies yeilds same result (equilibrium at intersection)

**Isoprofit Curves** - level curves of a firm’s profit (e.g., \( \pi^1(Q_1, Q_2) = c \))

\[ \frac{\partial \pi^1}{\partial Q_2} < 0 \text{ - because } Q_2 \uparrow \Rightarrow P \downarrow \text{ and } \pi^1(Q_1, Q_2) = P(Q_1 + Q_2) \cdot Q_1 \cdot C^1(Q_1) \]

\[ \frac{\partial \pi^1}{\partial Q_1} = 0 \text{ at } Q_1^* \text{ (best response; i.e., any } Q_1 \text{ on } R^1(Q_2) \) \]

Shape - because of first order condition shown above, curves peak on the best reply curve \( (R^1(Q_2)) \); for all \( Q_1 < Q_1^* \), \( \frac{\partial \pi^1}{\partial Q_1} > 0 \) (moving up to max); for all \( Q_1 > Q_1^* \), \( \frac{\partial \pi^1}{\partial Q_1} < 0 \Rightarrow 0 \) (moving away from max); max profit occurs at \( Q_m \) (monopoly output) so isoprofit curves get better as you move down to \( Q_m \)

**First Round** - looking at isoprofit curves, it would never make sense to play any quantity greater than \( Q_m \); regardless of the opponent’s output, any \( Q > Q_m \) will be on an inferior isoprofit curve \( . \) all \( Q > Q_m \) are strictly dominated

**Second Round** - look at opponent playing \( Q_m \); let \( Q^* \) be the firm’s best response (based on the best reply curve, \( R^1(Q_m) \)); all \( Q < Q^* \) are strictly dominated because they are on inferior isoprofit curves

**Third Round** - look at firm play \( Q^* \) from previous round; let \( Q' \) be the firm’s best response (based on the best reply curve); all \( Q > Q' \) are strictly dominated because they are on inferior isoprofit curves
More Rounds - this process continues indefinitely and gets a tighter and tighter bound on the intersection of the best reply curves; each round is more complicated to find the dominated strategies (that's why some theorists don't like iterative dominance); Note: didn't need the nonlinear assumption for this conclusion, just downward sloping $R^1$ and $R^2$ (but it makes the graphs easier)

Problem - infinite iterative dominance gets to equilibrium, but that's not what people do it real life (don't have that kind of time!); this problem is fairly simple and we can look at the graph and figure out the equilibrium will be at the intersection of the best reply curves; problem comes in games that have multiple intersections; each may be a Nash equilibrium, but how do we get there?

Fictional Referee - takes player's inputs and asks them to rechoose if the result is not an equilibrium

Bertrand Duopoly

Assumptions - similar to Cournot model except firms compete on price, not quantity; extensive form looks identical (just change $Q$ to $P$ in the decisions); will assume same constant marginal cost ($c$) for both firms to make this easier; also assume firms equally split the market if they charge the same price and firms cannot collude

Real World - makes sense when firms make $P$ decision and then produce (i.e., decide $Q$) based on orders (e.g., catalog sales)

Cost - $cQ'(P_1, P_2)$; quantity is function of price of both firms

Profit - $\pi(P_1, P_2) = P_iQ_i(P_1, P_2) - cQ_i(P_1, P_2)$

Best Reply - three cases:
- $P_1 > P_2$ - firm 1 sells nothing so $\pi^1 = 0$
- $P_1 = P_2$ - firms evenly split market so $\pi^1 = \pi^2 = \pi^1(P_1 + P_2, \infty)$ (profit if firm 1 charged price $P_1 + P_2$ and firm 2 charging more so it gets no sales)
- $P_1 < P_2$ - firm 1 sells everything and profits as if it were a monopolist; firm 1's best reply depends on relationship between $P_2$ and $P_m$
  - $P_2 > P_m$ - firm 1 should charge $P_m$ to get monopoly profit
  - $P_2 < P_m$ - firm 1 should charge slightly less than $P_2$

Problem - there is technically no best reply because function is not continuous at point of firm 1's best reply (wants the open dot to max profit, but at that point it drops to the solid dot)

Dominant Solvable - look at various cases to see what can be ruled out
- $P_1 > P_2 > MC$ - firm 1 wouldn't do this; it could do better be charging less than $P_2$
- $P_1 > P_2 = MC$ - firm 2 wouldn't do this; it could do better by charging more
- $P_1 > MC > P_2$ - firm 2 would lose money so it wouldn't do this
\[ P_1 = P_2 > MC \] - either firm will do better by lowering price
\[ P_1 = P_2 < MC \] - both firms losing money
\[ P_1 = P_2 = MC \] - neither firm can do anything to improve

**Equilibrium** - set of strategies each firm chooses with no incentive to change; in this case an equilibrium point is \( P_1 = P_2 = MC \)

**Lesson** - equilibrium can still exist when best reply doesn’t

**Non-Constant** \( MC \) - can’t rule out possibility that firm 2 sells with \( P_2 > P_1 \) because firm 1 may not be able to supply enter market at \( P_1 \) (see graph); this requires more complicated analysis and may find that there is no pure strategy equilibrium

**Differentiated Product Model** - more interesting results; may smooth out discontinuity so that best reply exists

**Stacklburg Duopoly**

**Assumptions** - similar to Cournot model except one firm decides output before the other (let’s say firm 1 goes first; so firm 2 knows \( Q_1 \) before deciding on \( Q_2 \))

**Cournot Equilibrium** - \( Q_1^C \) and \( Q_2^C \) are quantities produced by each firm in Cournot equilibrium; note that these quantities are the intersection of the two firms’ best response curves \( R^1(Q_2) \) and \( R^2(Q_1) \); each curve being where a firm maximizes profit given it’s opponents output; i.e., point were \( d\pi = 0 \); if firm 1 produces at \( Q_1 = Q_1^C \), then best reply for firm 2 is \( Q_2^C \); \( \cdot \cdot \) the Cournot equilibrium is also an equilibrium for the Stacklburg game (but not the only one)

**Other Equilibria** - firm 1 knows that firm 2 will play its best response to firm 1’s output; firm 1 can then view firm 2’s best response function \( R^2(Q_1) \) as it’s feasible set and try to maximize its profit; this occurs at the point were \( R^2(Q_1) \) is tangent to one of firm 1’s isoprofit lines (that’s the best firm 1 can do); if firm 1 selects this level of output firm 2 will respond with it’s best response and this is another equilibrium to the game (this is the Stacklburg equilibrium); Note: firm 1 prefers this new equilibrium because it is better off than in the Cournot equilibrium

**Credible Threat** - firm 2 would prefer the Cournot equilibrium; it may try to get there by threatening firm 1 and saying that it will produce \( Q_2^C \) regardless of what firm 1 produces in order to force firm 1 to produce \( Q_1^C \)

**Subgame Perfection** - once firm 1 has decided what it will do, the only logical choice for firm 2 is to proceed with it’s best reponse; any other action (e.g., playing \( Q_2^C \) because of a threat) would hurt firm 2; \( \cdot \cdot \) the threat to produce \( Q_2^C \) is not credible (firm 2 will not commit to something that is not optimal)

**Solution** - firm 2 can look for some type of commitment technology to convince firm 1 that firm 2 will produce \( Q_2^C \) regardless of what firm 1 does; real world examples:

- **Airlines** - airlines use agents with very limited bargaining power to make fares (threats) credible; otherwise, passengers would haggle over prices and based on subgame perfection, airlines would accept them

- **Cold War** - USA and USSR operated on strategy of mutually assured destruction (if either side attacked, the opponent would respond with a massive nuclear attack that would eventually destroy both sides); problem was that this wasn’t an optimal
response so it wasn’t a credible threat; *Dr. Stangelove* (fictional movie) talked about an automated response in order to make the threat credible

**Refinements** - theorists starting to incorporate probability that a player will respond irrationally (for all models, not just Cournot, Bertrand, & Stacklburg)

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**Total Differential vs. Total Derivative**

\[ z = f(x, y) \]

Total Differential: \[ dz = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy \]

Total Derivative with respect to \( x \): \[ \frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} \]