Nonlinear Dynamic Systems
Multiple Scales Analysis

The method of multiple scales considers an expansion that represents the response of the system to be a function of two or more independent variable (i.e. multiple time scales). For instance, the new independent time scales are related by \( \tau = t \) and

\[
\tau_n = \epsilon^n t \quad \text{for} \quad n = 1, 2, \ldots ,
\]

where \( t \) is the actual time, \( \epsilon \) is a small non-dimensional parameter, and each \( \tau_n \) represents a different time scale in the response of the system. This causes a change in the derivatives with respect to time

\[
\frac{d}{dt} = \frac{\partial}{\partial \tau} \frac{d\tau}{dt} + \frac{\partial}{\partial \tau_1} \frac{d\tau_1}{dt} + \frac{\partial}{\partial \tau_2} \frac{d\tau_2}{dt} = D_0 + \epsilon D_1 + \epsilon^2 D_2 ,
\]

\[
\frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + 2\epsilon^2 D_0 D_2 + \epsilon^2 D_1^2 .
\]

The common approach is that one assumes a solution in the form of an expansion

\[
x(\tau, \epsilon) = x_0(\tau, \tau_1, \tau_2) + \epsilon x_1(\tau, \tau_1, \tau_2) + \epsilon^2 x_2(\tau, \tau_1, \tau_2) ,
\]

where the number of terms in the expansion is equivalent to the number of independent time scales.

1 Quadratic restoring force

This section examines a system described by Eq. (4) and compares the results of a first order expansion to the results of a secorder multiple scales expansion.

\[
\ddot{x} + 2\zeta \dot{x} + \omega^2 x + k_2 x^2 = 0 ,
\]

where \( \zeta \) is the damping ratio, \( \omega \) is the system linear natural frequency, and \( k_2 \) is a nonlinear coefficient of a quadratic nonlinearity. To apply the method of multiple scales, it is convenient to introduce a change of variable in the above equation.
\[ \ddot{x} + 2\epsilon \mu \dot{x} + \omega^2 x + \epsilon \beta x^2 = 0, \quad (5) \]

where \( \zeta = \epsilon \mu \) and \( k_2 = \epsilon \beta \).

### 1.1 First order expansion: \( \mathcal{O}(\epsilon^1) \)

The assumed solution to Eq. (22) is written as a first order expansion

\[ x(\tau, \epsilon) = x_0(\tau, \tau_1) + \epsilon x_1(\tau, \tau_1), \quad (6) \]

where the independent time scales are defined as \( \tau = t, \tau_1 = \epsilon \tau \). It follows that the derivatives with respect to time become the following expansion terms of the partial derivatives with respect to the corresponding time scale.

\[
\begin{align*}
\frac{d}{dt} &= D_0 + \epsilon D_1, \quad (7a) \\
\frac{d^2}{dt^2} &= D_0^2 + 2\epsilon D_0 D_1. \quad (7b)
\end{align*}
\]

Substitutions into Eq. (22) give ......

\[
\left( D_0^2 + 2\epsilon D_0 D_1 \right) \left( x_0 + \epsilon x_1 \right) + 2\epsilon \mu \left( D_0 + \epsilon D_1 \right) \left( x_0 + \epsilon x_1 \right) + \omega^2 \left( x_0 + \epsilon x_1 \right) + \epsilon \beta \left( x_0 + \epsilon x_1 \right)^2 = 0 \quad (8)
\]

The next step is to separate terms into orders of epsilon \( \mathcal{O}(\epsilon) \)

\[
\begin{align*}
\mathcal{O}(\epsilon^0) : D_0^2 x_0 + \omega^2 x_0 &= 0, \quad (9a) \\
\mathcal{O}(\epsilon^1) : D_0^2 x_1 + \omega^2 x_1 &= -2D_0 D_1 x_0 - 2\mu D_0 x_0 - \beta x_0^2, \quad (9b)
\end{align*}
\]

where the terms of \( \mathcal{O}(\epsilon^2) \) and higher have been neglected as in the expansion defined by Eq. (6). The solution to the \( \mathcal{O}(\epsilon^0) \) equation is of the form

\[ x_0 = A(\tau_1)e^{i\omega \tau} + \bar{A}(\tau_1)e^{-i\omega \tau}, \quad (10) \]
where $A$ and $\bar{A}$ are complex conjugates. This solution is substituted into the $O(\epsilon^1)$, which requires the following terms

$$
D_0 x_0 = i\omega \left[ A(\tau_1)e^{i\omega \tau} - \bar{A}(\tau_1)e^{-i\omega \tau} \right]
$$

$$
D_0 D_1 x_0 = i\omega \left[ A(\tau_1)'e^{i\omega \tau} - \bar{A}(\tau_1)'e^{-i\omega \tau} \right],
$$

$$
x_0^2 = A(\tau_1)^2 e^{2i\omega \tau} + 2A(\tau_1)\bar{A}(\tau_1) + \bar{A}(\tau_1)^2 e^{-2i\omega \tau},
$$

(11a) (11b) (11c) (11d)

to obtain

$$
D_0^2 x_1 + \omega^2 x_1 = -2i\omega \left[ A(\tau_1)'e^{i\omega \tau} - \bar{A}(\tau_1)'e^{-i\omega \tau} \right] - 2\mu i\omega \left[ A(\tau_1)e^{i\omega \tau} - \bar{A}(\tau_1)e^{-i\omega \tau} \right] - \beta \left[ A(\tau_1)^2 e^{2i\omega \tau} + 2A(\tau_1)\bar{A}(\tau_1) + \bar{A}(\tau_1)^2 e^{-2i\omega \tau} \right]
$$

(12)

Next, we will eliminate the terms in Eq. (12) that cause an unbounded solution, these terms are commonly referred to as secular terms,

$$
-2i\omega \left[ A(\tau_1)' + \mu A(\tau_1) \right] e^{i\omega \tau} = 0.
$$

(13)

The polar form $A(\tau_1) = \frac{1}{2}a(\tau_1)e^{i\phi(\tau_1)}$ is now introduced into the above equation

$$
a' + ia\phi' + \mu a = 0
$$

(14)

Separating the above equation into real and imaginary components and then solving for $a$ and $\phi$ gives the following

$$
A(\tau_1) = \frac{1}{2}a_0 e^{(-\mu\tau_1 + i\phi_0)}
$$

(15)

where $a_0$ and $\phi_0$ are constants of integration. The solution to the $O(\epsilon^0)$ can now be written as

$$
x_0 = \frac{1}{2}a_0 \left[ e^{i(\omega\tau + \phi_0)} + e^{-i(\omega\tau + \phi_0)} \right] e^{-\mu\tau_1},
$$

(16)

Since the secular terms have been set to zero, the remaining $O(\epsilon^1)$ equation becomes
\[ D_0^2 x_1 + \omega^2 x_1 = -\beta \left[ A(\tau_1)^2 e^{2i\omega \tau} + 2A(\tau_1)\bar{A}(\tau_1) + A(\tau_1)^2 e^{-2i\omega \tau} \right], \quad (17) \]

\[ x_1 = \frac{\beta a_0^2}{\omega^2} \left[ -\frac{1}{2} + \frac{1}{12} \left( e^{2i(\omega \tau + \phi_0)} + e^{-2i(\omega \tau + \phi_0)} \right) \right] e^{-2\mu \tau_1}. \quad (18) \]

The total solution is written as

\[ x(\tau) = x_0(\tau, \tau_1) + \epsilon x_1(\tau, \tau_1) = \frac{1}{2} a_0 \left[ e^{i(\omega \tau + \phi_0)} + e^{-i(\omega \tau + \phi_0)} \right] e^{-\mu \tau_1} \]
\[ + \epsilon \frac{\beta a_0^2}{\omega^2} \left[ -\frac{1}{2} + \frac{1}{12} \left( e^{2i(\omega \tau + \phi_0)} + e^{-2i(\omega \tau + \phi_0)} \right) \right] e^{-2\mu \tau_1} \quad (19) \]

If we substitute the relationship for the time scale \( \tau_1 = \epsilon \tau \), we can write the final equation in terms of the original system parameters

\[ x(\tau) = \frac{1}{2} a_0 \left[ e^{i(\omega \tau + \phi_0)} + e^{-i(\omega \tau + \phi_0)} \right] e^{-\zeta \tau} + \frac{k_2 a_0^2}{\omega^2} \left[ -\frac{1}{2} + \frac{1}{12} \left( e^{2i(\omega \tau + \phi_0)} + e^{-2i(\omega \tau + \phi_0)} \right) \right] e^{-2\zeta \tau} \quad (20) \]

If an initial displacement is imposed \( (\vartheta_o) \) and the system is started from rest, the corresponding value for \( a_0 \) is found from the quadratic equation

\[ \frac{k_2}{3\omega^2} a_0^2 - a_0 + \vartheta_o = 0, \quad (21) \]

which provides the complication of multiple values for \( a_0 \). However, one can simply check which values of \( a_0 \) produce a more accurate result. A comparison between the \( \mathcal{O}(\epsilon^1) \) solution and numerical simulation is shown in Fig. 1.

### 1.2 Second order expansion: \( \mathcal{O}(\epsilon^2) \)

To apply a second order approach, it is convenient to modify Eq. (4) by introducing a change of variable

\[ \ddot{x} + 2\epsilon \mu \dot{x} + \omega^2 x + \epsilon^2 \beta x^2 = 0, \quad (22) \]

where \( \zeta = \epsilon \mu \) and \( k_2 = \epsilon^2 \beta \). The assumed solution to Eq. (22) is now written as a second order expansion
where the independent time scales are now defined as \( \tau = t, \tau_1 = \epsilon \tau, \tau_2 = \epsilon^2 \tau \). It follows that the derivatives with respect to time become the expansion terms of the partial derivatives with respect to the corresponding time scale as given in Eq. (2a) and Eq. (2b).

The substitution of Eq (23) and Eq. (2a,b) into Eq. (22) gives

\[
\left(D_0^2 + 2\epsilon D_0 D_1 + 2\epsilon^2 D_0 D_2 + \epsilon^2 D_1^2\right) \left(x_0 + \epsilon x_1 + \epsilon^2 x_2\right) + 2\epsilon \mu \left(D_0 + \epsilon D_1 + \epsilon^2 D_2\right) \left(x_0 + \epsilon x_1 + \epsilon^2 x_2\right) + \omega^2 \left(x_0 + \epsilon x_1 + \epsilon^2 x_2\right) + \epsilon^2 \beta \left(x_0 + \epsilon x_1 + \epsilon^2 x_2\right)^2 = 0 \quad (24)
\]

The next step is to separate terms into orders of epsilon \( \mathcal{O}(\epsilon) \)

\[
\mathcal{O}(\epsilon^0) : D_0^2 x_0 + \omega^2 x_0 = 0 , \quad (25a)
\mathcal{O}(\epsilon^1) : D_0^2 x_1 + \omega^2 x_1 = -2D_0 D_1 x_0 - 2\mu D_0 x_0 - \beta x_0^2 , \quad (25b)
\mathcal{O}(\epsilon^2) : D_0^2 x_2 + \omega^2 x_2 = -2D_0 D_1 x_1 - 2D_0 D_2 x_0 - D_1 x_0 - 2\mu D_0 x_1 - 2\mu D_1 x_0 - 2\beta x_0 x_1 . \quad (25c)
\]

where the terms of \( \mathcal{O}(\epsilon^3) \) and higher have been neglected as in the expansion defined by Eq. (6). The solution to the \( \mathcal{O}(\epsilon^0) \) equation is of the form
\[ x_0 = A(\tau_1, \tau_2)e^{i\omega \tau} + \bar{A}(\tau_1, \tau_2)e^{-i\omega \tau}, \]  

(26)

where \( A \) and \( \bar{A} \) are complex conjugates. This solution is substituted into the \( \mathcal{O}(\epsilon^1) \), which requires the following terms

\[
\begin{align*}
D_0 x_0 &= i\omega \left[ A(\tau_1, \tau_2)e^{i\omega \tau} - \bar{A}(\tau_1, \tau_2)e^{-i\omega \tau} \right] \\
D_0 D_1 x_0 &= i\omega \left[ D_1 A(\tau_1, \tau_2)e^{i\omega \tau} - D_1 \bar{A}(\tau_1, \tau_2)e^{-i\omega \tau} \right], \\
x_0^2 &= A(\tau_1, \tau_2)^2 e^{2i\omega \tau} + 2A(\tau_1, \tau_2)\bar{A}(\tau_1, \tau_2) + \bar{A}(\tau_1, \tau_2)^2 e^{-2i\omega \tau} \\
\end{align*}
\]

(27a)

(27b)

(27c)

to obtain

\[
\begin{align*}
D_0^2 x_1 + \omega^2 x_1 &= -2i\omega \left[ D_1 A(\tau_1, \tau_2)e^{i\omega \tau} - D_1 \bar{A}(\tau_1, \tau_2)e^{-i\omega \tau} \right] - 2\mu i\omega \left[ A(\tau_1, \tau_2)e^{i\omega \tau} - \bar{A}(\tau_1, \tau_2)e^{-i\omega \tau} \right] \\
&\quad - \beta A(\tau_1, \tau_2)^2 e^{2i\omega \tau} + 2A(\tau_1, \tau_2)\bar{A}(\tau_1, \tau_2) + \bar{A}(\tau_1, \tau_2)^2 e^{-2i\omega \tau}. \\
\end{align*}
\]

(28)

Next, we will eliminate the terms in Eq. (12) that cause an unbounded solution, these terms are commonly referred to as secular terms,

\[
-2i\omega \left[ D_1 A(\tau_1, \tau_2) + \mu A(\tau_1, \tau_2) \right] e^{i\omega \tau} = 0.
\]

(29)

The polar form \( A(\tau_1, \tau_2) = \frac{1}{2}a(\tau_1, \tau_2)e^{i\phi(\tau_1, \tau_2)} \) is now introduced into the above equation

\[
D_1 a(\tau_1, \tau_2) + i a(\tau_1, \tau_2)D_1 \phi(\tau_1, \tau_2) + \mu a(\tau_1, \tau_2) = 0
\]

(30)

Separating the above equation into real and imaginary components and then solving for \( a \) and \( \phi \) gives the following

\[
A(\tau_1) = \frac{1}{2} a_0(\tau_2) e^{-\mu \tau_1 + i\phi_0(\tau_2)}
\]

(31)

where \( a_0(\tau_2) \) and \( \phi_0(\tau_2) \) are constants of integration. The solution to the \( \mathcal{O}(\epsilon^0) \) can now be written as

\[
x_0 = \frac{1}{2} a_0(\tau_2) \left[ e^{i(\omega \tau + \phi_0(\tau_2))} + e^{-i(\omega \tau + \phi_0(\tau_2))} \right] e^{-\mu \tau_1},
\]

(32)
Since the secular terms have been set to zero, the remaining $\mathcal{O}(\epsilon^1)$ equation becomes

$$D_0^2 x_1 + \omega^2 x_1 = -\beta A(\tau_1, \tau_2)^2 e^{2i\omega \tau} + 2A(\tau_1, \tau_2)\bar{A}(\tau_1, \tau_2) + \bar{A}(\tau_1, \tau_2)^2 e^{-2i\omega \tau}, \quad (33)$$

The solution to the $\mathcal{O}(\epsilon^1)$ equation is of the form

$$x_1 = A_1(\tau_2)e^{i\omega \tau} + \bar{A}_1(\tau_2)e^{-i\omega \tau} + \frac{\beta a_0(\tau_2)^2}{\omega^2} \left[ -\frac{1}{2} + \frac{1}{12} \left( e^{2i(\omega \tau + \phi_0)} + e^{-2i(\omega \tau + \phi_0)} \right) \right] e^{-2\mu \tau}, \quad (34)$$

Substituting the $\mathcal{O}(\epsilon^0)$ and the $\mathcal{O}(\epsilon^1)$ equations into the $\mathcal{O}(\epsilon^2)$ equation results in

$$D_0^2 x_2 + \omega^2 x_2 = -2D_0 D_1 x_1 - 2D_0 D_2 x_0 - D_1 x_0 - 2\mu D_0 x_1 - 2\mu D_1 x_0 - 2\beta x_0 x_1. \quad (35)$$

$$D_0 x_1 = \quad (36a)$$
$$D_1 x_0 = \quad (36b)$$
$$D_0 D_2 x_0 = \quad (36c)$$
$$D_0 D_1 x_1 = \quad (36d)$$
$$x_0 x_1 = \quad (36e)$$

I will finish this some day....... But, not today.