GAMES OF SCHOOL CHOICE UNDER UNCERTAINTY

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This paper examines the Nash equilibrium characteristics of the preference revelation game induced by the Boston mechanism under the informational circumstances that arise in major school districts using this mechanism in the U.S. The results indicate that three significant findings of the previous literature fail to hold under the real-world informational setting due to the uncertainty created by the lotteries used in tie-breaking. First, under this setting, the set of Nash equilibrium outcomes under the Boston mechanism do not necessarily correspond to the set of stable assignments under students' true preferences. Second, switching to one of the alternative mechanisms called the student-optimal stable mechanism, such as in the recent transition in Boston, may result in efficiency losses in practice. Third, assuming that there is at least one sincere student who always reveals public school preferences truthfully, a strategic student who plays best response might weakly prefer the student-optimal stable mechanism to the Boston mechanism. An important policy implication is that the findings of the previous literature can not be used as arguments for or against replacing the Boston mechanism in these school districts.

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1. Introduction

School choice reforms and their effectiveness in improving the quality of the public school system in the U.S. remain a major topic of debate among policy makers and researchers. The main objective for most of these school choice reforms is to provide as equal access to quality education as possible for all students regardless of their socio-economic status. Along these lines, inter-district and intra-district school choice programs, which allow parents to choose schools outside of the neighborhood they reside, have become increasingly popular in the last decade[1].

Along with increased parental choice has come the need to implement 'well-behaving' public school-student assignment procedures. Public school assignment mechanisms have been evaluated along four desirable properties:

1. Explicit Rules: A public school assignment mechanism should have explicit rules in order to remove any ambiguity in assignment decisions. The absence of such explicit rules creates potential conflicts between school authorities and parents who question the fairness of school assignments, providing incentives for parents to seek legal action to overturn their school assignments[2].
2. Strategy-proofness: A preferred public school assignment mechanism avoids creating incentives for parents to play complicated games. Hence, truthful parental ranking of schools should be a dominant strategy. Strategy-proofness of the assignment mechanism is then desirable.

[1] According to the estimates from the 1999-2000 school year, 71 percent of the school districts in the West, 63 percent in the Midwest, 44 percent in the South and 19 percent in the Northeast employed these school choice programs in the U.S. (NCES, 2006).
[2] Abdulkadiroglu and Sonmez (2003) cite cases in Mississippi and Wisconsin where the assignment decisions were overruled due to the ambiguity created by the school assignment mechanism.

3. Stability: An assignment set is defined to be *stable* if there is no school-student pair (*i*,*s*) such that student *i* prefers school *s* to her current assignment and *either* school *s* prefers student *i* to at least one of the students assigned to it *or* school *s* has at least one empty seat. Absent stability, there exists 'justified envy' in the assignments providing incentives for parents to seek legal action to overturn school assignments.
4. Efficiency: For the public school assignment problem in the context of this paper[3], only the welfare of students is considered for Pareto efficiency, since the schools are regarded as objects to be consumed by students. Pareto efficient assignments are obviously desirable.

One of the most commonly used student assignment mechanisms is the Boston mechanism, so named because of its use until recently in Boston. This mechanism is still being used in other major school districts including Cambridge (MA), Charlotte (NC), Denver (CO), Hillsborough (Tampa, FL), Miami-Dade (FL), Minneapolis (MN), Seattle (WA) and Pinellas (St.Petersburg, FL)[4]. This mechanism has the virtue of removing the ambiguity in assignment decisions by imposing explicit rules. However, the mechanism has a major weakness: it is not strategy-proof (Abdulkadiroglu and Sonmez, 2003). In

[3] For the public school assignment problem discussed in this paper, priority categories mandated by the school districts are employed along with student preferences to determine the public school assignments. Since these rankings do not necessarily correspond to schools' preferences, only the students' preferences are considered for efficiency. On the contrary, there are cases such as the high school assignments in NYC where schools determine their own priorities. In that case, school preferences as well as student preferences might be taken into account for welfare considerations.

[4] In 2003, over one million students were enrolled in public schools within the boundaries of these school districts. The assignments for a significant number of public schools are determined using the Boston mechanism in these districts. For instance, in Denver, this mechanism is used for traditional public school assignments (approximately 79% of all public schools), whereas, in Seattle, the assignments for every public school in the district are decided with the use of the Boston mechanism.

other words, the Boston mechanism induces parents to play a difficult preference revelation game in these school districts, the details of which are discussed below.

Despite this weakness of the Boston mechanism, Ergin and Sonmez (2006) show that the Boston mechanism will result in stable assignments in equilibrium. They further show that the set of stable assignments under students' true preferences corresponds to the set of Nash equilibrium assignments under the Boston mechanism. This is perhaps surprising since, in equilibrium, preferences of students are not truthfully revealed in general. Nevertheless, the assignments satisfy the stability property in equilibrium. The crucial assumptions leading to these results are that the preference revelation game takes place under a complete informational setting where schools have strict priority rankings over students that are common knowledge.

In fact, in all of the major school districts using the Boston mechanism, public school preferences over students are determined by broad priority categories mandated by the school district (e.g. sibling already in the school). This generates the need to break the ties between students in the same priority categories before the assignment algorithm can be applied. In all of the aforementioned school districts, the ties between students are broken in some random fashion and the assignments are determined using the Boston algorithm *after* all of the student applications are received. Therefore, under the Boston mechanism, games of school choice, *in reality*, take place in the presence of uncertainty about schools' strict priority rankings over students.

One point of this paper is to show that, under the real-world informational setting, the stability property does not hold. Hence, the Boston mechanism might lead to unstable assignments and provide parents incentives to seek legal action to overturn their school assignments in these school districts. Furthermore, the results indicate that the stable assignments under students' true preferences are not necessarily Nash equilibrium outcomes.

Another mechanism that has been proposed and recently implemented is the student-optimal stable (SOS) mechanism[5]. Compared to the Boston mechanism, the SOS mechanism has the desirable feature of being strategy-proof[6]. In addition, it guarantees stable assignments, though Pareto efficiency is not guaranteed[7].

Ergin and Sonmez (2006) also show that even though neither the Boston mechanism nor the SOS mechanism guarantee Pareto efficiency, the assignments achieved by the latter will always weakly Pareto dominate the Boston mechanism assignments. Following this result, they state that a transition to the SOS mechanism may result in significant efficiency gains in the major school districts using variants of the Boston mechanism.

The second result of this paper indicates that, *in reality*, due to the uncertainty created by the tie-breaking, a transition to the SOS mechanism does not necessarily lead to a weak Pareto improvement in these school districts; there are even cases where such a transition results in Pareto inferior assignments. This implies that neither the transition in

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[5] After pointing out the weakness of the Boston mechanism, Abdulkadiroglu and Sonmez (2003) propose two alternative assignment mechanisms: the SOS mechanism and the top-trading cycles (TTC) mechanism. After the publication of this paper, Boston Public Schools contacted the authors to design a new public school assignment mechanism. In 2006, the SOS mechanism was implemented in Boston, replacing the Boston mechanism (Abdulkadiroglu et. al. (2005) and Abdulkadiroglu et. al. (2006)).

[6] See Dubins and Freeman (1981) and Roth (1982). Recently, Abdulkadiroglu et. al. (2007) examine the impact of two different tie-breaking methods on the efficiency and strategy-proofness of the SOS mechanism; single tie breaking where each student is given a random number to be used at every school and multiple tie-breaking where each student is assigned a different random number to be used at each school. Their main theoretical result indicates that a SOS mechanism that uses single tie-breaking is not dominated by any other mechanism that is strategy-proof for students.

[7] It has been well documented in the previous literature that given *strict* student preferences and *strict* school priorities, there exists no other stable assignment that Pareto dominates the assignment produced by the SOS mechanism; however overall Pareto efficiency is not guaranteed. Erdil and Ergin (forthcoming) show that when there are indifferences in school priorities as in the public school assignment problem discussed in this paper, there may exist another stable assignment that Pareto dominates the SOS outcome.

Boston guaranteed efficiency gains nor would such a transition in other major school districts using the Boston mechanism necessarily be beneficial in terms of efficiency[8].

Given the numerous aspects along which the SOS mechanism is superior to the Boston mechanism as shown in the previous literature, there has been an increasing curiosity among researchers as to why other school districts have not yet followed the footsteps of Boston Public Schools and replaced the Boston mechanism. Recently, Pathak and Sonmez (forthcoming) suggest the existence of important stakeholders who benefit from the Boston mechanism as a possible explanation to this puzzle. Specifically, their main result indicates that the school to which a strategic student who plays a best response is assigned under the Pareto-dominant equilibrium of the Boston mechanism is weakly better than her outcome under the SOS mechanism when there is at least one sincere student who always reveals her public school preferences truthfully. In other words, in the presence of sincere students, strategic players weakly prefer the Boston mechanism to the SOS mechanism if they can coordinate to achieve the Pareto-dominant assignment set under the Boston mechanism[9].

The third point of this paper is to show that the latter finding does not necessarily hold under the real-world informational setting. In other words, even if they manage to coordinate, strategic players may end up being assigned to worse schools under the

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[8] In the last section of their article, Ergin and Sonmez (2006) examine a case where the complete information assumption is violated. They discuss a scenario where there is uncertainty about the strict preference ordering of a student. Finding the Nash equilibrium given this *incomplete information* setting, they show that all Nash equilibrium outcomes are not necessarily stable and a student may be better-off under the Boston mechanism than under the SOS mechanism. This study enhances these results in two ways. First, identifying the real-world informational structure, the results obtained in this paper provide more practical policy implications. Furthermore, this study not only confirms these results under this setting, but also extends them in some important aspects indicated by the three main results mentioned earlier.

[9] It is a well-known fact that the games of school choice induced by the Boston mechanism might have multiple equilibria and the Pathak and Sonmez (fortcoming) result regards the equilibrium wherein the strategic players obtain their Pareto-dominant assignments among these Nash equilibria with some sincere players.

Boston mechanism than under the SOS mechanism. Therefore, the existence of important stakeholders who are strategic players is not a valid explanation to the aforementioned mystery in practice.

This paper demonstrates the failure of three important findings of the previous literature about the equilibrium characteristics of the Boston mechanism versus the SOS mechanism in practice. An important policy implication is that these findings must be carefully considered by the policy-makers if the Boston mechanism is to be abandoned, since they might provide misleading policy suggestions in these school districts.

The analysis proceeds as follows. Section 2 provides a background on the public school assignment problem, details the Boston and the SOS mechanisms as they are applied in major school districts, and shows how the resulting assignments would differ using an example. Section 3 evaluates the discussed Nash equilibrium properties of the preference revelation game induced by the Boston mechanism under the real-world informational setting, and Section 4 concludes.

2. Public School Assignment Problem and the Two Assignment Mechanisms

Open enrollment programs, which allow parents to send their children to the public schools outside of the neighborhood they reside, have become increasingly popular in the United States during the last two decades. Under the first-best setting, open enrollment programs allow parents to send their kids to any public school within the boundaries of a given region that contains, but is not limited to, the household's neighborhood. Therefore, under this scenario, public school assignments are trivial; each student is assigned to the public school of her choice within these boundaries. However, in practice, aside from these boundaries, parents are typically limited in their public school choices by constraints especially public school capacities. The presence of such constraints necessitates other parents' public school preferences to be taken into account in order to determine the public school assignment of a given student, which turns the public school assignment into a large-scale problem and obligates the use of centralized assignment mechanisms by the school districts.

In a public school assignment problem, there are *n* students and *k* public schools each of which has a given number of slots available. Equilibrium assignments depend on students' reported preferences, priorities of schools over students, and the assignment mechanism. It is assumed that each student has a utility function over the *k* public schools with strict preferences, which is here assumed to be common knowledge. Students first submit their preferences, i.e., a strict ranking of the *k* schools, and the assignments use the set of submitted (ordinal) rankings. Schools have priority rankings of students, based on broad priority categories mandated by the school district (e.g., residing in a walk zone), and lotteries are used to break the ties between the students in the same priority categories. Students know school capacities, the rules of the assignment mechanism, and the priority categories of schools when they submit their rankings. We focus on the fact that the outcomes of the lotteries are not known when students submit their preferences, while we also compare such equilibria to the case in the literature where schools have strict rankings of students as if lotteries were first publicly conducted. How the submitted preferences and school priorities interact to yield assignments depend on the rules of the assignment mechanism, the details of which we discuss next[10]. Students submit their preferences to maximize their expected utilities in Nash equilibrium.

[10] School priorities and the assignment mechanism are given so schools are not players in the game. This is in contrast to some two-sided matching problems as discussed in Gale and Shapley (1962).

2.1. The Boston Mechanism

The school assignment mechanism known as the Boston mechanism is currently being used in some major school districts such as Cambridge, Charlotte, Denver, Hillsborough County, Miami-Dade County, Minneapolis, Seattle and Pinellas County. It had also been used in Boston between 1999 and 2006. Under the Boston mechanism, a student who is not assigned to his first choice is considered for his second choice only after the students who ranked that student's second choice as their first choices. More specifically, all of these major school districts employ the following general scheme in their public school assignments[11];

- First step: School districts announce the assignment algorithm, the priority categories, and the way the lottery will be conducted[12] to break the ties between students in the same priority category. The major school districts using the Boston mechanism differ considerably in their choices and definitions of priority categories; however, the most commonly used are sibling and 'attendance zone' priorities. For instance, in Boston, the following priority categories are currently being used;

  1) Students who have siblings currently attending that school *and* who live in the 'walk zone' of the school.
  2) Students who have siblings currently attending that school.
  3) Students who live in the 'walk zone' of the school.

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[11] Documentation and more detailed information on the public school assignment procedures in the school districts listed are available upon request.

[12] The school districts differ significantly in the ways they use the priority categories along with the lottery outcome to rank the applicants. In Boston, the applicants for a given school are first ranked with respect to the priority categories and then the outcome of the lottery is used to rank those within the same priority category. In Miami-Dade, on the other hand, a weighted lottery is conducted where more random numbers are generated for those in higher priority categories. The rankings are then constructed using the best random number for each applicant.

4) Students who do not fall into the three categories above.

- Second step: Observing these, each student submits a ranking of her preferred schools. The outcome of the tie breaking is unknown at this time.
- Third step: Given the applicant pool, the lottery is conducted and each applicant is ranked according to the pre-specified priority categories and the outcome of the tie-breaking.
- Fourth step: The assignment of students based on the student preferences and the strict school priorities.
  - In the first round, only the first choices of students are considered. Based on the schools' priority rankings of students, the seats at each school are assigned one at a time.
  - In the n^{th} round, only the n^{th} choices of the students who could not be placed in the $(n-1)^{st}$ round are considered. The procedure is terminated when there are no unassigned students remaining.

The crucial point in this public school assignment procedure for this analysis is the timing of the lottery to break the tie between students in the same priority category. In all of the school districts listed, the students are required to submit their school preference rankings before the tie-breaking takes place. Therefore, there is uncertainty about schools' priority rankings over students at the time when the school choice game among students takes place. As demonstrated in the third section, this seemingly minor detail has serious consequences.

To illustrate how the Boston mechanism works, consider the following example by Roth (1982):

Example 1: Assuming complete information[13] and that the students truthfully reveal preferences, consider the following preferences and the priority rankings of the three students (i_1, i_2, i_3) and three schools (a, b, c) each of which has only one seat.

i_1: $b - a - c$	a: $i_1 - i_3 - i_2$
i_2: $a - b - c$	b: $i_2 - i_1 - i_3$
i_3: $a - b - c$	c: $i_2 - i_1 - i_3$

Applying the Boston mechanism to this example, we get the assignments (i_1, b), (i_2, c) and (i_3, a)[14]. Note that this example demonstrates how the Boston mechanism might induce students to misrepresent their preferences. If i_2 had misrepresented her preferences by listing b as her first choice, the school in which she has the highest priority, she would have been assigned to b instead of c and would have been better-off. Hence, the Boston mechanism is not strategy-proof in this case.

2.2. The Student-Optimal Stable (SOS) Mechanism

Unlike the Boston mechanism, none of the assignments are guaranteed until the assignment algorithm terminates using the SOS mechanism. The SOS mechanism is very similar to the solution to the college admissions problem by Gale and Shapley (1962), the Gale-Shapley deferred acceptance algorithm. The fourth step of the assignment procedure in the Boston mechanism is modified as follows:

- Step 1: Each student's first choice is considered. Each school puts all applicants into a queue unless the number of applicants is higher than its capacity, or rejects the ones ranked lower than its capacity in its priority ranking otherwise, while placing the rest in its queue.

[13] Under complete information, as in the previous literature, we assume that the students observe the strict priority rankings of schools and the true preferences of other students. This implies that when making their school choices, students know the applicant pool, each applicant's true preferences and the outcome of the tie-breaking in addition to the assignment procedure.
[14] Only the first choices are considered; given the priorities, i_3 is assigned to a and i_1 is assigned to b. i_2 is rejected from a and is assigned to c, since the only seat available in school b is occupied by i_1.

- Step k: The rejected applicants' next choices are considered. Comparing the new applicants with the applicants already in the queue, each school replaces the students on its queue based on its priority rankings. The process terminates when no student is rejected and each student is assigned to the school whose queue she belongs to when the algorithm terminates.

The main advantage of this approach over the Boston mechanism is that it is strategy-proof. It also implies stable assignments, but Pareto efficiency is not guaranteed (Roth, 1982). When applied to Example-1, the SOS mechanism yields the stable assignment set (i_1, *a*), (i_2, *b*) and (i_3, *c*)[15]. However, note that the matching (i_1, *b*), (i_2, *a*) and (i_3, *c*) is Pareto superior to the previous outcome. The first three columns of Table-1 summarize the key properties of the two assignment mechanisms under complete information.

3. Preference Revelation Game Induced by the Boston Mechanism

As illustrated in Example-1, the Boston mechanism induces the students to play a preference revelation game where the payoffs are determined by the preferences of the students over schools, the priorities of schools over students, and the rules of the mechanism. In this section, the following three Nash equilibrium properties of this game are evaluated under the real-world informational setting:

i. *Stability in equilibrium (Ergin and Sonmez, 2006)*: The set of Nash equilibrium outcomes of the preference revelation game induced by the Boston mechanism corresponds to the set of stable matchings under students' true preferences.

[15] At the end of step 1, i_2 gets rejected from *a*, i_3 is in the queue of *a* and i_1 is in the queue of *b*. i_2 goes to *b*; at the end of step-2, i_1 gets rejected from *b*, i_2 is in the queue of *b* and i_3 is in the queue of *a*. i_1 goes to *a*; at the end of the step-3, i_3 gets rejected from *a*, i_1 is in the queue of *a* and i_2 is in the queue of *b*. i_3 goes to *b*; at the end of the step-4, i_3 gets rejected from *b*, i_2 is in the queue of *b* and i_1 is in the queue of *a*. i_3 goes to *c*; at the end of the step-5, nobody gets rejected, i_1 is assigned to *a*, i_2 is assigned to *b* and i_3 is assigned to *c*, and the process terminates.

ii. *Transition to the SOS mechanism (Ergin and Sonmez, 2006)*: A transition from the Boston mechanism to the SOS mechanism would lead to unambiguous efficiency gains (i.e. weak Pareto improvements).

iii. *Sincere vs. strategic students in equilibrium (Pathak and Sonmez, forthcoming)*: In the presence of sincere students who always reveal their public school preferences truthfully, the school that a strategic student receives under the Pareto-dominant equilibrium (among the possibly multiple equilibria) of the Boston mechanism is weakly better than her outcome under the SOS mechanism. Furthermore, if there are multiple equilibria of the school choice game played by the strategic students under the Boston mechanism, a sincere student will receive the same assignment under all of these equlibria.

The key assumption behind these results is that the students have complete information about their relative priority positions at each school when making their school choices. In other words, the students are assumed to be acting as if they know the *results* of the tie-breaking lotteries before reporting their preferences. However, in reality, students have to take into account the uncertainty created by the lotteries when making their choices. The following subsections show that this uncertainty overturns the validity of the aforementioned equilibrium properties of the Boston mechanism.

3.1. Stability in Equilibrium

Example 2: Assume that there are three students (i_1, i_2, i_3) and three schools (a, b, c) each of which has only one seat. Assume further that i_1 and i_2 fall into the same priority category for a, and i_2 has higher priority than i_1 for b. Also assume that i_3 has lower priority than i_1 and i_2 for both a and b, but has a higher priority than the others for school *c*. Consider the following student preferences and school preferences corresponding to these priorities:

$i_1 : a - b - c$ $\quad$ $a: i_2 - i_1 - i_3$ or $i_1 - i_2 - i_3$

$i_2 : a - b - c$ $\quad$ $b: i_2 - i_1 - i_3$

$i_3 : b - c - a$ $\quad$ $c: i_3 - i_1 - i_2$

The utilities of the students from being assigned to each school are as follows:

	a	b	c
i_1	a_1	b_1	c_1
i_2	a_2	b_2	c_2
i_3	a_3	b_3	c_3

where
$$a_1 > b_1 > c_1$$
$$a_2 > b_2 > c_2$$
$$b_3 > c_3 > a_3$$

Notice that there are two states of nature; the one where the tie between i_1 and i_2 for school *a* is broken in favor of i_1, and the one where i_1 loses the lottery. There are six possible assignments in this example:

	i_1	i_2	i_3
A_1	a	b	c
A_2	a	c	b
A_3	b	a	c
A_4	b	c	a
A_5	c	a	b
A_6	c	b	a

Among these assignments, A_1 and A_3 are stable under students' true preferences and the pre-lottery priorities.

Since there are 3 schools to choose from, each student has 6 strategies, i.e. 6 ways to rank the schools. For each state of nature, we have six 6x6 matrices with the corresponding payoffs (utilities) as determined by the assignments of the Boston mechanism. Table-2 gives the expected payoff matrices under this scenario where the row player is i_1, the column player is i_2, the matrix player is i_3 and

$$d_j = \frac{1}{2}(a_j + b_j)$$
$$e_j = \frac{1}{2}(a_j + c_j) \quad j = 1, 2, 3.$$
$$f_j = \frac{1}{2}(b_j + c_j)$$

Consider the first implication of the stability property: all Nash equilibrium outcomes are stable. In order for this statement to hold, there should not exist a Nash equilibrium strategy set that results in an unstable assignment. Looking at Table-2, given students i_2 and i_3 report school *a* and school *b* as their first choices respectively (play *axx*[16] and *bxx* respectively), the strategy set of student i_1 yields the following expected utilities:

abc	*acb*	*bac*	*bca*	*cab*	*cba*
e_1	e_1	b_1	b_1	c_1	c_1

Therefore, if $e_1 > b_1$, then student i_1 will always report school *a* as her first choice and play *axx* given i_2 and i_3 play *axx* and *bxx* respectively. If i_1 plays *axx* and i_3 plays *bxx*, the strategy set of i_2 results in the following expected utilities:

abc	*acb*	*bac*	*bca*	*cab*	*cba*
e_2	e_2	b_2	b_2	c_2	c_2

Likewise, if $e_2 > b_2$, then student i_2 will always play *axx* given i_1 and i_3 play *axx* and *bxx* respectively. Finally, if both i_1 and i_2 play *axx*, i_3 will always be guaranteed a seat at her most preferred option, school *b*, if she reports school *b* as her first choice. Hence, given i_1 and i_2 play *axx*, i_3 will always play *bxx*.

[16] i_2 playing *axx* means that she either plays *abc* or *acb*.

This implies that, if the following conditions hold:

$$\frac{1}{2}(a_1 + c_1) > b_1 \tag{1}$$

$$\frac{1}{2}(a_2 + c_2) > b_2 \tag{2}$$

then i_1: *axx,* i_2: *axx* and i_3: *bxx* will be a subset of the set of all Nash equilibrium strategies for the overall game. There are two possible sets of assignments for this case:

1. If i_1 wins the lottery, the Boston mechanism will result in the following assignments: (i_1, *a*), (i_2, *c*), (i_3, *b*).
2. If i_2 wins the lottery, the Boston mechanism will result in the following assignments: (i_1, *c*), (i_2, *a*), (i_3, *b*).

Notice that the Nash equilibrium assignments obtained above are neither stable under the pre-lottery priorities and true preferences nor under the post-lottery priorities and true preferences; if i_1 wins the lottery, i_2 prefers *b* to her assignment (school *c*) and school *b* prefers i_2 to i_3. If i_2 wins the lottery, i_1 prefers *b* to her assignment (school *c*) and school *b* prefers i_1 to i_3. Therefore, given conditions (1) and (2), this example shows that all Nash equilibrium assignments are not necessarily stable under the real-world informational setting.

Now consider the second implication: all stable assignments under students' true preferences are Nash equilibrium outcomes. In our context, this implies that both A_1 and A_3 are Nash equilibrium assignments. In order to see if this condition is satisfied in this example, we need to check whether all strategy combinations that result in assignments A_1 and A_3 are Nash equilibrium strategies.

Assume that the conditions (1) and (2) are still satisfied. This implies that;

$$a_1 > d_1 > e_1 > b_1 > f_1 > c_1$$
$$a_2 > d_2 > e_2 > b_2 > f_2 > c_2$$
$$b_3 > f_3 > c_3 > e_3 > a_3$$

Finding the Nash equilibria of the overall game, one can observe[17] that the strategy set i_1: *axx,* i_2: *axx* and i_3: *bxx* is not only a subset of the set of Nash equilibrium strategies as shown earlier, but this set of strategies corresponds to the set of all Nash equilibrium strategies of the overall game as indicated by the bold-faced expected utilities in Table-2. Since this set of strategies yields the assignments A_2 or A_5 depending on the outcome of the lottery, neither A_1 nor A_3 can occur as a result of Nash equilibrium strategies. Therefore, the stable assignments under students' true preferences are not Nash equilibrium outcomes no matter how the tie is broken between i_1 and i_2.

Combining these two results, given conditions (1) and (2), this example demonstrates how the uncertainty created by the lottery changes the stability property obtained under the complete information assumption. The set of Nash equilibrium outcomes, {(i_1, *a*), (i_2, *c*), (i_3, *b*); (i_1, *c*), (i_2, *a*), (i_3, *b*)} and the set of stable assignments under students' true preferences, {(i_1, *a*), (i_2, *b*), (i_3, *c*); (i_1, *b*), (i_2, *a*), (i_3, *c*)} are two distinct sets. Therefore, the Boston mechanism, in practice, might result in unstable assignments providing parents incentives to seek legal action to overturn their assignments.

3.2. Transition to the SOS Mechanism

Example 3: Assume that there are four students (i_1, i_2, i_3, i_4) and four schools (*a, b, c, d*) each of which has only one seat. Assume further that i_2 and i_3 fall into the same priority category for *a*. Consider the following student preferences and school rankings:

[17] The derivation is given in the Appendix.

i_1: $b - a - c - d$ | a: $i_1 - i_3 - i_2 - i_4$ *or* $i_1 - i_2 - i_3 - i_4$
i_2: $a - b - c - d$ | b: $i_2 - i_1 - i_3 - i_4$
i_3: $a - c - b - d$ | c: $i_3 - i_1 - i_2 - i_4$
i_4: $c - d - b - a$ | d: $i_4 - i_3 - i_1 - i_2$

The utilities of the students from being assigned to each school are as follows:

	a	b	c	d
i_1	a_1	b_1	c_1	d_1
i_2	a_2	b_2	c_2	d_2
i_3	a_3	b_3	c_3	d_3
i_4	a_4	b_4	c_4	d_4

where
$$b_1 > a_1 > c_1 > d_1$$
$$a_2 > b_2 > c_2 > d_2$$
$$a_3 > c_3 > b_3 > d_3$$
$$c_4 > d_4 > b_4 > a_4$$

Under the Boston mechanism, in the case where all players report their most preferred schools as their first choices, i_3 has a 0.5 chance of being assigned to school *a* (if she wins the lottery) and 0.5 chance of being assigned to school *d* (if she loses). Therefore, the expected payoff from playing *axxx* for i_3 in the case where i_1, i_2 and i_4 report their most preferred schools as their first choices is $\frac{1}{2}(a_3 + d_3)$. On the other hand, if she does not reveal truthfully and report school *c* as her first choice, she will be assigned to school *c* no matter what the other players do. Given all other players reveal their first choices truthfully, i_3's best reply will be not to reveal truthfully and play *cxxx* if $\frac{1}{2}(a_3 + d_3) < c_3$[18]. Building on this intuition, we now construct a Nash equilibrium. Given that i_1 plays *bxxx*, and i_3 and i_4 play *cxxx*, i_2's best reply will be to play *axxx*, since

[18] Notice that given that all other players reveal their first choices truthfully, student i_3 will definitely be assigned to school *d* if she reports school *b* or school *d* as her first choice. Therefore, given all other players reveal their first choices truthfully, student i_3 will never report school *b* or school *d* as her first choice.

by doing so, she is guaranteed a seat at her favorite school. Furthermore, if i_2 and i_4 report schools *a* and *c* as their first choices respectively, and i_3 plays *cxxx*, there is no risk for i_1 to reveal her most preferred option truthfully and she will play *bxxx*. Finally, given that i_1 and i_2 reveal truthfully and i_3 plays *cxxx*, i_4 will always be assigned to school *d* no matter what she reveals; hence she is indifferent between all of her strategies. Hence, i_1: *bxxx*, i_2: *axxx*, i_3: *cxxx* and i_4: *cxxx* constitutes a set of Nash equilibrium strategies given $\frac{1}{2}(a_3 + d_3) < c_3$ which results in the assignments (i_1, *b*), (i_2, *a*), (i_3, *c*) and (i_4, *d*).

On the other hand, when applied to this example, the SOS mechanism will result in the assignments (i_1, *a*), (i_2, *b*), (i_3, *c*) and (i_4, *d*) if the tie is broken in favor of i_3 or the assignments (i_1, *b*), (i_2, *a*), (i_3, *c*) and (i_4, *d*) otherwise. However, note that the assignments achieved by the Boston mechanism weakly Pareto dominate the SOS mechanism outcomes if $\frac{1}{2}(a_3 + d_3) < c_3$.

This result is particularly important since a transition from the Boston mechanism to the SOS mechanism has currently been made in Boston. Even though student stable-optimal mechanism is superior to the Boston mechanism in terms of being strategy proof, the result obtained above shows that switching from the Boston mechanism to the SOS mechanism in the major school districts using the Boston mechanism does not guarantee a Pareto improvement; it may even cause a Paretian loss. The last three columns of Table-1 summarize the first two results obtained in this article by revising the properties of the two assignment mechanisms under the real-world informational setting.

3.3. Sincere vs. Strategic Students

Example 4: Assume that there are three students (i_1, i_2, i_3) and three schools (*a, b, c*) each of which has only one seat. Consider the following student preferences and priority rankings:

i_1: $b - a - c$	a: $i_1 - i_2 - i_3$ *or* $i_1 - i_3 - i_2$
i_2: $a - b - c$	b: $i_2 - i_1 - i_3$
i_3: $a - b - c$	c: $i_2 - i_1 - i_3$

Suppose that i_3 is a sincere player who always reveals her public school preferences truthfully whereas i_1 and i_2 are strategic players. There are two states of nature; the one where the tie between i_2 and i_3 is broken in favor of i_2, and the one where i_3 wins the lottery. Given that i_3 always plays *abc*, Table-3 provides the expected payoff matrix of this game under the Boston mechanism where the row player is i_1, the column player is i_2 and the payoffs are defined the same way as in Example-2.

Given that $e_2 > b_2$, there are two sets of Nash equilibrium in this game under the Boston mechanism characterized by i_1: *axx,* i_2: *bxx* and i_1: *bxx*, i_2: *axx*. The Boston mechanism then results in the following assignments in equilibrium:

	i_1: *axx,* i_2: *bxx*	i_1: *bxx,* i_2: *axx*
If the tie is broken in favor of i_2	(i_1, *a*), (i_2, *b*), (i_3, *c*)	(i_1, *b*), (i_2, *a*), (i_3, *c*)
If the tie is broken in favor of i_3	(i_1, *a*), (i_2, *b*), (i_3, *c*)	(i_1, *b*), (i_2, *c*), (i_3, *a*)

Notice that the former set of Nash equilibrium strategies provides expected payoffs of a_1 and b_2 whereas the latter produces b_1 and e_2 for students i_1 and i_2 respectively. Therefore, given that $e_2 > b_2$, the Pareto-dominant equilibrium strategy set for the two strategic students under the Boston mechanism is i_1: *bxx*, i_2: *axx*. On the other hand, when applied to this example, the SOS mechanism yields the assignments (i_1, *b*), (i_2, *a*), (i_3, *c*) if the tie is broken in favor of i_2 or (i_1, *a*), (i_2, *b*), (i_3, *c*) if i_2 loses the lottery.

Two points are worth noting. First, in the state of nature where the tie is broken in favor of i_3, the two Nash equilibria under the Boston mechanism yield different assignments for the sincere student. Hence, multiplicity is an issue for sincere students under the Boston mechanism in practice. Second, the schools that the strategic student i_2 are assigned to under the Pareto-dominant equilibrium outcome of the Boston mechanism are weakly worse than the SOS mechanism assignments. Therefore, even if the two strategic students coordinate to achieve the Pareto-dominant assignment set under the Boston mechanism, i_2 will weakly prefer the SOS mechanism to the Boston mechanism in this case.

Despite the convincing evidence that the Boston mechanism is dominated by the SOS mechanism along almost all of the desirable properties of a 'well-behaving' mechanism, the major school districts using the Boston mechanism have been reluctant to abandon this mechanism in favor of the alternative. The third result in this paper suggests that the existence of important stakeholders who are strategic players is not a valid explanation to this mystery, since the strategic players might weakly prefer the SOS mechanism in these school districts.

4. Conclusions

One of the most commonly used student assignment mechanisms with explicit rules is the Boston mechanism, so named because of its use until recently in Boston. Even though this mechanism is superior to some other pre-existing public school assignment mechanisms, it has a major weakness: it is not strategy-proof. In other words, under the Boston mechanism, some students may benefit from misrepresenting their true preferences. As a result, the Boston mechanism induces students to play a complicated preference revelation game where students' payoffs depend on others' revealed preferences, school priorities and the rules of the mechanism.

This paper demonstrates the failure of three important findings of the previous literature about the equilibrium characteristics of this preference revelation game induced by the Boston mechanism under the informational circumstances that arise in the major school districts using variants of this mechanism. First, in practice, the Boston mechanism might result in unstable assignments providing parents incentives to seek legal action to overturn their assignments. Second, a transition to one of the proposed alternatives called the student-optimal stable mechanism may result in efficiency losses under the real-world informational setting. Third, in the presence of sincere students who always reveal truthfully, the school a strategic student receives under the Pareto-dominant outcome of the Boston mechanism, in reality, might be weakly worse than the outcome under the student-optimal stable mechanism. An important policy implication of these illustrations is that such findings of the recent literature must be carefully considered by the policy-makers if the Boston mechanism is to be abandoned, since they might provide misleading policy suggestions in these school districts.

Appendix

Proposition: Given the conditions (1) and (2), i_1: *axx,* i_2: *axx* and i_3: *bxx* represents the set of all (pure-strategy) Nash equilibrium strategies of the game defined in Example-2.

Proof: Looking at Table-2, given the conditions (1) and (2), consider the following set of strategies that can not be Nash equilibrium:

1. Given i_3 plays *axx*, i_1 never plays *bxx* or *cxx* and i_2 never plays *bxx* or *cxx*.
2. Given i_3 plays *bxx*,

   i_1 never plays *cxx* $\Rightarrow$ i_2 never plays *bxx or cxx* $\Rightarrow$ i_1 never plays *bxx or cxx*.
3. Given i_3 plays *cxx*,

   i_2 never plays *bxx* $\Rightarrow$ i_1 never plays *bxx* $\Rightarrow$ i_2 never plays *bxx* or *cba* $\Rightarrow$ i_1 never plays *bxx* or *cxx* $\Rightarrow$ i_2 never plays *bxx* or *cxx*.
4. Given i_1 and i_2 never play *bxx* or *cxx*, i_3 never plays *cxx*.
5. Given i_1 plays *acb* and i_3 plays, i_2 never plays *acb* $\Rightarrow$ Given i_1 and i_2 never play *bxx* or cxx, i_3 never plays *axx* or *cxx*.

Therefore, the only possible set of Nash equilibrium strategies is i_1: *axx,* i_2: *axx* and i_3: *bxx*. We know from the previous discussion that i_1: *axx,* i_2: *axx* and i_3: *bxx* is a set of Nash equilibrium strategies. Therefore, i_1: *axx,* i_2: *axx* and i_3: *bxx* represents the set of all (pure-strategy) Nash equilibrium strategies of the game defined in Example-2.

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Table 1
The Key Equilibrium Properties of the Two Assignment Mechanisms

	Under Complete Information			Under Uncertainty		
		Guarantees			Guarantees	
	Strategy-Proof	Stable Assignments	Pareto Efficient Assignments	Strategy-Proof	Stable Assignments	Pareto Efficient Assignments
The Boston Mechanism	No	Yes	No	No	No	No
The SOS Mechanism	Yes	Yes	No[1]	Yes	Yes	No[2]

[1] Even though neither mechanism guarantees Pareto efficient assignments, the SOS mechanism assignments always weakly Pareto dominate the Boston mechanism assignments under complete information.
[2] The SOS mechanism assignments do not necessarily weakly Pareto dominate the Boston mechanism assignments under student uncertainty. There are even cases as discussed in this study where the Boston mechanism assignments weakly Pareto dominate the SOS mechanism assignments.

Table 2
The Expected Payoff Matrices of the Preference Revelation Game for Example-2

abc

	abc	acb	bac	bca	cab	cba
abc	(d_1,d_2,c_3)[1]	(d_1,e_2,f_3)[3]	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,c_2,b_3)	(a_1,c_2,b_3)
acb	(e_1,d_2,f_3)[2]	(e_1,e_2,b_3)[4]	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,c_2,b_3)	(a_1,c_2,b_3)
bac	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(b_1,c_2,a_3)	(b_1,c_2,a_3)
bca	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(b_1,c_2,a_3)	(b_1,c_2,a_3)
cab	(c_1,a_2,b_3)	(c_1,a_2,b_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)
cba	(c_1,a_2,b_3)	(c_1,a_2,b_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)

acb

	abc	acb	bac	bca	cab	cba
abc	(d_1,d_2,c_3)[1]	(d_1,d_2,c_3)[1]	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,c_2,b_3)	(a_1,c_2,b_3)
acb	(d_1,d_2,c_3)[1]	(d_1,d_2,c_3)[1]	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,c_2,b_3)	(a_1,c_2,b_3)
bac	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(b_1,c_2,a_3)	(b_1,c_2,a_3)
bca	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(b_1,c_2,a_3)	(b_1,c_2,a_3)
cab	(c_1,a_2,b_3)	(c_1,a_2,b_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)
cba	(c_1,a_2,b_3)	(c_1,a_2,b_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)

bac

	abc	acb	bac	bca	cab	cba
abc	**(e_1,e_2,b_3)**[4]	**(e_1,e_2,b_3)**[4]	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,c_2,b_3)	(a_1,c_2,b_3)
acb	**(e_1,e_2,b_3)**[4]	**(e_1,e_2,b_3)**[4]	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,c_2,b_3)	(a_1,c_2,b_3)
bac	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(b_1,c_2,a_3)	(b_1,c_2,a_3)
bca	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(b_1,c_2,a_3)	(b_1,c_2,a_3)
cab	(c_1,a_2,b_3)	(c_1,a_2,b_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(c_1,a_2,b_3)	(c_1,a_2,b_3)
cba	(c_1,a_2,b_3)	(c_1,a_2,b_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(c_1,a_2,b_3)	(c_1,a_2,b_3)

bca

	abc	acb	bac	bca	cab	cba
abc	**(e_1,e_2,b_3)**[4]	**(e_1,e_2,b_3)**[4]	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,c_2,b_3)	(a_1,c_2,b_3)
acb	**(e_1,e_2,b_3)**[4]	**(e_1,e_2,b_3)**[4]	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,c_2,b_3)	(a_1,c_2,b_3)
bac	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(b_1,c_2,a_3)	(b_1,c_2,a_3)
bca	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(b_1,c_2,a_3)	(b_1,c_2,a_3)
cab	(c_1,a_2,b_3)	(c_1,a_2,b_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(c_1,a_2,b_3)	(c_1,a_2,b_3)
cba	(c_1,a_2,b_3)	(c_1,a_2,b_3)	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(c_1,a_2,b_3)	(c_1,a_2,b_3)

cab

	abc	acb	bac	bca	cab	cba
abc	(d_1,d_2,c_3)[1]	(d_1,d_2,c_3)[1]	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)
acb	(d_1,d_2,c_3)[1]	(d_1,d_2,c_3)[1]	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)
bac	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(b_1,a_2,c_3)	(b_1,a_2,c_3)
bca	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(b_1,a_2,c_3)	(b_1,a_2,c_3)
cab	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(d_1,d_2,c_3)[1]	(a_1,b_2,c_3)
cba	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(b_1,a_2,c_3)	(a_1,b_2,c_3)

cba

	abc	acb	bac	bca	cab	cba
abc	(d_1,d_2,c_3)[1]	(d_1,d_2,c_3)[1]	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)
acb	(d_1,d_2,c_3)[1]	(d_1,d_2,c_3)[1]	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)
bac	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(b_1,a_2,c_3)	(b_1,a_2,c_3)
bca	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(b_1,a_2,c_3)	(b_1,a_2,c_3)
cab	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(d_1,d_2,c_3)[1]	(a_1,b_2,c_3)
cba	(b_1,a_2,c_3)	(b_1,a_2,c_3)	(a_1,b_2,c_3)	(a_1,b_2,c_3)	(b_1,a_2,c_3)	(a_1,b_2,c_3)

[1] Whoever wins the lottery is assigned to school *a* whereas the 'loser' is assigned to school *b*.
[2] If i_1 wins the lottery, assignments are (i_1, *a*), (i_1, *b*) and (i_3, *c*). If i_2 wins the lottery, assignments are (i_1, *c*), (i_2, *a*) and (i_3, *b*).
[3] If i_1 wins the lottery, assignments are (i_1, *a*), (i_2, *c*) and (i_3, *b*). If i_2 wins the lottery, assignments are (i_1, *b*), (i_2, *a*) and (i_3, *c*).
[4] If i_1 wins the lottery, assignments are (i_1, *a*), (i_2, *c*) and (i_3, *b*). If i_2 wins the lottery, assignments are (i_1, *c*), (i_2, *a*) and (i_3, *b*).

Table 3
Expected Payoff Matrix for Example-4

	abc	*acb*	*bac*	*bca*	*cab*	*cba*
abc	(a_1,b_2,c_3)	(a_1,c_2,b_3)	$\boldsymbol{(a_1,b_2,c_3)}$	$\boldsymbol{(a_1,b_2,c_3)}$	(a_1,c_2,b_3)	(a_1,c_2,b_3)
acb	(a_1,b_2,c_3)	(a_1,c_2,b_3)	$\boldsymbol{(a_1,b_2,c_3)}$	$\boldsymbol{(a_1,b_2,c_3)}$	(a_1,c_2,b_3)	(a_1,c_2,b_3)
bac	$\boldsymbol{(b_1,e_2,e_3)}$[1]	$\boldsymbol{(b_1,e_2,e_3)}$[1]	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(b_1,c_2,a_3)	(b_1,c_2,a_3)
bca	$\boldsymbol{(b_1,e_2,e_3)}$[1]	$\boldsymbol{(b_1,e_2,e_3)}$[1]	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(b_1,c_2,a_3)	(b_1,c_2,a_3)
cab	(c_1,d_2,d_3)[2]	(c_1,d_2,d_3)[2]	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(b_1,c_2,a_3)	(b_1,c_2,a_3)
cba	(c_1,d_2,d_3)[2]	(c_1,d_2,d_3)[2]	(c_1,b_2,a_3)	(c_1,b_2,a_3)	(b_1,c_2,a_3)	(b_1,c_2,a_3)

[1] Whoever wins the lottery is assigned to school *a* whereas the loser is assigned to *c*.
[2] Whoever wins the lottery is assigned to school *a* whereas the loser is assigned to *b*.