The following exercises are designed to help you master the matrix algebra presented in class. Asterisks indicate a type of (or an operation that could be part of a) question that could appear on the first exam.

1. A study involves \( n = 4 \) participant (\( P_1 \) to \( P_4 \)) and two variables \( x_1 \) and \( x_2 \). The data are as follows: \( P_1: x_1 = 10 \) and \( x_2 = 12 \); \( P_2: x_1 = 8 \) and \( x_2 = 14 \); \( P_3: x_1 = 12 \) and \( x_2 = 10 \); \( P_4: x_1 = 14 \) and \( x_2 = 8 \).

   a. *Arrange the data on the variables \( x_1 \) and \( x_2 \) as vectors \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \).

   b. *Write out \( \mathbf{x}_1' \) and \( \mathbf{x}_2' \).

   c. *Arrange the data for \( x_1 \) and \( x_2 \) as a matrix \( \mathbf{X} \) with subjects as rows and variables as columns.

   d. *What are the row and column dimensions of \( \mathbf{X} \)?

   e. *Write out \( \mathbf{X}' \)

   f. *What is the value of \( X_{32} \)? \( X_{41} \)?

   g. *Express the result of multiplying each score on \( x_1 \) by 2 as a vector \( \mathbf{x}_3 \).

   h. *Express the sum of the variables \( x_1 \) and \( x_2 \) as a vector \( \mathbf{x}_4 \).

   i. *Express twice \( x_1 \) plus three times \( x_2 \) as a vector \( \mathbf{x}_5 \).

   j. Let \( \mathbf{d}' = [1/4 \quad 1/4 \quad 1/4 \quad 1/4] \). *Calculate \( \mathbf{d}' \mathbf{X} \). How can the elements of \( \mathbf{d}' \mathbf{X} \) be interpreted?
k. \( b' = [2 \quad 3] \). *Calculate \( Xb \). How can the elements of \( Xb \) be interpreted?

l. *Calculate \( d'Xb \). How can \( d'Xb \) be interpreted?

m. Let \( 1' \) be a \( 1 \times 4 \) vector in which each element is 1. Write out \( \bar{x}_i1 \). What are the elements of \( x_i - \bar{x}_i1 \)? Calculate \( \frac{1}{3}(x_i - \bar{x}_i1)'(x_i - \bar{x}_i1) \). What have you calculated? Calculate \( \frac{1}{3}(x_i - \bar{x}_i1)'(x_i - \bar{x}_i1) \). What have you calculated?

2. Suppose the same subjects are observed on two additional variables \( y_1 \) and \( y_2 \). These data are as follows: \( P_1: y_1 = 1 \) and \( y_2 = 3 \); \( P_2: y_1 = 3 \) and \( y_2 = 7 \); \( P_3: y_1 = 2 \) and \( y_2 = 5 \); \( P_4: y_1 = 4 \) and \( y_2 = 9 \). Collect these data in a \( 4 \times 2 \) matrix \( Y \).

   a. *Are \( X \) and \( Y \) equal?
   b. *What is the value of \( X + Y \)?
   c. *What is the value of \( 2Y \)?
   d. *What is the value of \( X - Y \)?

3. *Calculate \( XY \)

4. *In order to compute \( AB \) what must be true about \( A \) and \( B \)?

5. Suppose two variables have covariance matrix

\[
S = \begin{bmatrix} 10 & 2 \\ 2 & 12 \end{bmatrix}.
\]

   a. *What is the correlation between the two variables?
Let $D = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{12} \end{bmatrix}$.

b. *What kind of matrix is $D$?

c. Find $D^{-1}$.

d. Calculate $D^{-1}SD^{-1}$. What have you calculated?

e. If you have the correlation matrix $R$ for the two variables and calculate $DRD$, what have you calculated?

6. Suppose we have three variables $x_1$, $x_2$, and $x_3$. Symbolically the correlation matrix between these three variables is

$$R = \begin{bmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{bmatrix}$$

a. *Is $R$ a square matrix?

b. *Write $R'$?

6. *Write a $4 \times 4$ identity matrix.

7. *Suppose three variables $(x_1, x_2, \text{ and } x_3)$ are three variables with variances $s_1^2$, $s_2^2$, and $s_3^2$ and covariances $s_{12}$, $s_{13}$, and $s_{23}$. Use the matrix expressions for the variance of a linear combination to derive scalar expressions for the variance of each the following linear combinations:

a. $x_1 - 2x_2$

b. $-x_1 + 3x_3$
c. \( x_1 + x_2 + x_3 \)

d. \( x_1 + 2x_2 + x_3 \)

8. *Suppose three variables \((x_1, x_2, x_3)\) have means \(\bar{x}_1\), \(\bar{x}_2\), and \(\bar{x}_3\). Use the matrix expressions for the mean of a linear combination to derive scalar expressions for the mean of each the following linear combinations:

a. \( x_1 - 2x_2 \)

b. \(-x_1 + 3x_3\)

c. \(x_1 + x_2 + x_3\)

d. \(x_1 + 2x_2 + x_3\)

9. In simple linear regression we use the linear model

\[
y_i = \alpha + bx_i + e_i
\]

where \(i\) is an index for the participants. If \(y\) is an \(n \times 1\) vector of \(y\) scores, \(1\) an \(n \times 1\) vector of ones, \(x\) an \(n \times 1\) vector of \(x\) scores, \(e\) an \(n \times 1\) vector of unknown error scores, \(\beta' = [\alpha \,
\beta]\) the regression parameters collected into a vector, and \(X = [1 \,
x]\) an \(n \times 2\) matrix, then in matrix terms the model may be written as

\[
y = X\beta + e \tag{1}
\]

Suppose the \(x\) and \(y\) data are
a. Substitute in equation 1, showing the numeric elements of $y$ and $X$ and the symbolic elements of $\beta$ and $e$.

The matrix formula for calculating $\hat{\beta}$ is

$$
\hat{\beta} = (X'X)^{-1} X'y.
$$

b. Calculate $\hat{\beta}$.