Limits and Continuity

Sequence - function from positive integers to real numbers; e.g., \( x_n = n+1, \ n = 1, 2, \ldots \)
Limit - sequence has a limit if it converges to a real number \( L \)
\[
\lim_{n \to \infty} x_n = L
\]
Example - \( x_n = 1/n; \lim_{n \to \infty} x_n = 0 \)

Rigorous Definition
\( f(x) \) has limit (or tends to) \( A \) as \( x \) tends to \( a \), and write \( \lim_{x \to a} f(x) = A \), if for each number \( \varepsilon > 0 \)
there exists a number \( \delta > 0 \) such that \( |f(x) - A| < \varepsilon \) for every \( x \) with \( 0 < |x - a| < \delta \)
In English - \( \lim_{x \to a} f(x) = A \) means that we can make \( f(x) \) as close to \( A \) as we want for all \( x \)
sufficiently close to (but not equal to) \( a \)
Using It - if asked to use the definition to prove a limit exists, you first assume any \( \varepsilon > 0 \) and
solve \( |f(x) - A| < \varepsilon \) for \( x \). Then use \( 0 < |x - a| < \delta \) to get a value for \( \delta \) in terms of \( \varepsilon \).

Theorems
1) If a sequence \( \{x_n\} \) is non-decreasing (\( \forall \ n \ x_{n+1} \geq x_n \)) and bounded from above (\( \exists L \) s.t. \( L \geq x_n \ \forall \ n \)), then the sequence \( \{x_n\} \) must converge

2) If a sequence \( \{x_n\} \) is non-increasing (\( \forall \ n \ x_{n+1} \leq x_n \)) and bounded from below (\( \exists L \) s.t. \( L \leq x_n \ \forall \ n \)), then the sequence \( \{x_n\} \) must converge

3) If a sequence is not monotonic, but has bounds, then the sequence may not converge, but it has convergent subsequences

   Example: \( x_n = 1 \) when \( n \) is even and -1 when \( n \) is odd; can't find a \( \delta \) to satisfy definition for \( \varepsilon < 1 \), but the subsequences are bounded at 1 and -1

   3a) if all the subsequences have the same limit, then the sequence has a limit

   Example: \( x_n = 1/n \) when \( n \) is even and -1/n when \( n \) is odd; converges to 0

4) If a sequence is not bounded, it will diverge (but can't say a sequence that is bounded necessarily converges... see #3)
Rules for Limits
If \( \lim_{x \to a} f(x) = A \) and \( \lim_{x \to a} g(x) = B \), then

a) \( \lim_{x \to a} A = A \)
b) \( \lim_{x \to a} (f(x) \pm g(x)) = A \pm B \)
c) \( \lim_{x \to a} (f(x) \cdot g(x)) = A \cdot B \)
d) \( \lim_{x \to a} (f(x) / g(x)) = A / B \) (if \( B \neq 0 \))
e) \( \lim_{x \to a} (f(x))^{pq} = A^{pq} \) (if \( A^{pq} \) is defined)
f) If functions \( f \) and \( g \) are equal for all \( x \) close to \( a \) (but not necessarily at \( x = a \)), then
\[ \lim_{x \to a} f(x) = \lim_{x \to a} g(x) \] whenever either limit exists.

Special Cases
Don't Exist - vertically asymptotic functions (\( \pm \infty \))
One-Sided - value depends on which side you approach the limit from
Infinite Limits - horizontally asymptotic functions

Vector Notation
\[ x = (x_1, x_2, \ldots, x_k) \]
Sequence of vectors - converge when \( |L - x_n| \) gets smaller

- Euclidean Distance - \( d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots} \)
- Taxi Distance - \( d(x,y) = |x_1 - y_1| + |x_2 - y_2| + \ldots \)
- ??? Distance - \( d(x,y) = \max(|x_1 - y_1|, |x_2 - y_2|, \ldots) \)

Neighborhood
Neighborhood of \( x \) is a region around that point with certain distance, \( \mathcal{K}(x, \varepsilon) = \{ y : d(x,y) < \varepsilon \} \)
i.e., a circle centered on \( x \) with radius \( \varepsilon \)

Limit Point - number \( x \) is a limit point of a set \( S \) if every \( \varepsilon \) neighborhood of \( x \) contains a point of \( S \)
other than \( x \)

Finite sets never have limit points

Interior Point - \( x \) is interior to \( S \) if \( \exists \varepsilon > 0 \), such that if \( y \in \mathcal{K}(x, \varepsilon) \) then \( y \in S \) (i.e., \( \mathcal{K}(x, \varepsilon) \subset S \))

Every interior point is a limit point, but not the other way around

Open Set - \( S \) is an open set if every element is an interior point

Closed Set - a set is closed if it contains all of its limit points (points on border are limit points of open sets even though the points aren't in the set)

Special Cases - only two sets can be both open and closed at the same time (\( \emptyset \) and \( U \))
If \( S \) is open, then \( S^c \) is closed

Limits of Functions (using neighborhoods)
\( f(x) \) has limit \( L \) at \( a \) if for each number \( \varepsilon > 0 \) there exists a number \( \delta > 0 \) such that if \( x \in \mathcal{K}(a, \delta) \) then \( f(x) \in \mathcal{K}(L, \varepsilon) \)
Continuity
Continuous - graph of the function has no breaks; formal definition:
\( f \) is continuous at \( x = a \) if \( \lim_{x \to a} f(x) = f(a) \)

Conditions:
1) function \( f \) must be defined at \( x = a \)
2) the limit of \( f(x) \) as \( x \) tends to \( a \) must exist
3) this limit must be exactly equal to \( f(a) \)

If only condition 1 isn't satisfied, it is a “removable” discontinuity

Some continuous functions
\( f(x) = c \) (a constant)
\( f(x) = x \)

Polynomials (they’re a sum of continuous functions)
\( R(x) = P(x)/Q(x) \) (where \( P(x) \) and \( Q(x) \) are polynomials and \( Q(x) \neq 0 \))

Intermediate Value Theorem
Let \( f \) be a continuous function for all \( x \). Let \( f(x_0) = a \) and \( f(y_0) = b \) where \( a < b \), then for any \( c \) between \( a \) and \( b \), \( \exists \ x \) between \( x_0 \) and \( y_0 \) such that \( f(x) = c \)

Proof 1 (outline) -
a) Create two sequences by if \( f[(x_0 + y_0)/2] < c \) then
\( x_1 = (x_0 + y_0)/2 \), else \( y_1 = \)

b) Show the sequences converge at \( c \)

Proof 2 (outline) -
a) Define \( A = \{ x : f(x) \geq c \ \text{and} \ x_0 \leq x \leq y_0 \} \) and \( B = \{ x : f(x) \leq c \ \text{and} \ x_0 \leq x \leq y_0 \} \)
b) Show \( A \) and \( B \) are closed sets
c) Show \( A \cap B \neq \emptyset \)
d) \( \exists \ x \in A \cap B \ldots f(x) \leq c \) and \( f(x) \geq c \) so \( f(x) = c \)

Properties of Continuous Functions
If \( f \) and \( g \) are continuous at \( a \), then
a) \( f + g \) and \( f - g \) are continuous at \( a \)
b) \( f \cdot g \) and \( f/g \) (if \( g(a) \neq 0 \)) are continuous at \( a \)
c) \( [f(x)]^{p/q} \) is continuous at \( a \) if \( [f(x)]^{p/q} \) is defined

d) \( f(g(x)) \) is continuous at \( a \) if both \( f(x) \) and \( g(x) \) are continuous at \( a \) (composites)

Limits of Continuous Functions
Just plug in value rather than taking the limit

Continuity and Differentiability
If \( f \) is differentiable at \( x = a \), then \( f \) is continuous at \( x = a \)