ASSOUAD NAGATA DIMENSION ON THE INTEGERS

1. Basic Facts and Definitions

Definition 1.1. A metric space $X$ is said to be of Assouad-Nagata dimension $\leq n$, i.e., $\dim_{AN} X \leq n$, if there exists a constant $c$ so that $\forall r > 0$ we can construct a cover $U = \{U_0, U_1, \ldots, U_n\}$ of $X$ such that each $U_i$ is $r$-disjoint and uniformly bounded by $cr$.

Definition 1.2. A metric space $X$ has Assouad-Nagata dimension $= n$ if $\dim_{AN} X \leq n$ is true and $\dim_{AN} X \leq n - 1$ is false.

Remark 1.3. Three points:
(i) Each $U_i$ is a family of sets.
(ii) $r$-disjoint means that any two disjoint sets in $U_i$ have distance at least $r$.
(iii) A family of sets is called uniformly bounded by $D$ if every set of the family has diameter less than $D$.

Definition 1.4. For different values of $r$ the map $D^n_r : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $D^n_r (r) = cr$ which gives the bound of the diameters is formally called an $n$-dimensional control function. In Assouad Nagata dimension the $n$-dimensional control function is a dilation.

Definition 1.5. A bi-Lipschitz function $f : (X, d_X) \to (Y, d_Y)$ satisfies the property
$$\mu d_X(x, y) \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y)$$
for some $\mu, \lambda > 0$.

Proposition 1.6. Suppose that $X$ and $Y$ are metric spaces and that $\dim_{AN} X \leq n$. If $f : X \to Y$ is bi-Lipschitz and onto then $\dim_{AN} Y \leq n$.

Let $U$ a covering of $X$. Use $f(U)$ to create a covering of $Y$.

Remark 1.7. $\dim_{AN}$ is bi-Lipschitz invariant.

Proposition 1.8. For a metric space $(X, d)$, the following are equivalent:
(i) $\dim_{AN} X \leq n$
(ii) $\exists c_1$ such that $\forall r > 0, \exists$ a cover $U_r$ of $X$ with $r$-multiplicity at most $n + 1$ and mesh at most $c_1 r$.
(iii) $\exists c_2 > 0$ such that $\forall r > 0, \exists$ a cover $V_r$ of $X$ with multiplicity at most $n + 1$, mesh at most $c_2 r$ and Lebesque number at least $r$.

For the proof of the above see [7]

Definition 1.9.
- $\text{mesh}(U_r) = \sup \{\text{diam}(A)| A \in U_r\}$
- $\text{multiplicity}(U_r) = \max \{|A_i| \in U_r, \text{such that } \cap_i A_i \neq \emptyset\}$

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- $r$ - multiplicity $(U_r) = \max\{\#A_i \text{ intersecting a ball } B_r(x)\}$
- Lebesgue number $(U_r) = \inf\{\sup\{\text{dist}(x, X - U) : U \in U_r\} : x \in X\}$

**Proposition 1.10.** If $X = A \cup B$, then $\dim_{AN}X \leq \max\{\dim_{AN}A, \dim_{AN}B\}$.

The proof is really straightforward.

**Definition 1.11.** A metric space $X$ is said to be of asymptotic Assouad-Nagata dimension $n$, i.e. $asdim_{AN}X = n$, if $n$ is the smallest natural number for which $\exists c, d$ constants such that $\forall r > 0$ there exists a cover $U = \{A_0, A_1, \ldots, A_n\}$ of $X$ such that each $A_i$ is a family of $r$-disjoint sets and $\text{diam}(A_{j}) \leq cr + d$.

**Remark 1.12.** In asymptotic Assouad-Nagata dimension the dimension control function is linear and not just a dilation.

**Proposition 1.13.** For a discrete metric space $X$, $asdim_{AN}X = \dim_{AN}X$. In particular, if $X$ is a finitely generated group with the word metric, then $asdim_{AN}X = \dim_{AN}X$.

For a detailed proof of the above see [5]

2. FINITELY GENERATED GROUPS AS METRIC SPACES

**Definition 2.1.** Let $G = \langle S | R \rangle$, $|S| < \infty$. Define the following word norm on $G$:

$$||x|| = \min\{l(w) : x = w = s_{i_1}s_{i_2}\ldots s_{i_k}, s_{i_j} \in S\}$$

where $k = l(w)$ is the length of the word $w$.

**Definition 2.2.** Define the word metric on $G$ by $d_w(x, y) = ||x^{-1}y||$.

**Proposition 2.3.** $d_w$ is left $G$-invariant, $d_w(x, y) = d_w(gx, gy)$.

Proof.

$$d_w(gx, gy) = ||(gx)^{-1}gy|| = ||x^{-1}g^{-1}gy|| = ||x^{-1}y|| = d_w(x, y) \Box$$

**Proposition 2.4.** $(G, d_w)$ is a discrete metric space.

Proof. $d_w(x, y) \in \mathbb{Z}$.

**Corollary 2.5.** With $d_w$, $asdim_{AN}G = \dim_{AN}G$.

**Proposition 2.6.** If $G$ is finitely generated and $\text{diam}(G) < M$ then $G$ is finite.

Proof. For any $g \in G$, $||g|| \leq M$. If $|S| = k$ then $|G| \leq (2k)^M$.

**Proposition 2.7.** If $G$ is finitely generated and $\text{diam}(G) < M$ then $asdim_{AN}G = 0$.

Proof. Let $U_0 = G$. Consider $e = 1, d = M$. Then for all $r > 0$, $U_0$ is a cover of $G$, $U_0$ is trivially $r$-disjoint, and $\text{diam}(U_0) \leq D^0_2(r) = 1r + M$.

**Corollary 2.8.** If $G$ is finitely generated and $\text{diam}(G) < M$ then $\dim_{AN}G = 0$. See 1.13.

**Proposition 2.9.** If $G$ is finite then $asdim_{AN}G = \dim_{AN}G = 0$.

Proof. Let $A_0 = G$. 


Proposition 2.10. If $\operatorname{asdim}_{AN}G = 0$, and $G$ finitely generated, then $G$ is finite.

Proof. Let $r \geq 1$ and consider the appropriate covering. Let $G \subset U_0 = \{A_0, A_1, \ldots \}$. $|A_0| = k < \infty$ because, $\operatorname{diam}(A_0) \leq c_1 + d = c + d$. As a result, if $A_0 \supset G$, then $G$ is finite, as desired. Assuming that this is not the case, there exist $x \in A_0$ and $s \in S \cup S^{-1}$ such that $xs \notin A_0$. If $xs \in A_n$, then we have that $A_0$ and $A_n$ are at most $1$-disjoint, contradiction. Thus, $U_0 = \{A_0\}$ and $G$ is finite. □

Definition 2.11. A map $\| \cdot \| : G \to \mathbb{R}$ is said to be a proper norm if it satisfies the following conditions:

i) $\|g\|_G = 0$ if and only if $g$ is the neutral element in $G$.

ii) $\|g\|_G = \|g^{-1}\|_G$ for every $g \in G$.

iii) $\|g \cdot h\|_G \leq \|g\|_G + \|h\|_G$ for all $g, h \in G$

iv) For every $k > 0$ the number of elements in $G$ such that $\|g\|_G \leq k$ is finite.

Proposition 2.12. Suppose $G$ is an infinite, finitely generated group with a proper norm. Consider $d : G \times G \to \mathbb{R}^+$ the induced metric. Then $\operatorname{asdim}_{AN}(G, d) > 0$.

Proof. Assume not. Let $k = \min\{|s_i|; s_i \in S \cup S^{-1}\}$

Let $r = k + 1$. Then there exists a cover of $G$ containing only one family of sets, $U = \{A\}$, with $A = \{A_1, A_2, \ldots \}$. Suppose $A_i \supset G$ for some $i$. Then $\operatorname{diam}(G) \leq \operatorname{diam}(A_i) \leq D$. Since $G$ is discrete $G$ has to be finite. Contradiction. Thus $G$ is not fully contained in any of the $A_i$’s. In particular $G$ is not contained in $A_1$. So there exists some $x \in A_1$ with $xs \notin A_1$ where $s \in (S \cup S^{-1})$. Then, if $xs \in A_j$, $A_1$ and $A_j$ are at most $r$-disjoint, and so they are not $r$-disjoint. Contradiction. □

Remark 2.13. Since $\mathbb{Z} < 1 >$ is an infinite, free group, with one generator, from the proposition above we have that $\operatorname{asdim}_{AN}(\mathbb{Z}, d_w) \neq 0$. Further, we would like to know precisely the $\operatorname{asdim}_{AN}\mathbb{Z}$. That result is an immediate corollary of the following theorem.

Theorem 2.14. If $T$ is a tree then $\operatorname{asdim}_{ANT} \leq 1$.

Proof. Fix $e$ to be the origin of the tree. Let $d > 0$. Consider $A_k = \{x \in T : kd \leq d(x, e) < (k + 1)d\}$ the “Annulus” of radius $(kd, (k + 1)d)$ for all $k \in \mathbb{N}$ Consider the Gromov product on the tree with $e$ as base point:

$$(x|y) = \frac{1}{2}(d(x, e) + d(y, e) - d(x, y))$$

Define the relation:

$$x \sim y \iff (x|y) \geq (k - \frac{1}{2})d$$

Recall that the Gromov product on the tree satisfies the relation:

$$(x|y) \geq \min \{ (x|z), (y|z) \}$$

since the tree is $0$-hyperbolic. This proves that the relation is transitive. The other two properties are trivial thus we have an equivalence relation. Let $U_{i, k}$ the equivalence classes in $A_k$. Name:

$$U_0 = \{U_{i, k}, i \in I_k, k \in \mathbb{N} \text{ even}\}$$

$$U_1 = \{U_{j, k}, j \in J_k, k \in \mathbb{N} \text{ odd}\}$$

□
Lemma 3.2. These families are \( d \)-disjoined. We will prove it for \( \mathcal{U}_k \). Let \( U_{1,k}, U_{2,l} \) be two sets of \( \mathcal{U}_k \). If \( k \neq l \) then clearly
\[
d(U_{1,k}, U_{2,l}) \geq d(A_k, A_l) = |k - l|d
\]
and since \( k \neq l \) we have \( |k - l| \geq 1 \) since both are integers. Thus \( d(U_{1,k}, U_{2,l}) \geq d \).

Remark 3.1. Thus far, we have restricted our investigation to the norm that is
\[
\text{asdim} Z \leq 1.
\]

Proof. By the previous proposition, \( \text{asdim}_\mathbb{Z} = 1 \) and by our theorem, \( \text{asdim}_\mathbb{Z}(\mathbb{Z}, d_w) \leq 1 \) so \( \text{asdim}_\mathbb{Z}(\mathbb{Z}, d_w) = 1 \).

3. Monotone norms on \( \mathbb{Z} \)

Remark 3.1. Thus far, we have restricted our investigation to the norm that is
most natural to apply to finitely generated groups. However, this restriction has
been self-imposed and there is no technical reason to avoid other types of norms.
We now generalize our results for proper, monotone norms.

Lemma 3.2. If \( p : \mathbb{R} \rightarrow \mathbb{R}^+ \) is an increasing function then
\( B_1(x, p(t)) = B_2(x, t) \)
for all \( x \in \mathbb{R} \) and \( t \in \mathbb{R}^+ \) where \( B_1(x, p(t)) = \{ y \in \mathbb{Z} : p(x - y) \leq p(r) \} \) and
\( B_2(x, t) = \{ y \in \mathbb{Z} : |x - y| \leq t \} \).

Proof. Notice that \( y \in B_1(x, p(r)) \Leftrightarrow p(x - y) \leq p(r) \). Since \( p \) is increasing we get
\( p(x - y) \leq p(r) \Leftrightarrow |x - y| \leq r \). Clearly \( |x - y| \leq r \Leftrightarrow y \in B_2(x, r) \). Combining the
above we have the wanted inclusions.

Lemma 3.3. If \( p : \mathbb{R} \rightarrow \mathbb{R}^+ \) is increasing then the families:
\[
\mathcal{U}_0 = \{ B_1(4kt, p(t)) : k \in \mathbb{Z} \}
\]
\[
\mathcal{U}_1 = \{ B_1((4k + 2)t, p(t)) : k \in \mathbb{Z} \}
\]
are \( p(t) \)-disjoined and \( 2p(t) \)-bounded, where \( B_1(x, s) \) as the previous lemma. Fi-

Finally, they form a cover of \( \mathbb{Z} \).

Proof. From the previous lemma we have that:
\[
\mathcal{U}_0 = \{ B_2(4kt, t) : k \in \mathbb{Z} \}
\]
\[
\mathcal{U}_1 = \{ B_2((4k + 2)t, t) : k \in \mathbb{Z} \}
\]
Clearly $\mathcal{U}_0$ and $\mathcal{U}_1$ cover $\mathbb{Z}$. Given $a \in \mathbb{Z}$, there exists an integer $k$ such that $4kt \leq a < 4(k+1)t$. Then, either $4kt \leq a < 4kt+t$ and $a \in B_2(4kt, t)$, $(4k+2)t-t \leq a < (4k+2)t + t$ and $a \in B_2((4k+2)t, t)$ or $4(k+1)t - t \leq a < 4(k+1)t$ so $a \in B_2(4(k+1)t, t)$.

Also they are $2p(t)$-bounded since:

$$\text{diam}(B_1(4kt, p(t))) \leq 2p(t)$$

Finally it is easy to show that $\mathcal{U}_0$ and $\mathcal{U}_1$ are $p(t)$-disjoined. We will prove it for $\mathcal{U}_0$ and the proof is the same for $\mathcal{U}_1$. Let $B_1(4k_1t, p(t))$ and $B_1(4k_2t, p(t))$ with $k_1 \neq k_2$ such that they are not $p(t)$-disjoined. Then, since both of them are compact, there exist $x \in B_1(4k_1t, p(t))$ and $y \in B_1(4k_2t, p(t))$ such that $p(|x - y|) < p(t)$. Since $p$ is increasing we get $|x - y| < t$. But then

$$|4k_1t - 4k_2t| \leq |4k_1t - x| + |x - y| + |y - 4k_2t|$$

(A)

Since $x \in B_1(4k_1t, p(t))$ we get $x \in B_2(4k_1t, t)$ and thus $|x - 4k_1t| \leq t$. Similarly $|y - 4k_2t| \leq t$. Thus (A) becomes:

$$|4k_1t - 4k_2t| \leq 3t$$

which leads to $|k_1 - k_2| \leq \frac{3}{4}$. Since $k_1, k_2 \in \mathbb{Z}$ we have $k_1 - k_2 = 0$ and thus $k_1 = k_2$ which is a contradiction. Thus $\mathcal{U}_0$ is $p(t)$-disjoined.

\[\square\]

**Proposition 3.4.** Suppose $p : \mathbb{R} \to \mathbb{R}^+$ is an increasing, proper, non-bounded norm. Consider the induced metric $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^+$:

$$d(x, y) = p(|x - y|)$$

Then asdim$_{AN}(\mathbb{Z}, d) = 1$

**Proof.** Let $r > 0$. Since $p$ is not bounded there exists an $a \in \mathbb{Z}$ such that $p(a-1) \leq r$ and $p(a) > r$. Consider the two families

$$\mathcal{U}_0 = \{B_1(4ka, p(a)) : k \in \mathbb{Z}\}$$

$$\mathcal{U}_1 = \{B_1((4k + 2)a, p(a)) : k \in \mathbb{Z}\}$$

From the previous lemma we have that $\mathcal{U}_0, \mathcal{U}_1$ are a covering of $\mathbb{Z}$. Also they are $p(a)$-disjoined. Since $p(a) \geq r$ they are also $r$-disjoined. Finally they are $2p(a)$-bounded. Notice that

$$2p(a) \leq 2(p(a-1) + p(1)) \leq 2p(a-1) + 2p(1) \leq 2r + 2p(1)$$

name $c = 2$, $d = 2p(1)$. Then the families are $cr + d$ disjoined. Since $r$ was chosen arbitrarily and $c, d$ are constants from the definition of asymptotic Assouad - Nagata dimension we have asdim$_{AN}(\mathbb{Z}, d) \leq 1$.

From a previous proposition([2,12]) asdim$_{AN}(\mathbb{Z}, d) \geq 1$ which concludes the proof.

\[\square\]

**Proposition 3.5.** Suppose that $p : \mathbb{R} \to \mathbb{R}^+$ is a decreasing norm. Then the induced metric on $\mathbb{Z}$ is not proper.

**Proof.** Suppose that the induced metric was proper. Consider the ball $B(x, p(r)) = \{y \in \mathbb{Z} : p(x-y) < p(r)\}$. Since $p$ is decreasing we have that $B(x, p(r)) \supseteq A = \{y \in \mathbb{Z} : |x-y| > r\}$. But $B_1(x, p(r))$ is compact. Since it is a metric space.
$B_1(x, p(r))$ is bounded and thus finite since the metric is proper and $\mathbb{Z}$ is discrete. On the other hand $A$ is clearly infinite. Contradiction.

**Remark 3.6.** Since we only deal with proper norms we can now say that our norm is monotone and mean that it is increasing.

**Corollary 3.7.** Suppose that $p : \mathbb{R} \to \mathbb{R}^+$ is a norm which is constant in $\mathbb{Z}$ (and 0 at 0). Then the induced metric is not proper.

**Proof.** Let $p(x) = d$ for all $x \neq 0$ in $\mathbb{Z}$. Then the ball $B(0, 2d) = \mathbb{Z}$, is infinite and thus the norm cannot be proper.  

**References**


