Analyzing Games

Strategic Tensions - listed throughout book, summarized here:

First - clash between individual and group interests (prisoners' dilemma); can be resolved with commitment devices (threats, legally binding contracts, etc.) which will change payoffs or other part of the strategic setting

Second - strategic uncertainty; rationalizability doesn't always lead to a unique strategy profile; even if it does, rationalizability only requires players' behavior and beliefs be consistent with rationality, which doesn't mean the beliefs are correct; can be resolved through institutions, rules, norms of behavior and culture that facilitate coordination in society

Third - inefficient coordination; Nash equilibrium (so neither side has a unilateral incentive to deviate) that is not Pareto optimal

Strictly Ordered Game - clear ranking among alternatives; lots of different ways to set this up (72), but number of insights are limited

Three Basic Insights -
- Dominant Strategy for 1 (or both) players
- Multiple Equilibria -
- No Equilibrium -

Static Games - all players' actions are taken simultaneously and independently; also called one-shot games

Dominance

Strictly Dominated Strategy - something else is better in every contingency (every opponent's strategy); A2 is strictly dominated by A2 if payoff for A1 > payoff for A2 for all opponent's strategies B1, B2, ..., Bn

Weakly Dominated - same as strictly, but can have some payoffs = (others are >)

Rationality Requirement - minimum requirement for player to be rational is to not play a strictly dominated strategy

Pure Strategy Domination - one strategy is better than another

Mixed Strategy Domination - combination of strategies is better than the dominated strategy; harder to find because probabilities don't have to be 50/50 and could be more than just two strategies; trick is to look for alternating patterns of large and small numbers in payoff matrix

Example - play A1 half the time and A2 half the time; expected payoff for player 1 is 0.5(10) + 0.5(0) = 5 for both of player 2's strategies; that means A3 is strictly dominated

Formally - ∃p such that px_i + (1 - p)y_i > z_i for i = 1, 2, ..., n (all of opponent's strategies)

When Exist?

- If both > 0, A1 dominates A2
- If both < 0, A2 dominates A1

px_1 + (1 - p)y_1 > z_1 ⇒ p(x_1 - y_1) > z_1 - y_1
px_2 + (1 - p)y_2 > z_2 ⇒ p(x_2 - y_2) > z_2 - y_2
Assume x_1 > y_1 and x_2 < y_2; (works other way too):
\[ p > \frac{z_1 - y_1}{x_1 - y_1} \quad \text{and} \quad p < \frac{z_2 - y_2}{x_2 - y_2} \]

\[ . \quad \therefore \frac{z_1 - y_1}{x_1 - y_1} < \frac{z_2 - y_2}{x_2 - y_2} \]

A mixed strategy \( (p \) for \( A_1 \) and \( (1 - p) \) for \( A_2) \) dominates \( A_3 \)

**Expectations** - have to be taken with utilities, not payoffs; problem with that is payoffs ($) are observable and utilities aren't; solution is to use payoffs and assume risk neutral

**Risk Aversion** - built into utility values so any unhappiness resulting from randomness is already built in (i.e., can't argue a sure \((4, 4)\) is better than an expected \((5, 5)\))

**Principal-Agent Problem** - if owner has lots of workers, it's usually OK to assume risk neutral owner

**Iterated Dominance** - process of eliminating strategies that are strictly dominated; go through for each player looking for a dominated strategy; if one is found for any player, go through the sub-game (with the dominated strategies removed) and look for more strictly dominated strategies; repeat until there are no more strictly dominated strategies

**Simple Game** - only eliminate dominated strategies

**Example** - no dominant strategy for player 1, but can see that player 2 has a dominant strategy \((B_2)\); \[\ldots\] game is simplified to having player 1 pick between the 0 and 10 of \( A_1 \) and \( A_2 \)

**Iterative** -

**Example** - can remove last row because \( A_3 \) is strictly dominated by mixed strategy involving \( A_1 \) and \( A_2 \) (50-50 as in previous example); now look for another strategy that's strictly dominated \((B_1)\) is by \( B_2 \)

**Debate** - some theorists don't like iterative process because it's too complicated to look for dominated strategies

**Rationalizable Strategies** - set of strategies that survive iterated dominance

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**Efficiency**

**More Efficient** - strategy \( s \) is more efficient than \( s' \) if all the players prefer (or are indifferent between) the outcome of \( s \) to the outcome of \( s' \) and the preference is strict for at least one player

**Pareto Efficient** - \( s \) is Pareto efficient (or Pareto optimal) if there is no other strategy profile that is more efficient (i.e., cannot improve payoff to any player without hurting other players)

**Why Care** - efficiency doesn't lead us to an equilibrium, but allows us to talk about the relative quality of multiple equilibria (or even a single one)

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**Best Response**

**What To Play** - depends on beliefs of what opponent will do

**Beliefs** - can be decided either by dominance criteria or by probabilities; theorists argue this point; some says dominant strategy is all there is; others argue for beliefs about opponent based on the game (e.g., iterative dominated strategies)

**Best Response** - lists all of a player's best choices given each of the opponents' choices; response is to belief, not to opponent so theorists argue the name (reply, reaction, etc.)
Formally - $BR_i(\mu_i)$ or $R_i(\mu_i)$ is set of best responses $s_i \in S_i$ such that $u_i(s_i, \mu_i) \geq u_i(s_i', \mu_i)$ for every $s_i' \in S_i$.

Finding It - assume opponent will pick 1 strategy then find best reply; move to next strategy for opponent and pick the best reply to that; this generates a reaction function or best response.

Strategic Form Only - only use best reply function for strategic form; extensive form shows order of play so there could actually be a response.

Example - supposed player 1’s belief is that player 2 will play L with probability 1/3, C with probability 1/2, and R with probability 1/6; look at player 1’s payoffs:

\[ u_1(U, \mu_2) = (1/3)2 + (1/2)0 + (1/6)4 = 8/6 \]
\[ u_1(M, \mu_2) = (1/3)3 + (1/2)0 + (1/6)1 = 7/6 \]
\[ u_1(D, \mu_2) = (1/3)1 + (1/2)3 + (1/6)2 = 13/6 \]

\[ \therefore BR_1(1/3, 1/2, 1/6) = \{D\} \]

Dominance vs. Best Response - for any finite game, the set of best responses for a player will be a subset of the set of strategies that are not strictly dominated; for a finite, two-player game the sets will be the same.

Analysis Steps -

1. Look for strategies that are best responses to simplest beliefs (pure strategies); these strategies obviously cannot be strictly dominated.
2. Look for strategies that are strictly dominated by other pure strategies.
3. Look for strategies that are strictly dominated by mixed strategies.

Nash Equilibrium

Nash Equilibrium - set of strategies, one for each player, $(s_1^*, s_2^*, \ldots, s_n^*)$ where $s_i^* \in S_i'$ such that $\forall i, u_i(s_i^*, s_i^*) \geq u_i(s_i, s_i^*) \forall s_i \in S_i'$.

English - each player’s strategy is a best response to the best responses of the other players; no player has an incentive to change his choice.

Finding It - for each player underline his best response to each of the opponents’ strategies; any cell (set of strategies) that has both payoffs underlined is a Nash equilibrium.

Nash ≠ Optimal - Nash equilibrium is not necessarily Pareto optimal (e.g., prisoners’ dilemma).

Strong Nash Equilibrium - some theorists try to expand Nash equilibrium to look at coalitions of players not being able to improve by deviating.

Problems - (1) when strong Nash equilibrium exists, problem is trivial, (2) how do players form coalitions? if side payments to get players to cooperate, that changes the game, (3) cost of coordination; if it’s too high, strong Nash won’t happen.

Strict Nash Equilibrium - uses $>$ instead of $\geq$; argument is that it can’t be equilibrium is player can pick a different strategy with the same payoff.

Problem - rarely exists.

Infinite Strategies - if game has infinite strategy spaces, compute best response mapping ($BR_i$, as function of opponents’ strategies); solve system of equations to

Pareto Optimal

Nash Eq. (also dominant eq.)
find where best response functions intersect (see duopoly examples or mixed-strategy Nash equilibrium below)

**Nash Theorem** - if (1) $S^i$ is compact and convex for every $i$ and $U^i$ is (2) jointly continuous in $s_i$ and $s_{-i}$ (i.e., everybody's strategies) and is (3) quasiconcave in $s_i$ (i.e., own strategy) then a Nash equilibrium exists

- $S^i$ **Bounded** - if not bounded, could have possibility of no best reply
- $S^i$ **Compact** - if not compact, could also have possibility of no best reply
- $S^i$ **Convex & $U^i$ Quasiconcave** - best replies are convex sets
- $U^i$ **Jointly Continuous** - guarantees best reply changes with respect to opponent's strategy in well behaved way; appeal to fixed point theorem (shows if system of equations has solution) to show that best replies intercept

**Checking 2 & 3** - find best reply functions, show they're well behaved and intersect; the intersection is the Nash equilibrium

**Mixed-Strategy Nash Equilibrium**

**Loosen Assumptions** - can still have Nash equilibrium without all these assumptions

- **Discrete Strategies** - if $S^i$ is discrete it isn't convex by definition, but can still have Nash equilibrium

- **Mixed Strategies** - all games with finite strategies will have a Nash equilibrium with mixed strategies (probability distribution over strategy space)

- **Strategy Space** - [0, 1]... this is compact (closed & bounded) and convex (line connecting any two points is still in the interval)

- **General Rule** - given finite number of strategies, any probability distribution over those strategies will be compact and convex

- **Expected Payoff** - $E(U^i) = apq + bp(1 - q) + c(1 - p)q + d(1 - p)(1 - q) = (a - b - c + d)pq + (b - d)p + (a + c)q + d$

  Continuous function of $p$ & $q$; also, if we fix $q$, this is linear in $p$ (which is quasiconcave)

**Mixed-Strategy Nash Equilibrium** - no player can change his probabilities and do better; **2 key criteria** (solve problems [i.e., find probabilities] with the first one, then check the second one based on the probabilities found in the first one):

1. If two (or more) strategies are played with positive probability by a player, then the expected value of those strategies are equal to each other
2. The expected payoff of strategies with positive probability must be at least as great as ($\geq$) the expected value from strategies with zero probability

**Expected Payoff**

Player 1

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<thead>
<tr>
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<th>H</th>
<th>T</th>
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<tbody>
<tr>
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<td>$a_2, x_1$</td>
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<tr>
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Player 2

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<td>$d_1$</td>
<td>$d_2$</td>
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<td>$d_4$</td>
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</tbody>
</table>

Assume $p_1 & p_2 > 0; p_3 & p_4 = 0$

**Criteria 1**

$\sum_{j=1}^{4} a_j q_j = \sum_{j=1}^{4} b_j q_j \geq \sum_{j=1}^{4} c_j q_j$

**Criteria 2**

$\sum_{j=1}^{4} a_j q_j = \sum_{j=1}^{4} b_j q_j \geq \sum_{j=1}^{4} d_j q_j$

**Weak Nash** - since the criteria outlined above requires $\geq$ (vs. $>$), mixed-strategy Nash equilibrium is a form of weak Nash equilibrium (not strict equilibrium)
Matching Pennies - has no Nash equilibrium; play with mixed strategy (e.g., player 1 picks H with probably $p$ and T with probability $1 - p$

Best Reply for Player 1 - look at boundary points first
- $q = 1 \Rightarrow$ best reply is H ($p = 1$)
- $q = 0 \Rightarrow$ best reply is T ($p = 0$)

In Between - compare expected payoffs for player 1 using H and T
- $EV_H^1 = 1q - 1(1 - q) = 2q - 1$
- $EV_T^1 = -1q + 1(1 - q) = 1 - 2q$

At $q$ near 0, $EV_T^1 > EV_H^1$, so player 1 should play T; At $q$ near 1,
- $EV_H^1 > EV_T^1$, so player 1 should play H
- $EV_H^1$ and $EV_T^1$ converge at $q = 0.5$ at which point player 1 is indifferent between H and T

Best Reply for Player 2 - similar argument to player 1
- $p = 1 \Rightarrow$ best reply is T ($q = 0$)
- $p = 0 \Rightarrow$ best reply is H ($q = 1$)

Nash Equilibrium - best replies intersect at $p = q = 0.5$

General Case - payoffs for mixed strategies are linear in probability so will always have best replies like this (stay on one value and at some $p$ or $q$, player is indifferent)

Expected Value - find $EV_i^k$, expected value of player $i$ playing strategy $k$

What to Play - assign probability 1 to single strategy with largest expected value; otherwise split probability equally between all strategies that tie for largest expected value

Find Split - to find where player is indifferent, set expected values equal to each other and solve for opponent's probability

Coordination Game - has two pure strategy Nash equilibria and one mixed strategy Nash equilibria

Purification - turning a mixed-strategy into a pure strategy by conditioning on some "irrelevant" piece of information not known to the opponent (or opponent knows info, but doesn't know how the info is being used); randomness introduced by nature (not by the player) so the player doesn't bear the cost of randomization

Poker Example - have a hand [3D, 3H, QC, QD, *]; if the fifth card is > 7, bet; if it's ≤ 7 don't bet

Debate - some theorists don't like mixed-strategy equilibrium idea, but applied economists like any equilibrium because they look at an equilibrium and predict how it'll change (with comparative statics or other tool)

Empirical Data - can never confirm an equilibrium (never have enough info), but can test direction of change of an equilibrium (e.g., if we do X, what happens to the equilibrium)
Mixed-Strategy Results - comparative statics of mixed-strategy equilibria generally aren't intuitive

If $a_1 \uparrow$, typical intuition says player 1 is more likely to play row 1.

But assume there is a mixed-strategy equilibrium for each player.

Criteria 1 says:

\[
\begin{align*}
  a_1q + a_2(1 - q) &= a_3g + a_4(1 - q) \\
  b_1p + b_2(1 - p) &= b_3p + b_4(1 - p)
\end{align*}
\]

Solving for $p$ and $q$ shows $q = \frac{a_4 - a_2}{a_1 + a_4 - a_2 - a_3}$ and $p = \frac{b_4 - b_2}{b_1 + b_4 - b_2 - b_3}$

∴ $a_1 \uparrow \Rightarrow q \downarrow$; row player's probabilities doesn't depend on his payoffs; goal of mixed-strategy equilibrium is to make opponent indifferent so $p$ only depends on opponent's payoffs.

Critics - OK, you get non-obvious results with mixed-strategy equilibrium, but they may not be useful results.

Local vs. Global Results - a local result depends on derivatives in the immediate area (e.g., stay with a mixed-strategy), but the global result could be totally different; not uncommon to have local & global comparative statics move in different directions.

Equilibrium Correspondence - looks at equilibrium versus some payoff (e.g. probability of going to war $[(1 - p)(1 - q)]$ versus cost of going to way for player 1 $[a_1]$).

Local Result - stay in mixed-strategy equilibrium; $a_1 \uparrow \Rightarrow$ increased chance of war (not intuitive result).

Global Result - change $a_1$ enough and we get global result: $a_1 \uparrow \Rightarrow$ no war.

Slutsky - to find local comparative statics "typically the way we do it is we differentiate the hell out of everything".

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Slutsky - Chapter 9 is "reasonably incomprehensible" and congruous is a "word he just made up".

Caution - don't be swayed by arguments based on large differences in payoffs; can be fooled into "what is reasonable", but it doesn't matter because a transformation can always change the maximum and minimum payoffs to be as close together (or far apart) as you want (based on Von Neumann-Morgenstern expected utility theory); example looking at payoffs for row player only and assuming (A1,B1) is Nash equilibrium; seems reasonable to play A2 in order to eliminate risk from irrational opponent (or mistake); after all, difference in payoff is very small (10 vs. 9.999), but difference from mistake is huge; problem is you can use a transformation $V = aU + b$ to get to the less dramatic game next to it ($\varepsilon$ is some small number); in fact, you could get the payoffs to be 0.99 instead of 0 (it'll make $\varepsilon$ even smaller).

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### Player 1

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<tr>
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### Player 2

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</tr>
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