1. A particle of mass $m$ is constrained to travel along the path shown in Figure 1, which is described by the following function

$$y(x) = -5x^2 + 10x^4,$$  \hspace{1cm} (1)

where $x$ is defined as the horizontal location of the particle. Develop the equation of motion for the system, find the fixed points for the system, classify the stability of each fixed point, and sketch the neighboring trajectories. Explain how the stable and unstable fixed point(s) match or do not match your intuition?

Figure 1: Graph of the curved path $y(x)$ in a vertical gravitational field.
Figure 2 is a schematic of a double pendulum undergoing parametric excitation (i.e. time periodic movement of the pivot point). The first pendulum is subjected to a harmonic displacement, of amplitude $A$ and frequency $\Omega$, oriented along an angle ($\alpha$) which is at an incline with the horizontal. Angular oscillations for the first pendulum, with an effective length of $L_1$, are described by $\theta_1$. The corresponding angular motions of the second pendulum, with an effective length of $L_2$, are described by $\theta_2$. Derive the equations of motion for this system.

Figure 2: Schematic of a double pendulum subject to parametric excitation at a tilted angle.
3. Use the method of isoclines to construct a phase portrait for the van der Pol equation

\[ \ddot{x} + \epsilon (x^2 - 1) \dot{x} + x = 0, \]  

using the following value $\epsilon = 0.5$. Plot the vector fields for $x$ vs. $\dot{x}$ over the range of $-5 < x < 5$ and $-5 < \dot{x} < 5$. Find the fixed point(s) for the system and classify their stability.

4. The drag force on a body immersed in a fluid, moving a very low Reynolds numbers, is commonly approximated with proportional damping. The equation of motion for a viscously damped pendulum undergoing free vibrations is given by Eq. 3. Plot the phase space diagram, using any method you would like, over the range of $-3\pi < \theta < \pi$ and $-4 < \dot{\theta} < 4$. Label the separatrix and the different basins of attraction. Does the system have fixed point attractors, periodic attractors, chaotic attractors or some combination of each type of attractor?

\[ \ddot{\theta} + \zeta \dot{\theta} + \omega^2 \sin \theta = 0. \]  

with parameters $\zeta = 0.05$ and $\omega = 1$ [rad/s].

Figure 3: Schematic of a pendulum undergoing free vibrations.
5. The contact of a rigid sphere with a perfectly elastic surface is often referred to as Hertzian contact [1–3]. The typical model for the motion of a vibration system in intimate contact with the surface is given by

\[ m \ddot{x} + c \dot{x} + kx + \frac{4}{3} E \sqrt{R} x^{3/2} = F_s + F_o \cos \Omega t, \]

(4)

Using the parameters listed below, simulate the response of the system and plot: 1) displacement vs. time; 2) a power spectrum of the displacement signal (PSD); 3) the phase space of the system (i.e. displacement vs. velocity); and 4) a Poincaré section for the system (i.e. periodic samples of the displacement vs. velocity).

\( m = 259.95 \times 10^{-6} \, \text{[Kg]} \), \( c = 0.0470 \, \text{[Ns/m]} \), \( k = 160.56 \, \text{[N/m]} \), \( E = 19 \times 10^9 \, \text{[N/m}^2\text{]} \)
\( R = 2 \, \text{[\mu m]} \), \( F_s = 100 \, \text{[\mu N]} \), \( F_o = 20 \, \text{[\mu N]} \), \( \Omega = 120 \, \text{[rad/s]} \)

Figure 4: Schematic of a spherical indenter undergoing oscillatory motion while in contact with a flat material surface.

References

