Approximate Probabilistic Optimization Using Exact-Capacity-Approximate-Response-Distribution (ECARD)

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There are two major barriers in front of probabilistic structural design. First, uncertainties associated with errors in structural and aerodynamic modeling and quality of construction are not well characterized as statistical distributions and insufficient information may lead to large errors in probability calculations. Second, probabilistic design is computationally expensive because repeated stress calculations (typically finite element analysis) are required for updating probability calculation as the structure is being re-designed. Targeting these two barriers, we propose a probabilistic design optimization method, where the probability of failure calculation is confined to failure stresses, to take advantage of the fact that statistical characterization of failure stresses is required by Federal Aviation Administration (FAA) regulations. The stress distribution is condensed into a representative single value thereby eliminating the need for expensive stress distribution calculation, so a probabilistic optimization problem is transformed into a semi-deterministic optimization problem. Since the procedure starts from the deterministic optimum, a small number of iterations is expected, and a reliability analysis is required only once in each iteration. The proposed method provides approximate sensitivity of failure probability with respect to design variables, which is essential in risk allocation. The method is demonstrated with (i) a beam problem with two failure modes, and (ii) ten-bar truss problem. In the ten-bar truss problem, risk is allocated between the truss elements, while risk is allocated between different failure modes in the beam example.

Nomenclature

β	=	reliability index
c.o.v.	=	coefficient of variation

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c_f, c_σ = coefficients of variation of the failure stress and the stress, respectively.

Δ^* = relative change in characteristic stress σ^* corresponding to a relative change of Δ in stress σ

Δ = relative change in stress

$F(\ )$ = cumulative distribution function of the failure stress

$f(\ )$ = probability density function of the failure stress

k = correction factor for characteristic stress

μ_f, μ_σ = mean values of the failure stress and the stress, respectively.

$s(\ )$ = probability density function of the stress

σ_f = failure stress

σ^* = characteristic stress

$\sigma_p{}^*$ = characteristic stress for previous design

σ = stress

P_f^{approx} = approximate probability of failure of probabilistic design

$P_f{}^p$ = probability of failure at previous design

P_f = actual probability of failure

P_{fd} = probability of failure at given deterministic design

W_d = weight of deterministic design

W = weight of probabilistic design

I. Introduction

There are two major barriers in front of probabilistic (or reliability-based) structural design. First, uncertainties associated with errors in structural and aerodynamic modeling and quality of construction are not well characterized as statistical distributions, and it has been shown that insufficient information may lead to large errors in probability calculations (e.g., Ben-Haim and Elishakoff[1], Neal, *et al.*[2]). Due to this fact, many engineers are reluctant to pursue probabilistic design. The second barrier to the application of probabilistic structural optimization is computational expense. Probabilistic structural optimization is expensive because repeated stress calculations (typically FEA) are required for updating probability calculation as the structure is being re-designed. Targeting these two main barriers, we propose an approximate method that substantially reduces the number of expensive probabilistic stress calculations. In the proposed method, the probabilistic calculation is confined only to failure stress, which is often well characterized. The stress probability distribution is updated only periodically.

Traditionally, reliability-based design optimization (RBDO) is performed based on a double-loop optimization scheme, where the outer loop is used for design optimization while the inner loop performs a sub-optimization for reliability analysis, using methods such as First-Order Reliability Method (FORM). Since this traditional approach is computationally expensive, even prohibitive for problems that require complex finite element analysis (FEA), alternative methods have been proposed by many researchers. For instance, Lee and Kwak[3], Kiureghian *et al.*[4], Tu *et al.*[5], Lee *et al.*[6], and Qu and Haftka[7] used inverse reliability measures to reduce the computational expense of the inner loop. To alleaviate the computational cost further, single loop approaches were also proposed. This can be achieved by replacing the probabilistic optimization with sequential deterministic optimization (often using inverse reliabilty measures see Du and Chen[8] and Ba-abbad *et al.*[9]). However, these methods do not easily lend themselves to allocating risk between failure modes in a structure where many components can fail[11]. We note, however, that most of the computational expense is associated with repeated stress calculation. So we propose an approximate probabilistic design approach that reduces the number of expensive stress calculations. That is, we approximate the probabilistic optimization that separates the uncertainties which can be evaluated inexpensively and those whose effects are expensive to evaluate. We boil down the stress distribution to a single characteristic stress by utilizing the inverse cumulative distribution of the failure stress, and we propose an inexpensive approximation of that characteristic stress. We call the proposed approximate probabilistic design approach exact-capacity-approximate-response-distribution or ECARD.

The remainder of the paper is organized as follows. Section II proposes an approximate method that allows probabilistic design based only on probability distribution of failure stresses. The application of the method to a beam problem and ten-bar truss problem are presented in Sections III and IV. Finally, the concluding remarks are listed Section V.

II. Exact-Capacity Approximate-Response-Distribution Probabilistic Structural Design

Structural failure, using most failure criteria, occurs when a stress, σ, at a point exceeds a failure stress, σ_f. Both the stress and the failure stress often show uncertainty due to the randomness in system parameters. In such a case, the safety of the system can be estimated using a probability of failure. When the failure stress is random but the stress σ is deterministic, the probability of failure, P_f, is defined as

$$P_f = \mathrm{Prob}\left(\sigma \geq \sigma_f\right) = F\left(\sigma\right) \tag{1}$$

where F is the cumulative distribution function (CDF) of the failure stress σ_f. On the other hand, when both the stress and the failure stress are random, the probability of failure is calculated by integrating Eq. (1) for all possible values of the stress σ

$$P_f = \int_{-\infty}^{\infty} F(\sigma) s(\sigma)\, d\sigma \tag{2}$$

where $s(\sigma)$ is the probability density function (PDF) of the stress. The above integral can be evaluated using either analytical integration, Monte Carlo simulation (MCS), or first-/second-order reliability method (FORM/SORM).

It is clear from Eq. (2) that accurate estimation of probability of failure requires accurate assessments of the probability distributions of the stress, σ, and the failure stress, σ_f. For the failure stress σ_f, the FAA requires aircraft builders to perform characterization tests in order to construct a statistical model, and then to select failure allowables (A-basis or B-basis values) based on this model. Hence, the failure stress is often characterized reasonably well statistically. On the other hand, the PDF of the stress, s (σ), is poorly known, because it depends on the accuracy of various factors, such as structural and aerodynamic calculations, the knowledge of the state of the structure, damage progression, flight conditions and pilot actions.

By using the intermediate value theorem[12], Eq. (2) can be re-written as

$$P_f = F(\sigma^*)\int_{-\infty}^{\infty} s(\sigma)\,d\sigma = F(\sigma^*) \tag{3}$$

where the second equality is obtained by using the fact that the integral of $s(\sigma)$ is one. Equation (3) states that the effect of (the poorly characterized) probability density of the stress can be boiled down to a single characteristic stress, σ^*. This value can be obtained by (a) calculating the probability of failure from Eq. (2) with estimated $s(\sigma)$ and (b) using the inverse transformation of the CDF of the failure stress from

$$s^* = F^{-1}(P_f) \tag{4}$$

Instead of using the estimated $s\ (\sigma)$, it is equally possible to use historical data on the probability of failure of aircraft structural components to replace step (a). That is, given an estimate of the probability of failure, we can obtain the characteristic stress σ^* that corresponds to this historical aircraft accident data when airplanes are designed using the deterministic FAA process. In addition, when the probability of failure changes from P_f to P_f^*, the change in the characteristic stress can be represented using the relative change D^* (see Figure 1).

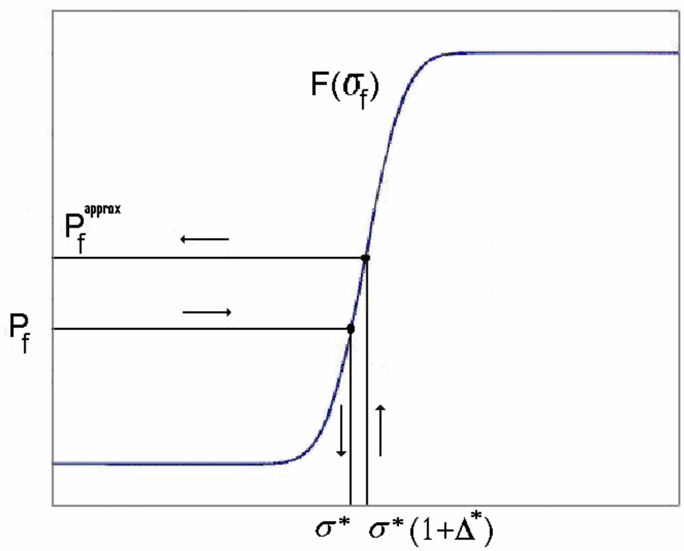


Figure 1. Calculation of characteristic stress σ^* from the given probability of failure and CDF of the failure stress

In this paper, we assume that the probabilistic design starts from a known deterministic optimum. The probabilistic design will improve upon the deterministic design by reducing the structural safety margin on some components while increasing it for others. We assume that the structural redesign changes the stress distribution by simple scaling of σ to $\sigma(1+\Delta)$ as shown in Figure 2. The changed random stress $\sigma(1+\Delta)$ will produce a new probability of failure, P_f^{approx}, and will have a new characteristic stress, $\sigma^*(1+\Delta^*)$. The key idea of the proposed approximate probability distribution is that the new characteristic stress can be approximated without recourse to the expensive probabilistic analysis. After redesign, the relation between the probability of failure and the characteristic stress is given as

$$P_f^{approx} = F\left[\sigma^*(1+\Delta^*)\right] \quad \text{or} \quad \sigma^*(1+\Delta^*) = F^{-1}(P_f^{approx}) \tag{5}$$

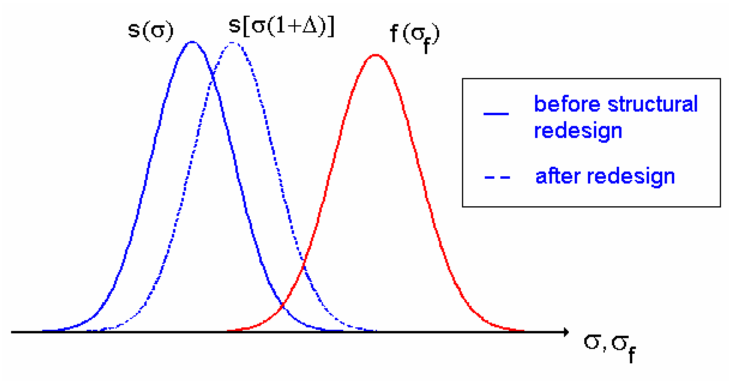


Figure 2. Stress distribution s(σ) before and after redesign in relation to failure-stress distribution f(σ_f).

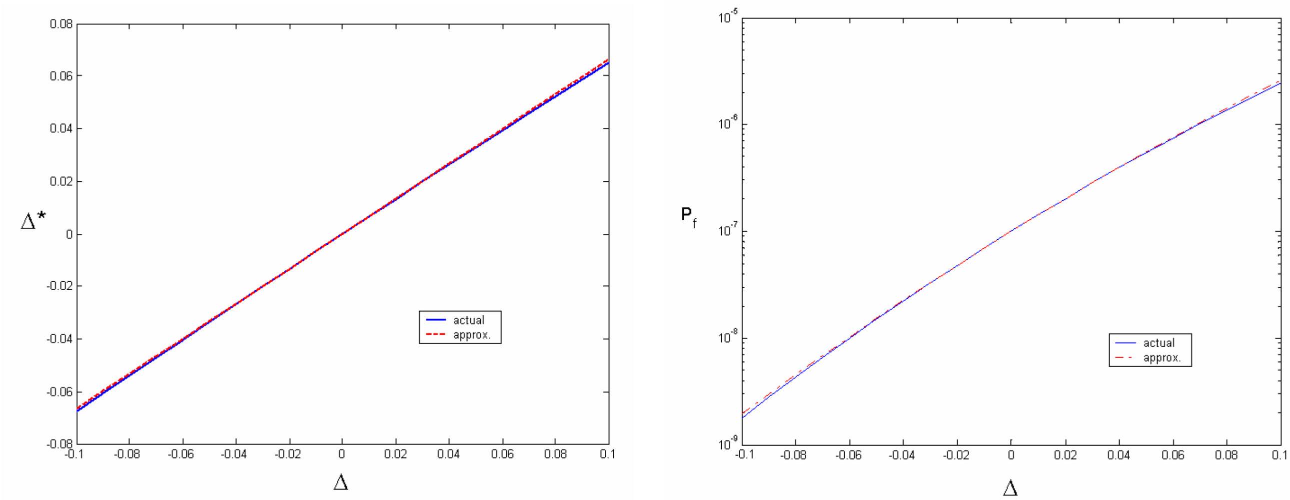
We assume that the relative change in the characteristic stress, Δ^*, is proportional to the relative change in the stress, Δ. That is,

$$\Delta^* = k\,\Delta \tag{6}$$

where k is a proportionality constant that depends on the mean and coefficient of variation of the stress and the failure stress. It is the sensitivity of the characteristic stress change with respect to the stress change. In this paper, we call it the *correction factor*. The above assumption in linearity is reasonable when Δ is relatively small (as we will see next). Since the probabilistic design starts from the optimum deterministic design, the relative change in stress, Δ, will be small in general.

Probabilistic optimization can be viewed as risk allocation between different failure modes or different structural members. This allocation requires the sensitivity of failure probability with respect to design variables, which is missing from the standard sequential optimization and reliability assessment (SORA) method[8]. In the proposed approximate probabilistic optimization, this sensitivity information is approximately presented in the correction factor. The change in the failure probability is represented in Δ^*, while the change in design variables is represented in Δ.

We will demonstrate that a linear relationship between Δ and Δ^* works well using a typical transport aircraft structure that has the probability of failure around 10^{-7}. We consider lognormally distributed failure stress with mean value of $\mu_f = 100$ and coefficient of variation of $c_f = 8\%$, and normally distributed stress with coefficient of variation of $c_\sigma = 20\%$. The mean value of the stress is calculated as 42.49 to have a probability of failure of 10^{-7}. Figure 3(a) shows the relation between Δ and Δ^*. We can see that the linearity assumption is quite accurate over the range $-10\% \le \Delta \le 10\%$. Figure 3(b) shows the effect of the Δ^* approximation on the probability of failure.



(a) (b)

Figure 3. (a) Comparison of approximate and exact Δ and Δ* and (b) the resulting probabilities of failure for lognormal failure stress (with μ_f=100 and c_f=8%) and normal stress (with μ_σ=42.49 and c_σ=20%)

Given the deterministic design as an initial design, we can perform an approximate probabilistic re-design as follows.

1. Calculate the probability of failure at the given design or previous design, $P_f^{\,p}$ (using FORM or MCS)
2. Calculate characteristic stress σ_0^* from Eq. (3), using the inverse CDF of the $P_f^{\,p}$ and the mean and c.o.v of the failure stress.
3. Calculate deterministic stresses σ_0 for the initial design using the mean values of all input variables.
4. Calculate correction factor k:
   a. If using FORM: Perturb the response by some Δ (here 0.01) and obtain failure probabilities for both responses. Then use inverse CDF of the probability of failures to calculate characteristic stress responses for both of them. Calculate Δ^* for each characteristic response using

$$\Delta^* = \frac{\sigma^*}{\sigma_p{}^*} - 1 \tag{7}$$

   Calculate the correction factor:

$$k = \frac{\Delta^*}{\Delta} \tag{8}$$

   b. If using MCS: Perturb the random response samples up and then down by $\Delta_1 = -\Delta_2$ (here 5%). Then use inverse CDF of the probability of failures to calculate the characteristic responses. Calculate Δ^* for each response using Eq. (7), and compute the corresponding two correction factors k_1 and k_2 using Eq. (8). Using Eq. (9), calculate the correction factor as the average of k_1 and k_2

$$k = \tfrac{1}{2}(k_1 + k_2) \tag{9}$$

5. Perform re-design by solving the exact-capacity-approximate-response-distribution (ECARD) optimization given in Eq. (10), where new deterministic stresses σ are calculated using the mean values of all random variables.

$$\begin{aligned} &\min_{\overline{\mathrm{x}}} \; W\left(\overline{\mathrm{x}}\right) \\ &\text{s.t.} \;\; P_f{}^{approx}\left(\overline{\mathrm{x}}\right) \le P_{fd} \end{aligned} \tag{10}$$

where

$$\Delta = \frac{\sigma}{\sigma_p} - 1 \tag{11}$$

$$\Delta^* = k\,\Delta \tag{12}$$

$$\sigma^* = F^{-1}\left(P_f{}^p\right) \tag{13}$$

$$P_f{}^{approx} = F\left[\sigma^*\left(1+\Delta^*\right)\right] \tag{14}$$

$$\sigma = \sigma\left(\overline{\mathrm{x}}\right), \text{ and } \;\sigma_p = \sigma_p\left(\overline{\mathrm{x}}\right) \tag{15}$$

6. Check weight change compared to the previous iteration and error in probability of failure estimate (P_f-P_f^*) to their pre-specified tolerances for convergence, If converged in Step 8, STOP. Otherwise, GO TO Step 1 and CONTINUE.

Thus the method uses exact representation of the failure stress distribution (capacity) and an approximate modeling of the stress distribution (response), hence the name exact-capacity-approximate-response distribution (ECARD) method. The accuracy of ECARD to locate the true optimum depends on the magnitudes of errors involved in the approximations. For instance, Figure 3 showed that the approximation works well if the changes in the stresses due to redesign are small. Also, the accuracy in estimating the correction factor *k* affects the accuracy of the approximate method. If accuracy is not sufficient, the approximate method will lead to a design likely to be nearer to the true optimum and the failure probability close to the required ones. Then, the approximate optimum can be used as the new starting point and the approximate optimization can be performed in such an iterative way until the sufficient accuracy in probability of failure is reached. The result may still be somewhat sub-optimal because of the approximate nature of the sensitivity of probability of failure. Iterative use of approximate method is discussed in more detail in the following section.

III. Application of ECARD to Beam Problem

Our first demonstration example is a cantilever beam problem, where risk is allocated between different failure modes.

Problem description

The cantilever beam design problem is analyzed by many researchers including Wu et al. [10], Qu et al.[7], Ba-abbad et al[9]. The cantilever beam depicted in Figure 4 has two failure modes: stress failure and excessive displacement.

The minimum weight design is sought by varying the width w and thickness t of the beam. The applied loads F_X and F_Y along with the elastic modulus E and failure stress σ_f are random variables. All random variables are assumed normally distributed with mean and coefficient of variation values as listed in Table 1. The beam width w and thickness t are modeled as deterministic variables.

Table 1. The mean and coefficient of variation of the random variables. *All variables follow normal distribution.*

Random variable	Mean	Coefficient of variation
F_X (lb)	500	20%
F_Y (lb)	1,000	10%
E (psi)	2.9×10^7	5%
σ_f (psi)	40,000	5%

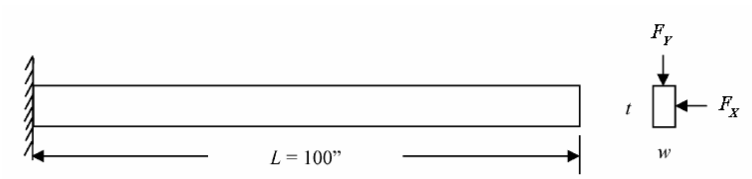


Figure 4. Cantilever beam: geometry and loading

The limit-state functions corresponding to stress failure mode can be written as

$$g_1 = \sigma_f - \left(\frac{600}{wt^2} F_Y + \frac{600}{w^2 t} F_X \right) \equiv C_1 - R_1 \tag{16}$$

where C_1 and R_1 are the capacity and response parameters of g_1. Similarly, the limit-state functions corresponding to displacement failure mode can be written as

$$g_2' = D_0 - \frac{4L^3}{Ewt} \sqrt{\left(\frac{F_Y}{t^2} \right)^2 + \left(\frac{F_X}{w^2} \right)^2} \tag{17}$$

Or, the limit-state function can be re-written in a more convenient form for our approximate method as

$$g_2 = \frac{D_0 Ewt}{4L^3} - \sqrt{\left(\frac{F_Y}{t^2} \right)^2 + \left(\frac{F_X}{w^2} \right)^2} \equiv C_2 - R_2 \tag{18}$$

where C_2 and R_2 are the capacity and response parameters of g_2, L is the beam length of 100 inches and the critical displacement D_0 is taken as 2.2535 inches.

A. Deterministic optimization

Deterministic optimization problem for minimum weight can be written as

$$
\begin{aligned}
&\min_{w,t}\ A = wt \\
&\text{s.t.}\ \ k_{c,1}\sigma_f - \left(\frac{600}{wt^2}S_{FL}F_Y + \frac{600}{w^2t}S_{FL}F_X\right) \geq 0 \\
&\qquad k_{c,2}\frac{D_0Ewt}{4L^3} - \sqrt{\left(\frac{S_{FL}F_Y}{t^2}\right)^2 + \left(\frac{S_{FL}F_X}{w^2}\right)^2} \geq 0
\end{aligned}
\tag{19}
$$

where S_{FL} is the safety factor for loads, and $k_{c,1}$ and $k_{c,2}$ are knockdown factors for allowables in the first and the second limit-state functions, respectively. For demonstration, we take S_{FL}=1.5 (the load safety factor used in aircraft design), and $k_{c,1}$ and $k_{c,2}$ as equal to 1.0. Usually these knockdowns are obtained from A-basis or B-basis values as we will discuss in our second demonstration example. However, here the use of $k_{c,1}$=$k_{c,2}$=1.0 led to probability of failure that is similar to the one used in past studies, so these values were selected.

The deterministic optimization problem in Eq. (19) is solved using the Sequential Quadratic Programming tool of MATLAB (using the function *fmincon*). The results of deterministic optimization are listed in Table 2. The probabilities of failure are calculated using FORM and MCS. Using FORM, the probability of failure corresponding to the stress failure mode, P_{f1d}, is 9.301×10^{-5}, while the probability of failure corresponding to the displacement failure mode, P_{f2d}, is 2.652×10^{-3} where the subscript '*d*' stands for the deterministic design. Note that the last column is the system probability of failure, P_F, which is approximated as the sum of the probabilities of failure corresponding to the two different failure modes P_{f1} and P_{f2}. Using MCS, the probabilities of failures are only slightly different. Moreover, the MCS was performed with 10^6 samples and coefficient of variation associated with probability of failures for MCS is approximately 10% for P_{f1d} and 2 % for P_{f2d}. Note that the FORM solution is exact for the stress probability of failure, since the limit-sate function is linear and the random variables are random.

Table 2. Deterministic optimum of the beam problem

Width (in)	Thickness (in)	Area (in^2)	P_{f1d}		P_{f2d}		P_{Fd}	
			FORM	*MCS*	*FORM*	*MCS*	*FORM*	*MCS*
2.275	4.414	10.042	9.301×10^{-5}	9.822×10^{-5}	2.652×10^{-3}	2.659×10^{-3}	2.745×10^{-3}	2.756×10^{-3}

B. Probabilistic optimization

In this section, we perform probabilistic optimization. The probabilistic optimization problem is formulated as

$$
\begin{aligned}
&\min_{w,t}\ A = wt \\
&\text{s.t.}\ \ P_F \approx P_{f1} + P_{f2} \leq 0.0027
\end{aligned}
\tag{20}
$$

As seen from Eq. (20) the system probability of failure, P_F, is approximated as the sum of the probabilities of failure corresponding to the two different failure modes P_{f1} and P_{f2}. This approximation is Ditlevsen's first-order upper bound, so the system failure probability is estimated conservatively. The probabilities of failure P_{f1} and P_{f2} can be calculated using FORM or MCS. When MCS is used, system probability can be calculated without the approximation. However, we still use the approximation in order to have a probabilistic optimum to compare with the ECARD result.

The probabilistic optimization problem is also solved using the *fmincon* function of MATLAB. The results of probabilistic optimization using FORM are listed in Table 3. MCS results in Table 4 are based on 10^6 random samples for every iteration of *fmincon* function of MATLAB. When we compare probabilistic optimum to deterministic optimum, we see that weight can be reduced by 6%, while reducing the system failure probability by 2%. The reduction in both the weight and the probability of failure is obtained by the more efficient risk allocation of the probabilistic design. The deterministic design leads to smaller failure probability of failure for the stress failure mode than the displacement failure mode, while the situation is reversed for the probabilistic design.

Table 3. Probabilistic optimum of the beam problem using FORM

Width (in)	Thickness (in)	Area (in^2)	P_{f1}	P_{f2}	P_F
2.620	3.601	9.436	2.326×10^{-3}	3.738×10^{-4}	2.70×10^{-3}

Table 4. Probabilistic optimum of the beam problem using MCS

Width (in)	Thickness (in)	Area (in^2)	P_{f1}^+	P_{f2}^+	P_F^+
2.651	3.559	9.437	2.369×10^{-3}	3.310×10^{-4}	2.70×10^{-3}

Figure 5 shows the deterministic design, probabilistic design, probability constraint contour and objective function contour. Notice that the deterministic design almost satisfies the probability constraint (with 2% discrepancy), but it is 6% heavier than the probabilistic design.

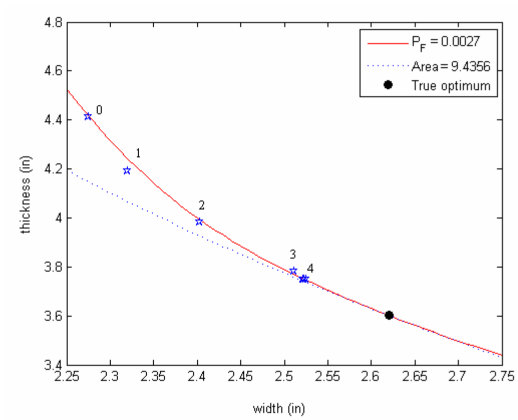


Figure 5. Deterministic, probabilistic and approximate optimum design using FORM

C. Probabilistic optimization using ECARD

Approximate probabilistic optimization problem can be formulated based on Eq. (20) as

$$\begin{aligned} &\min_{w,t} \;\; A = wt \\ &\text{s.t.} \;\; P_{FS}{}^{approx} = P_{f1}{}^{approx} + P_{f2}{}^{approx} \le 0.0027 \end{aligned} \tag{21}$$

where P_{f1}^{approx} and P_{f2}^{approx} are, respectively, approximations of P_{f1} and P_{f2}. The approximate failure probabilities are calculated as

$$P_{f1}^{approx} = F_{C1}\left[\sigma_1^*\left(1+k_1\Delta_1\right)\right] \quad \text{and} \quad P_{f2}^{approx} = F_{C2}\left[\sigma_2^*\left(1+k_2\Delta_2\right)\right] \tag{22}$$

where F_{C1} is the CDF of C_1, F_{C2} is the CDF of C_2. The characteristic stresses σ_1^* and σ_2^* are calculated as

$$\sigma_1^* = F_{C1}^{-1}\left(P_{f1}^p\right) \quad \text{and} \quad \sigma_2^* = F_{C2}^{-1}\left(P_{f2}^p\right) \tag{23}$$

The correction factor k_1 is a function of mean and c.o.v. of the response R_1 and the capacity C_1

Table 5 lists the designs attained in the iterations of the approximate optimization using FORM. We see that after 6th iteration, the approximate optimization converges to an approximate optimum close to the true optimum. Comparing to the true optimum, the approximate optimum is about 0.2% heavier while having the same probability of failure, P_F. However, it is still 4% lighter than the deterministic optimum.

Table 5. Iterations of approximate ECARD probabilistic optimization for the cantilever beam problem using FORM. *The quantities with 'approx' correspond to their approximate values.*

Iter.	w (in)	t (in)	Area (in^2)	P_{F1}	P_{F1}^{approx}	P_{F2}	P_{F2}^{approx}	P_{FS}	P_{FS}^{approx}
0	2.275	4.414	10.042	9.30E-05	9.30E-05	2.65E-03	2.65E-03	2.75E-03	2.75E-03
1	2.32	4.193	9.728	4.87E-04	5.77E-04	3.51E-03	2.12E-03	4.00E-03	2.70E-03
2	2.402	3.983	9.569	1.05E-03	1.26E-03	1.99E-03	1.44E-03	3.04E-03	2.70E-03
3	2.511	3.784	9.503	1.50E-03	1.79E-03	7.00E-04	9.13E-04	2.20E-03	2.70E-03
4	2.525	3.749	9.464	1.85E-03	1.90E-03	7.36E-04	8.04E-04	2.59E-03	2.70E-03
5	2.521	3.75	9.456	1.92E-03	1.92E-03	7.94E-04	7.81E-04	2.72E-03	2.70E-03
6	2.522	3.75	9.457	1.91E-03	1.91E-03	7.84E-04	7.86E-04	2.70E-03	2.70E-03
7	2.522	3.75	9.457	1.92E-03	1.91E-03	7.86E-04	7.85E-04	2.70E-03	2.70E-03

Table 6 lists the designs attained in the iterations of the approximate optimization using MCS. We see that after 5th iteration, the approximate optimization converges to an approximate optimum close to the true optimum. Comparing to the true optimum, the approximate optimum is about 0.49% heavier while having the same probability of failure for the system as deterministic optimum design, P_F.

Table 6. Iterations of approximate ECARD probabilistic optimization for the cantilever beam problem using MCS. *The quantities with 'approx' correspond to their approximate values.*

Iter.	w (in)	t (in)	Area (in^2)	P_{F1}	P_{F1}^{approx}	P_{F2}	P_{F2}^{approx}	P_{FS}	P_{FS}^{approx}
0	2.2752	4.4137	10.04205	0.00008	0.00008	0.00278	0.00278	0.00275	0.00275
1	2.4454	3.9723	9.713902	0.00047	0.00045	0.00073	0.00225	0.00120	0.00270
2	2.4888	3.8490	9.579486	0.00116	0.00158	0.00074	0.00112	0.00190	0.00270
3	2.5223	3.7723	9.514996	0.00165	0.00170	0.00067	0.00100	0.00232	0.00270
4	2.5080	3.7840	9.490243	0.00192	0.00184	0.00097	0.00086	0.00289	0.00270
5	2.5006	3.8007	9.503981	0.00177	0.00178	0.00098	0.00092	0.00275	0.00270

IV. Application of ECARD to Ten-bar Truss Problem

Our second demonstration example is a ten-bar truss problem (see Figure 6). First, we present deterministic optimization of the problem. Then, probability of failure calculation using Monte Carlo simulations is discussed. Finally, probabilistic optimization is performed using ECARD, and the accuracy and efficiency of the method is evaluated.

A. Deterministic optimization

The problem definition for the ten-bar truss problem is taken from Haftka and Gurdal[17] (page 237). The minimum weight design is obtained by varying the cross section areas of the truss members, which are to subject to stress constraints and minimum gage constraints. The allowable stress of an element can be related to the mean value of the failure stress via

$$\sigma_{allow} = k_{dc}\,\bar{\sigma}_f \tag{24}$$

where k_{dc} is a knockdown factor as a function of the failure stress distribution and number of coupon tests. In this example, we use $k_{dc} = 0.8$.

The truss is made of aluminum with a given weight density and elasticity modulus listed in Table 7. The joints 2 and 4 are subjected to vertical loads as shown in the figure. Note that the loads P_1 and P_2 equal to load safety factor S_{FL} times their design values P_{d1} and P_{d2} as given in Eq. (25).

$$P_1 = S_{FL} P_{d1}, \qquad P_2 = S_{FL} P_{d2} \tag{25}$$

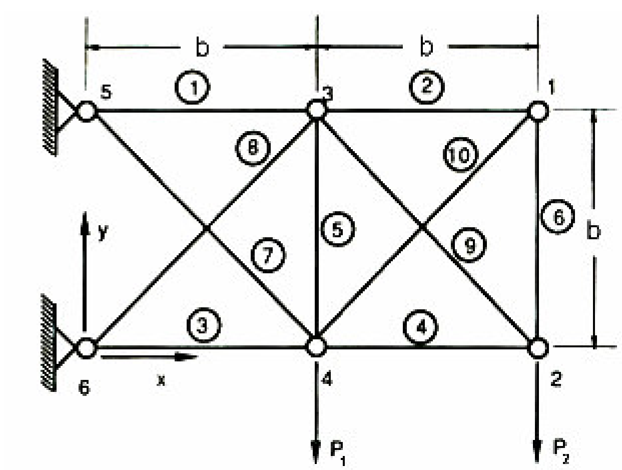


Figure 6. Ten-bar truss example

Table 7.Input data for truss problem

Parameters	Values
P_1	100 kips

P_2	100 kips
b	360 inches
S_{FL}	1.5
P_{d1}	66.7 kips
P_{d2}	66.7 kips
k_{dc}	0.87
Density, ρ	0.1 lb/in^3
Modulus of Elasticity, E	10^4 ksi
Maximum allowable stress	25 ksi*
Minimum gauge constraints	0.1 in^2

*for element 9, maximum allowable stress is 75 ksi

The deterministic minimum weight design formulation can be written as

$$\begin{aligned} &\min_{A_i} \; W = \sum_{i=1}^{10} \rho L_i A_i \\ &\text{s.t.} \; \frac{N_\text{i}\left(P_1, P_2, \mathbf{A}\right)}{A_i} = \sigma_i \le \left(\sigma_{allow}\right)_i \end{aligned} \tag{26}$$

where L_i, N_i, and A_i are, respectively, the length, axial force, and cross-sectional area of element i. **A** is 10×1 cross section area vector, σ and σ_{allow} are the stress and allowable stress in an element, respectively. Analytical solution for the forces in members is given in Appendix B. The results of deterministic optimization are listed in Table 8.. The probabilities of failure of the elements, given in the last column of Table 8 are calculated using separable Monte Carlo simulations[19]. Probability of failure calculation is discussed in the following section.

Table 8.Deterministic optimum of ten-bar truss problem

Element	Area (in^2)	Weight (lb)	Stress (ksi)	P_{fd}
1	7.900	284.4	25.0	2.13E-03
2	0.100	3.6	25.0	1.06E-02
3	8.100	291.6	-25.0	4.80E-04
4	3.900	140.4	-25.0	2.19E-03
5	0.100	3.6	0.0	4.04E-04
6	0.100	3.6	25.0	1.07E-02
7	5.798	295.2	25.0	1.69E-03
8	5.515	280.8	-25.0	1.89E-03
9	3.677	187.2	37.5	5.47E-13
10	0.141	7.2	-25.0	1.07E-02
Total	---	1497.6	--	4.08E-02

[(a)] Pf values are calculated via Separable Monte Carlo simulations with sample size of 1,000,000. Calculation of probabilities of failure is discussed in the next section.

B. Probability of failure calculation using Monte Carlo simulations (MCS)

Failure of an element is predicted to occur when the stress in the element is larger than its failure stress. That is, the limit-state function for an element can be written as

$$g = \left(\sigma_f\right)_{true} - \sigma_{true} \tag{27}$$

where the subscript 'true' stands for the true value of the relevant quantity, which is different from its calculated (or predicted) value due to errors. Introducing the errors, Eq. (27) can be re-written as

$$g = \left(1 - e_f\right)\left(\sigma_f\right)_{calc} - \left(1 + e_\sigma\right)\sigma_{calc} \tag{28}$$

Here, e_f is the error in failure prediction, e_σ is the error in stress calculation. We formulated the errors such that positive errors correspond to a conservative decision. Hence, the sign in front of error in stress is positive, while the sign in front of the error in failure stress is negative. Even though our stress calculation is exact, we pretend that we have error, e_σ, considering the analysis of a more complicated structure where the stresses are calculated using finite element analysis, FEA. The calculated stress can be written in a compact form as

$$\sigma_{calc} = \sigma_{FEA}\left[\left(1+e_{P1}\right)P_1, \left(1+e_{P2}\right)P_2, \left(1+e_{\mathbf{A}}\right)\mathbf{A}\right] \tag{29}$$

where $\sigma_{FEA}[\ ]$ are calculated stresses using finite element analysis, e_{P1} and e_{P2} are errors in loads P_1 and P_2, and $e_{\mathbf{A}}$ is 10×1 error vector corresponding to 10×1 cross section area vector, **A**. The limit-state function can be arranged in separable form (i.e., in a form that allows the use of separable MCS) as

$$g = \left(\sigma_f\right)_{calc} - \frac{\left(1+e_\sigma\right)}{\left(1-e_f\right)}\sigma_{FEA}\left[\left(1+e_{P1}\right)P_1, \left(1+e_{P2}\right)P_2, \left(1+e_{\mathbf{A}}\right)\mathbf{A}\right] \equiv \left(\sigma_f\right)_{calc} - R_{calc} \tag{30}$$

where R_{calc} stands for the calculated stress response. In addition to errors, variabilities are also present in the limit-state function in that the terms $\left(\sigma_f\right)_{calc}$, P_1, P_2 and **A** are random variables that involve variabilities. These errors and variabilities are considered random variables. The distribution types and probabilistic parameters of errors and variabilities are listed in Table 9. The probabilities of failure of the elements are calculated using separable MCS. Separable MCS requires smaller number of simulations compared to crude MCS for the same level of accuracy. For detailed analysis of advantages of separable MCS, the reader is referred to Smarslok *et al.*[19].

Table 9. Error and variabilities in ten-bar truss problem

Uncertainties	Distribution type	Mean	Scatter
Errors			
e_σ	Uniform	0.0	± 5%
e_{P1}	Uniform	0.0	± 10%
e_{P2}	Uniform	0.0	± 10%
$e_{\mathbf{A}}$ (10×1 vector)	Uniform	0.0	± 3%
e_f	Uniform	0.0	± 20%
Variabilities			
P_1, P_2	Extreme type I	P_d=100 kips/S_{FL}	10% c.o.v.

A (10×1 vector)	Uniform	$\bar{\mathbf{A}}$ (10×1 vector)	4% bounds
$\left(\sigma_f\right)_{calc}$	Lognormal	$1/k_{dc}\times 25$ ksi or $1/k_{dc}\times 75$ ksi	8% c.o.v.

After calculating the probabilities of failure of all the elements, P_f, the system failure probability, P_F, is approximated as

$$P_F \approx \sum_{i=1}^{10} \left(P_{fd}\right)_i \tag{31}$$

Note that Eq. (31) is Ditlevsen's first-order upper bound, so the system failure probability is estimated conservatively.

C. Probabilistic optimization

Given a deterministic design as a starting point, the probabilistic optimization problem can be formulated such that the weight of the structure is minimized, while maintaining the same system probability of failure as stated in Eq. (32)

$$\begin{aligned} &\min_{A_i} \; W = \sum_{i=1}^{10} \rho L_i A_i \\ &\text{s.t.} \; \sum_{i=1}^{10} \left(P_f\right)_i \le \sum_{i=1}^{10} \left(P_{fd}\right)_i \end{aligned} \tag{32}$$

where P_f is the element probability of failure for the probabilistic design, while P_{fd} is the element probability of failure for the deterministic design. Results of the probabilistic optimization is shown in Table 10. Overall weight was reduced by 6% (90.47 lbs) while maintaining the total probability of failure of the original deterministic design. Used a sample size of 10,000 and converged after 59 iterations.

Table 10. Probabilistic optimum of ten-bar truss problem using Separable MCS

Elements	Deterministic Areas	Probabilistic Areas	Deterministic Pf	Probabilistic Pf
1	7.900	7.1920	2.14E-03	5.88E-03
2	0.100	0.3243	1.04E-02	3.07E-03
3	8.100	7.1620	5.07E-04	8.26E-03
4	3.900	3.7010	2.41E-03	2.15E-03
5	0.100	0.4512	3.66E-04	3.18E-05
6	0.100	0.3337	1.07E-02	2.14E-03
7	5.798	5.1697	1.56E-03	1.02E-02
8	5.515	4.9782	1.92E-03	3.75E-03
9	3.677	3.5069	4.10E-13	4.70E-13
10	0.141	0.4325	1.06E-02	5.46E-03
Totals:	1497.6 Ibs	1407.13 Ibs	4.10E-02	4.10E-02

D. Probabilistic optimization using ECARD

Approximate probabilistic optimization problem can be written based on Eq. (32) as

$$\begin{aligned} &\min_{A_i} \; W = \sum_{i=1}^{10} \rho L_i A_i \\ &\text{s.t.} \; \sum_{i=1}^{10} \left(P_f^{\,approx} \right)_i \le \sum_{i=1}^{10} \left(P_{fd} \right)_i \end{aligned} \tag{33}$$

where $P_f^{\,approx}$ is the approximate probability of failure as a function of σ^* as we defined earlier in Eq. (5). If σ^* approximation did not involve any errors, then the probabilistic design obtained from Eq. (33) would be the true probabilistic optimum. On the other hand, since the accuracy of σ^* approximation is not perfect then solution of Eq. (33) provides a design close to the true optimum (depending, of course, on the accuracy of the approximation). The true optimum, however, can be reached through iterations (semi-deterministic optimizations) by updating the σ^* approximation at the end of each iteration and by using the approximate probabilistic optimum design obtained at the end of each iteration as starting point for the next iteration.

The number of iterations needed to reach true optimum depends on the accuracy of σ^*. The number of MCS performed in Step 4 of probabilistic optimization procedure as given on page 6 affects the accuracy of σ^* approximation. If we use a small number of MCS in Step 4, we can cut from the computational cost. However, using a small number of MCS reduces our confidence in the mean and the c.o.v. of the response R and thereby reduces the accuracy of σ^* approximation, so the number of iterations to reach the true optimum increases. Therefore, the number of MCS in Step 4 is problem dependent and must be chosen accordingly.

Table 11 shows the results of approximate probabilistic optimization and progress towards the true optimum shown in Table 10. For this example problem, the approximate method needs only four iterations (ECARD optimizations) to converge close to the true optimum. We see that at the end of the 4th iteration the weight difference compared to the previous iteration is 0.03% and the system probability of failure is the same as deterministic system probability of failure. In addition, the errors in element failure probability approximations are less than 2%. Since that the probability of failure of the element #9 is very small, the error in its probability of failure is not taken into consideration. As expected the mean stresses in the light weight elements decrease while the mean stresses in the heavier elements increase. This reflects that it is weight efficient to have higher safety factor to low-weight elements than to high-way elements in the overall risk allocation.

We have solved optimization problem in Eq. (32) in section C, we had to calculate the actual probabilities (using MCS) many times during the problem solution process. So the computational expense was onerous. However, our approximate probabilistic design requires calculation of the actual probabilities of failure of the elements five times, thus the computational expense is greatly reduced.

Table 11. Results of approximate probabilistic optimization and progress towards the true optimum.

Element	Determ. Des.	iter_01	iter_02	iter_03	iter_04
		AREAS (in^2)			
1	7.900	7.4487	7.4787	7.4841	7.4849
2	0.100	0.1000	0.1000	0.1000	0.1000
3	8.100	7.0752	7.0406	7.0401	7.0402

4	3.900	3.9382	3.9666	3.9710	3.9716
5	0.100	0.1000	0.1000	0.1000	0.1000
6	0.100	0.1000	0.1000	0.1000	0.1000
7	5.798	5.0457	5.0440	5.0442	5.0441
8	5.515	5.3538	5.3873	5.3941	5.3951
9	3.677	3.8416	3.9657	3.9873	3.9908
10	0.141	0.1314	0.1310	0.1309	0.1309
Weight (lb)	1497.6	1407.16	1415.94	1417.71	1418.00
MEAN STRESSES (ksi)					
1	16.6667	17.7656	17.7047	17.6934	17.6918
2	16.6667	14.4790	14.0059	13.9276	13.9147
3	-16.6667	-18.9868	-19.0693	-19.0688	-19.0684
4	-16.6667	-16.5606	-16.4537	-16.4378	-16.4354
5	0.0000	4.4620	4.7514	4.7913	4.7977
6	16.6667	14.4790	14.0059	13.9276	13.9147
7	16.6667	18.9660	18.9510	18.9472	18.9468
8	-16.6667	-17.3456	-17.2576	-17.2391	-17.2363
9	25.0000	24.0093	23.2747	23.1515	23.1313
10	-16.6667	-15.5797	-15.1199	-15.0471	-15.0354
APPROXIMATE P_F					
1	2.13E-03	5.65E-03	5.26E-03	5.21E-03	5.20E-03
2	1.06E-02	2.16E-03	2.11E-03	2.10E-03	2.10E-03
3	4.80E-04	7.44E-03	7.51E-03	7.51E-03	7.50E-03
4	2.19E-03	1.97E-03	1.77E-03	1.74E-03	1.74E-03
5	4.04E-04	4.04E-04	1.72E-03	1.86E-03	1.88E-03
6	1.07E-02	2.17E-03	2.09E-03	2.09E-03	2.09E-03
7	1.69E-03	1.23E-02	1.20E-02	1.20E-02	1.20E-02
8	1.89E-03	3.59E-03	3.22E-03	3.17E-03	3.16E-03
9	5.47E-13	3.09E-14	2.50E-15	1.67E-15	6.66E-16
10	1.07E-02	5.17E-03	5.21E-03	5.22E-03	5.23E-03
SYSTEM	4.08E-02	4.08E-02	4.08E-02	4.08E-02	4.08E-02
ACTUAL P_F					
1	2.13E-03	5.53E-03	5.26E-03	5.21E-03	5.20E-03
2	1.06E-02	3.09E-03	2.25E-03	2.13E-03	2.11E-03
3	4.80E-04	6.95E-03	7.51E-03	7.51E-03	7.50E-03
4	2.19E-03	1.96E-03	1.77E-03	1.74E-03	1.74E-03
5	4.04E-04	1.72E-03	1.86E-03	1.88E-03	1.88E-03
6	1.07E-02	3.09E-03	2.23E-03	2.11E-03	2.09E-03
7	1.69E-03	1.21E-02	1.20E-02	1.20E-02	1.20E-02
8	1.89E-03	3.49E-03	3.22E-03	3.17E-03	3.16E-03
9	5.47E-13	2.77E-14	2.44E-15	1.59E-15	1.48E-15
10	1.07E-02	7.14E-03	5.50E-03	5.27E-03	5.24E-03
SYSTEM	4.08E-02	4.51E-02	4.16E-02	4.10E-02	4.08E-02

V. Concluding remarks

An exact-capacity approximate-response-distribution (ECARD) probabilistic optimization method that dispenses with most of the expensive structural response calculations (typically done via finite element analysis) was proposed in this paper. ECARD was demonstrated with two examples. First, probabilistic optimization of a cantilever beam was performed, where risk was allocated between the different failure modes. Then, probabilistic optimization of a ten-bar truss problem was performed, where risk was allocated between truss members. From the results obtained in these two demonstration problems, we reached to the following conclusions.

1. In both problems, ECARD converged to near optima of that allocated risk between failure modes much more efficiently than the deterministic optima. The differences between the true and approximate optima were due to the errors involved in probability of failure estimations, which led to errors in the derivatives of probabilities of failure with respect to design variables that is required in risk allocation problems.
2. The approximate optimum required six inexpensive ECARD iterations and seven probability of failure calculations for the beam example to locate the approximate optimum. In ten-bar truss example, four ECARD iterations were required and probabilities of failure of the elements are calculated five times to locate the approximate optimum. This represents substantial reduction in the number of probability calculation required for full probabilistic optimization.

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Appendix A. Calculation forces in members

Analytical solution to ten-bar truss problem is given in Elishakoff *et al.*[18]. The member forces satisfy the following equilibrium and compatibility equations. Note: Values with "*" are incorrect in the reference. The correct expressions are:

$$N_1 = P_2 - \frac{1}{\sqrt{2}}N_8,\quad N_2 = -\frac{1}{\sqrt{2}}N_{10},\quad N_3 = -P_1 - 2P_2 - \frac{1}{\sqrt{2}}N_8,\quad N_4 = -P_2 - \frac{1}{\sqrt{2}}N_{10},$$

$$N_5 = -P_2 - \frac{1}{\sqrt{2}}N_8 - \frac{1}{\sqrt{2}}N_{10},\quad N_6 = -\frac{1}{\sqrt{2}}N_{10},\quad N_7 = \sqrt{2}\left(P_1 + P_2\right) + N_8,\quad N_8{*} = \frac{b_1 a_{22} - a_{12} b_2}{a_{11} a_{22} - a_{12} a_{21}}$$

$$N_9 = \sqrt{2}P_2 + N_{10},\quad N_{10}{*} = \frac{a_{11} b_2 - a_{21} b_1}{a_{11} a_{22} - a_{12} a_{21}} \tag{A1}$$

where

$$a_{11}{*} = \left(\frac{1}{A_1} + \frac{1}{A_3} + \frac{1}{A_5} + \frac{2\sqrt{2}}{A_7} + \frac{2\sqrt{2}}{A_8}\right),\quad a_{12}{*} = a_{21}{*} = \frac{1}{A_5},\quad a_{22}{*} = \left(\frac{1}{A_2} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} + \frac{2\sqrt{2}}{A_9} + \frac{2\sqrt{2}}{A_{10}}\right),$$

$$b_1{*} = \left[\frac{\sqrt{2}P_2}{A_1} - \frac{P_1 + 2P_2}{A_3} - \frac{P_2}{A_5} - \frac{2\sqrt{2}\left(P_1 + P_2\right)}{A_7}\right],\quad b_2{*} = \left[\frac{-\sqrt{2}P_2}{A_4} - \frac{\sqrt{2}P_2}{A_5} - \frac{4P_2}{A_9}\right] \tag{A2}$$

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