Nonlinear Prices
pp. 175-177

**Two-Part Tariff** - fixed charge plus linear charge (vs. standard Ramsey price in last section which only had a linear charge); could always set fixed charge to zero so this is more flexible than single linear charge.; expect greater total surplus

**Example** -
Suppose \( TC(Q) = F + cQ \)

Suppose \( E = \frac{F}{N} \) (fixed cost divided by potential number of identical consumers)

Set first-best price: \( p^* = p^* = c \) (marginal cost)
This outcome can be sustained if no customers are forced off the network (i.e., no customers think \( E \) is too high)

**Tradeoff** - higher \( E \) allows regulator to set lower \( p \), but could lose customers

**Two Commodities** - essentially have two commodities: access and product (will get Ramsey prices again)

**Model** - assume monopoly supplier of single product; assume product quality is uniform, but consumers vary on how they value the product (e.g., electricity may be valued more by consumer in FL in the summer to run the air conditioner)

Cost Conditions: \( a = \) MC of providing access; assume same for all customers (e.g., cost to connect power line to a house)
\( c = \) constant MC of providing product to consumer who is connected (e.g., producing 1 KW-hr of electricity)
\( F = \) fixed cost of production

Demand Conditions: \( U(Q, \theta) = \) gross value to customer of type \( \theta \) from consuming \( Q \)
units of product
\( U_0 > 0 \) (more is better)
\( U_\theta > 0 \) (labeling convention; \( \theta \uparrow \Rightarrow \) higher valuation of product)
\( U(Q, \theta) - pQ - E = \) net utility (consumer surplus)
\( q(p, \theta) = \) demand function for type \( \theta \) consumer
This is derived by solving: \( \max_{Q} U(Q, \theta) - pQ - E \) s.t. budget

Formal result: \( \frac{\partial U(Q, \theta)}{\partial Q} \bigg|_{Q=q(p, \theta)} = p \)

Good enough for what we need (used to simplify K-T conditions on next page): \( U_\theta (\cdot) = p \)

(Nothing new; this is similar to what we did on bottom of p.3 & p.8 in previous set of notes)
Consumers: \( \theta \in [\underline{\theta}, \overline{\theta}] \) (continuity of consumer types)

\[ g(\theta) = \text{density of } \theta \]

\( N = \text{max number of potential customers; normalize to 1 so we can think of } \% \text{ of consumers who want access} \)

\( \theta_0 = \text{lowest valuation of } \theta \text{ that gets access (may be greater than } \underline{\theta} ) \)

\[ \text{calculated by solving: } U(q(p, \theta_0), \theta_0) - pq(p, \theta_0) - E = 0 \]

Consumers with \( \theta < \theta_0 \) will have CS < 0 so they choose not to have access and get CS = 0

Assume \( \overline{\theta} \) is large enough that some consumers will want access

Assume no price discrimination (if regulator knows \( \theta \), he can charge different prices to different consumers)

**Regulator's Problem** - same as before, except have new decision variable, \( E \):

\[
\max_{p,E} \text{CS} + \pi \quad \text{s.t. } \pi \geq 0
\]

Trick is filling in formulas for CS and \( \pi \):

\[
\text{CS} = \int_{\theta_0}^{\overline{\theta}} [U(q(p, \theta), \theta) - pq(p, \theta) - E] g(\theta) d\theta
\]

(Expected CS if \( g(\theta) \) is a belief; actual CS if \( g(\theta) \) is distribution of known \( \theta \))

\[
\pi = \int_{\theta_0}^{\overline{\theta}} [(p - c)q(p, \theta) - a + E] g(\theta) d\theta - F
\]

**Lagrangian** - \( \ell = \text{CS} + (1 + \lambda)\pi \)

\[
\ell = \int_{\theta_0}^{\overline{\theta}} [U(q(p, \theta), \theta) - pq(p, \theta) - E] g(\theta) d\theta + (1 + \lambda) \left[ \int_{\theta_0}^{\overline{\theta}} [(p - c)q(p, \theta) - a + E] g(\theta) d\theta - F \right]
\]

**Kuhn-Tucker Conditions** - need to look at derivates wrt both \( p \) and \( E \); need Leibniz Rule because \( \theta_0 \) is a function of \( p \) and \( E \)

\[
\ell_p = \int_{\theta_0}^{\overline{\theta}} [U(q(\cdot),p) - pq(\cdot) - q(\cdot)] g(\theta) d\theta - \left[ U(q(p, \theta_0), \theta_0) - pq(p, \theta_0) - E \right] g(\theta_0) \frac{\partial \theta_0}{\partial p} + (1 + \lambda) \left[ \int_{\theta_0}^{\overline{\theta}} [(p - c)q(p, \theta) + q(\cdot)] g(\theta) d\theta - [(p - c)q(p, \theta_0) - a + E] g(\theta_0) \frac{\partial \theta_0}{\partial p} \right] = 0
\]

Simplification:

\[
S_0 = [U(q(p, \theta_0), \theta_0) - pq(p, \theta_0) - E] = \text{CS of } \theta_0 \text{ type consumer (= 0 by definition)}
\]

\[
\pi_0 = [(p - c)q(p, \theta_0) - a + E] = \text{firm profit from } \theta_0 \text{ type consumer}
\]

\[
\ell_p = \int_{\theta_0}^{\overline{\theta}} [U(q(\cdot),p) - pq(\cdot) - q(\cdot)] g(\theta) d\theta + (1 + \lambda) \left[ \int_{\theta_0}^{\overline{\theta}} [(p - c)q(p, \theta) + q(\cdot)] g(\theta) d\theta \right] - S_0 g(\theta_0) \frac{\partial \theta_0}{\partial p} - (1 + \lambda) \pi_0 g(\theta_0) \frac{\partial \theta_0}{\partial p} = 0 \quad \text{(colors explained on next page)}
\]
\[ \ell_E = \bar{\pi} \left[ \int_{\theta_o}^{\bar{\pi}} \frac{d\theta}{1 + \lambda} \right] \left[ \int_{\theta_o}^{\bar{\pi}} f(\theta)d\theta \right] - S_o g(\theta_o) \frac{\partial \theta_o}{\partial E} - (1 + \lambda) \pi_o g(\theta_o) \frac{\partial \theta_o}{\partial E} = 0 \]

More Simplification:
- \( S_o = 0 \) by definition (highlighted at top of page); left this in just to make sure derivative was taken correctly
- \( U_q(\cdot) = p \) by definition (bottom of p.1)
- \( Q \equiv \int_{\theta_o}^{\bar{\pi}} q(p, \theta) g(\theta)d\theta = \) total consumption
- \( Q_p \equiv \int_{\theta_o}^{\bar{\pi}} q_p(p, \theta) g(\theta)d\theta = \) derivative of total consumption wrt \( p \) while holding \( \theta \)
  constant (even though technically \( \theta \) changes with \( p \))
- \( G(\theta) \equiv \int_{\theta_o}^{\bar{\pi}} g(\theta)d\theta = \) cumulative distribution \( \Rightarrow \int_{\theta_o}^{\bar{\pi}} g(\theta)d\theta = 1 - G(\theta_o) \)

\[ \ell_p = -Q + (1 + \lambda) \left[ Q + (p - c)Q_p - \pi_o g(\theta_o) \frac{\partial \theta_o}{\partial p} \right] = 0 \]
\[ \ell_E = \lambda \left[ 1 - G(\theta_o) \right] - (1 + \lambda) \pi_o g(\theta_o) \frac{\partial \theta_o}{\partial E} = 0 \]

Put \((1 + \lambda) \pi_o g(\theta_o) \frac{\partial \theta_o}{\partial p}\) terms on RHS (also cancel the \( Q \) in the \( \ell_p \) equation):

\[ \ell_p : \lambda Q + (1 + \lambda)(p - c)Q_p = (1 + \lambda) \pi_o g(\theta_o) \frac{\partial \theta_o}{\partial p} \]
\[ \ell_E : \lambda \left[ 1 - G(\theta_o) \right] = (1 + \lambda) \pi_o g(\theta_o) \frac{\partial \theta_o}{\partial E} \]

Now have RHS that are fairly similar; just need to find relationship between \( \frac{\partial \theta_o}{\partial p} \) & \( \frac{\partial \theta_o}{\partial E} \)

**Claim** - \( \frac{\partial \theta_o}{\partial p} = \frac{\partial \theta_o}{\partial E} q(p, \theta_o) \)

**Proof**: "kind of" totally differentiate \( U(q(p, \theta_o), \theta_o) - pq(p, \theta_o) - E = 0 \)

TD wrt \( p \) & \( \theta_o \) (\( E \) constant):

\[ \left[ U_q(\cdot)q_q(\cdot) - pq(\cdot)q(\cdot) \right] dp + \left[ U_q(\cdot)q_{a_o}(\cdot) + U_{a_o}(\cdot) - pq_{a_o}(\cdot) \right] d\theta_o = 0 \]

Recall \( U_q(\cdot) = p \)

\[-q(\cdot)dp + U_{a_o}(\cdot)d\theta_o = 0 \Rightarrow \frac{d\theta_o}{dp} = \frac{q(p, \theta_o)}{U_{a_o}(q(p, \theta_o), \theta_o)} \]

TD wrt \( E \) & \( \theta_o \) (\( p \) constant):

\[ -1dE + \left[ U_q(\cdot)q_{a_o}(\cdot) + U_{a_o}(\cdot) - pq_{a_o}(\cdot) \right] d\theta_o = 0 \]
\[
[-1]dE + U_{\theta_0}(\cdot)d\theta_0 = 0 \quad \Rightarrow \quad \frac{d\theta_0}{dE} = \frac{1}{U_{\theta_0}(q(p, \theta_0), \theta_0)}
\]

\[
\therefore \quad \frac{d\theta_0}{dp} = \frac{q(p, \theta_0)}{U_{\theta_0}(q(p, \theta_0), \theta_0)} = \frac{d\theta_0}{dE} q(p, \theta_0)
\]

Apply this to the \( \ell_p \) equation:

\[
\ell_p : \quad \lambda \bar{Q} + (1 + \lambda)(p - c)\bar{Q}_p = \left[(1 + \lambda)\pi_0 g(\theta_0) \frac{\partial \theta_0}{\partial E}\right] q(p, \theta_0)
\]

Note term in brackets is equal to the LHS of the \( \ell_E \) equation:

\[
\ell_p : \quad \lambda \bar{Q} + (1 + \lambda)(p - c)\bar{Q}_p = \lambda \left[1 - G(\theta_0)\right] g(p, \theta_0)
\]

"We're down to a relatively simple equation."

Now try to get this to look like the Ramsey Rule (i.e., get price margin on LHS):

\[
(1 + \lambda)(p - c)\bar{Q}_p = \lambda \left[1 - G(\theta_0)\right] g(p, \theta_0) - \bar{Q}
\]

Divide by \( 1 + \lambda \):

\[
(p - c)\bar{Q}_p = \frac{\lambda}{1 + \lambda} \left[1 - G(\theta_0)\right] g(p, \theta_0) - \bar{Q}
\]

Divide by \( \bar{Q} \) (factor \(-1\) from RHS):

\[
(p - c)\frac{\bar{Q}_p}{\bar{Q}} = - \frac{\lambda}{1 + \lambda} \left[1 - G(\theta_0)\right] \frac{q(p, \theta_0)}{\bar{Q}}
\]

Define \( \bar{\bar{Q}} \equiv \frac{1}{1 - G(\theta_0)} \bar{Q} = \frac{1}{1 - G(\theta_0)} \int_{\theta_0} q(p, \theta)g(\theta)d\theta = "\text{mean truncated consumption}"

(average of those consuming; the integral is the total consumption [defined on top of previous page]; the fraction is front is \( 1 \) over the total number of consumers)

\[
(p - c)\frac{\bar{Q}_p}{\bar{Q}} = - \frac{\lambda}{1 + \lambda} \left[1 - q(p, \theta_0) \right]
\]

Multiply LHS by \( p / p \):

\[
\frac{p - c}{p} \frac{\bar{Q}_p}{\bar{Q}} = - \frac{\lambda}{1 + \lambda} \left[1 - q(p, \theta_0) \right]
\]

Define \( \bar{\varepsilon} = \frac{\bar{Q}_p}{\bar{Q}} \) (price elasticity of demand for those who buy) and \( \bar{\lambda} = \frac{\lambda}{1 + \lambda} \):

\[
\frac{p - c}{p} \bar{\varepsilon} = \bar{\lambda} \left[1 - q(p, \theta_0) \right]
\]

Interpretation -

1. \( q(p, \theta_0) > 0 \) and \( \bar{Q} > 0 \) so the term in parentheses is less than \( 1 \) so price is closer to MC than standard Ramsey rule; "The marginal price \( p \) is closer to MC \( c \) when a two-part tariff is feasible than when the regulator must set only a single unit price (no entry fee)."
(2) $\frac{p-c}{p}$ declines as $q(p, \theta_0)$ increases toward $\bar{Q}$

If there is not much difference between customers (i.e., the purchase by the minimum value customer is close to the average purchase), then the regulator can charge higher entry fee ($E$) and get lower unit price ($p$)

(2a) if consumers are identical (as in example on p.1), then unit price equals marginal cost

**Multiple Two Part Tariffs** - water bill is typical example (although it usually has increasing price); example here has decreasing price:

<table>
<thead>
<tr>
<th>Unit Price</th>
<th>Range of Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>$(Q_0, Q_1]$</td>
</tr>
<tr>
<td>$p_2$</td>
<td>$(Q_1, Q_2]$</td>
</tr>
<tr>
<td>$p_3$</td>
<td>$(Q_2, Q_3]$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$p_n$</td>
<td>$(Q_{n-1}, Q_n]$</td>
</tr>
</tbody>
</table>

**Total Outlay** - amount consumer pays

$$E + p_i Q \quad \text{if } Q \in (Q_{i-1}, Q_i]$$

**Implementing** - this looks like a complicated price schedule so regulated firms could present an alternative and let consumers decide which they want to use: $(E, p_1), (E_2, p_2)$ or $(E_3, p_3)$

where $E_2 = E + (p_1 - p_2)Q_1$ and $E_3 = E_2 + (p_3 - p_2)Q_2$

**Best Rate** - there's a slight difference here; if there is no uncertainty about consumption these two methods are the same; otherwise, the first method guarantees the consumer the "best deal" (lowest total outlay); usually firms list the latter (simple to read) and charge with the former (best deal)... e.g. Spring recently started doing this with cell phone charges


**Basics** - said regulator can always do better (in terms of social welfare) by offering more increments

**Model** - heterogeneous customers with independent demands $Q(p, \theta)$

$Q_p(\cdot) < 0$ (downward sloping)

$Q_\theta(\cdot) > 0$ (labeling convention; consumers with higher values of $\theta$ value product more)
Proposition 1 (Two-Part vs. Linear) - suppose a monopoly supplier (making nonnegative profit) is charging a single uniform price for its product, \( p > MC \). Then a Pareto dominating optional two-part tariff is feasible.

“Proof” (sort of... using same logic as Willig’s proof, but looking at a graphical proof of a specific case for the intuition)

Assume:

- \( TC(Q) = cQ \) (not critical, but makes things simpler)
- 2 customer types: \( \theta_2 > \theta_1 \)
- Original price is \( p \)
- Linear demand (for simplicity)

Introduce optional two-part tariff:

\[ p_2 \in (c, p_1) \quad \text{(i.e.,} \ c < p_2 < p_1) \]

\[ E = (p_1 - p_2) \cdot Q(p_1, \theta_2) \]

(fixed charge offsets lost revenue from current sales; price guarantees firm and type 2 consumer are better off on additional sales)

Note from graph: \( E = d + e + f \) (rectangle with height \( p_1 - p_2 \) and width \( Q(p_1, \theta_2)\))

\( \theta_1 \) Type:

Using \( p_1 : CS = a \)

Using \( (p_2, E) \) : \( CS = a + e + f - (d + e + f) = a - d \)

\[ \therefore \text{choose } p_1; \ \theta_1 \text{ Type is no worse off than before} \]

\( \theta_2 \) Type:

Using \( p_1 : CS = a + b \)

Using \( (p_2, E) \) : \( CS = a + b + c + d + e + f - (d + e + f) = a + b + c \)

\[ \therefore \text{choose } (p_2, E); \ \theta_2 \text{ Type is better off than before} \]

Firm:

No change from \( \theta_1 \) Type so ignore them

Using \( p_1 : \pi_{\theta_1} = (p_1 - c)Q(p_1, \theta_2) \)

Using \( (p_2, E) \) : \( \pi_{\theta_2} = (p_2 - c)Q(p_2, \theta_2) \)

Should be able to see those in the graph, if not this helps:

Lost revenue from \( Q(p_1, \theta_2) \) units = \( (p_1 - p_2)Q(p_1, \theta_2) = d + e + f \)

Gained revenue from entry fee = \( E = d + e + f \) (offsets the loss above)

Gained profit from new sales = \( (p_2 - c)[Q(p_2, \theta_2) - Q(p_1, \theta_2)] = g \)

\[ \therefore \text{offering } (p_2, E) \text{ to } \theta_2 \text{ Types makes firm better off than before} \]

Since \( \theta_1 \) Types are no worse off and \( \theta_2 \) Types and firm are strictly better off, this is a Pareto improvement

Proposition 2 (Adding Two-Part) - suppose the same setting with types \( \theta \in [\theta_1, \theta_2] \) and the following set of optional two-part tariffs: \( (p_1, E_1), (p_2, E_2), \ldots, (p_n, E_n) \), where

\[ p_1 > p_2 > \ldots > p_n, \ E_1 < E_2 < \ldots < E_n. \]

Further suppose \( p_n > c \) (MC) and some strictly positive output demanded at the \( (p_n, E_n) \) tariff. A Pareto dominating optional two-part tariff of the following type can always be designed:

\[ p_{n+1} \in (c, p_n) \quad \text{(i.e.,} \ c < p_{n+1} < p_n) \]

\[ E_{n+1} = (p_n - p_{n+1}) \cdot Q(p_n, \theta) \]

**Non-Linear Pricing** - limit of adding more increments is fully non-linear pricing; a different price charged for each fraction of output

**Model** -
\[ p(q, \theta) = \text{willingness of type } \theta \text{ consumer already purchasing } q \text{ units to pay for an additional unit of output (i.e., inverse demand curve)} \]
\[ p_a(\cdot) > 0 \] (labeling convention; consumers with higher values of \( \theta \) value product more)

Assume \( \theta \sim \text{Uniform}(0,1) \)

\[ R(q) = \text{total charge to (outlay by) customer for } q \text{ units of output} \]

Assume \( TC(Q) = cQ \) so \( c = \text{constant marginal cost of production} \)

\( q^*(\theta) = \text{optimal consumption of type } \theta \text{ consumer} \)

solves \( \max_q U(q, \theta) - R(q) \), where \( U(q, \theta) = \int_0^q p(\tilde{q}, \theta) d\tilde{q} \)

(i.e., \( \max \text{CS} = \text{area under demand curve} - \text{total outlay by consumer} \))

**NOTE:** correction from the notes in class which wrote: \( \max_q U(q, \theta) = \int_0^q p(\tilde{q}, \theta) d\tilde{q} - R(q) \)

**Second Order Conditions** - ignoring these (and corner solutions) to save time so we can cover more topics

**Solution** - \( p(q^*(\theta), \theta) = P(q^*(\theta)) \) (i.e., \( \text{MB} = \text{MC} \))

**Graph** - \( P(q) \) can be upward sloping (standard supply curve), but anything with greater slope than \( p(q, \theta) \) works; we’ll get to the conditions that guarantee that later

\[ q^*(\theta) = \text{is a weakly increasing function of } \theta \]

Proof is “obvious” (consumers who value good more will buy more)

**Marginal Customer Type** - type of customer just willing to buy \( q \) units of output

\[ \theta^*(q) = \min_{\theta} \{ \theta \mid q^*(\theta) \geq q \} \]

**Lemma 2** - \( \theta^*(q) \) is weakly increasing in \( q \)

Proof is “obvious”

**Weighted Social Welfare** - change up (introduced on Problem Set 4)

We were using: \( \max_{p,E} \text{CS} + \pi \) s.t. \( \pi \geq 0 \)

Now we’ll use a weighted social welfare function: \( \max_{p,E} \alpha \text{CS} + (1 - \alpha)\pi \)

As long as we restrict \( \alpha \in [0, \frac{1}{2}] \) (so CS is not weighted more than \( \pi \)) this will guarantee nonnegative profit to the firm; if more weight is given to CS, firm profits will be driven negative and the problem is essentially unbounded

**Benefit** - removes problem of having to solve for lagrange multiplier
**Regulator’s Problem** - think of increments of output:

\[
W = \int_{0}^{\infty} \left[ \alpha \int_{\theta(q)}^{\infty} [p(q, \theta) - P(q)] g(\theta) d\theta + (1 - \alpha) \int_{\theta(q)}^{\infty} [P(q) - c] g(\theta) d\theta \right] dq
\]

Sub \( \bar{\theta} = 1 \) and \( g(\theta) = 1 \) from \( \theta \sim \text{Uniform}(0,1) \) assumption

\[
W = \int_{0}^{\infty} \left[ \alpha \int_{\theta(q)}^{\infty} (p(q, \theta) - P(q)) d\theta + (1 - \alpha) \int_{\theta(q)}^{\infty} [P(q) - c] d\theta \right] dq
\]

**Marginal Customer Theorem** - if \( \theta^*(q) > 0 \) (i.e., interior solution), then the price schedule that maximizes \( W \) given \( \theta^*(q) \) is \( P(q) = p(q, \theta^*(q)) \) (i.e., charge marginal willingness to pay of the marginal customer)

"Proof:" (sort of)

If you charge a higher price, the \( \theta^*(q) \) type won’t buy

If you charge a lower price, the \( \theta^*(q) \) type won’t be the marginal customer (plus the problem is set up with more emphasis on profit so it makes sense that the regulator should charge as much as possible)

**Trick** - it’s mathematically easier to solve by focusing on \( \theta^*(q) \), rather than \( P(q) \)

\[
\max_{\theta^*(q)} W = \int_{0}^{\infty} \left[ \alpha \int_{\theta^*(q)}^{\infty} [p(q, \theta) - P(q)] + (1 - \alpha) [P(q) - c] \right] d\theta dq
\]

s.t. \( \frac{d\theta^*(q)}{dq} \geq 0 \)

\( P(q) = p(q, \theta^*(q)) \)

Temporarily ignore the first constraint (we’ll deal with it later) and sub the second into the objective so we have an unconstrained optimization problem

**Point-wise Optimization** - original problem (integral over all \( q \)) is optimal on average, but we’re going to find an optimal solution point by point (for each \( q \))

"That’s disturbing" (Fernando)

So for each \( q \), we’re choosing the optimal marginal consumer (JC)

\[
\max_{\theta^*(q)} \int_{\theta^*(q)}^{\infty} \left[ \alpha [p(q, \theta) - p(q, \theta^*(q))] + (1 - \alpha) [p(q, \theta^*(q)) - c] \right] d\theta
\]

\[
\frac{\partial(\cdot)}{\partial \theta^*(q)} = \int_{\theta^*(q)}^{\infty} (1 - 2\alpha) p_\theta(q, \theta^*(q)) d\theta - \left\{ \alpha [p(q, \theta^*(q)) - p(q, \theta^*(q))] + (1 - \alpha) [p(q, \theta^*(q)) - c] \right\} = 0
\]

where \( p_\theta(q, \theta^*(q)) = \frac{\partial p(q, \theta)}{\partial \theta} \bigg|_{\theta = \theta^*(q)} \)

\[
\frac{\partial(\cdot)}{\partial \theta^*(q)} = \int_{\theta^*(q)}^{\infty} (1 - 2\alpha) p_\theta(q, \theta^*(q)) d\theta - (1 - \alpha) [P(q) - c] = 0
\]

Constant \( \text{wrt} \ \theta \)
\[
\frac{\partial (\cdot)}{\partial \theta^* (q)} = (1-2\alpha)p_\theta (q, \theta^* (q))[1-\theta^* (q)] - (1-\alpha)[P(q) - c] = 0
\]

Move \( P(q) - c \) to one side (trying to get this to look like a Ramsey Rule):

\[
(1-\alpha)[P(q) - c] = (1-2\alpha)p_\theta (q, \theta^* (q))[1-\theta^* (q)]
\]

Divide by \( 1-\alpha \):

\[
P(q) - c = \frac{1-2\alpha}{1-\alpha} p_\theta (q, \theta^* (q))[1-\theta^* (q)]
\]

Spell out \( p_\theta \):

\[
P(q) - c = \frac{1-2\alpha}{1-\alpha} \frac{\partial p}{\partial \theta^*} [1-\theta^* (q)]
\]

Divide by \( P(q) \):

\[
\frac{P(q) - c}{P(q)} = \frac{1-2\alpha}{1-\alpha} \frac{1}{\frac{\partial p}{\partial \theta^*}} \frac{1-\theta^* (q)}{P(q)}
\]

Flip the last terms:

\[
\frac{P(q) - c}{P(q)} = \frac{1-2\alpha}{1-\alpha} \frac{\partial (1-\theta^*)}{\partial p} \frac{1}{\frac{\partial p}{1-\theta^* (q)}}
\]

Let \( k = \frac{1-2\alpha}{1-\alpha} \) and \( \varepsilon^* = \frac{\partial (1-\theta^*)}{\partial p} \frac{P(q)}{1-\theta^* (q)} < 0 \)

\( \varepsilon^* \) is the "extent of the market" - fraction of customers who buy at least \( q \) units

\[
\frac{P(q) - c}{P(q)} = \frac{k}{\varepsilon^*}
\]

**English** - the proportionate market for the \( q^\text{th} \) unit is inversely proportional to the extent of the market (i.e., charge higher market for units that more people buy)

**Independent Goods** - result is sort of like viewing each level of output as an independent good; only have to worry about demand for each specific quantity of the good (i.e., only look up and down for each quantity on the graph)

**Example** - if \( P(q_1) \uparrow \), then \( \theta_1 \) type won't want to buy; in graph, \( \theta_2 \) type is now the marginal customer

**Characteristics of the Optimal Pricing Schedule**

(1) \( \alpha \uparrow \Rightarrow \frac{P(q) - c}{P(q)} \downarrow \)

**Proof**: On right side only \( k \) is a function of \( \alpha \) : need to show \( k \) is decreasing with \( \alpha \)

\[
\frac{\partial k}{\partial \alpha} = \frac{(1-\alpha)(-2) - (1-2\alpha)(-1)}{(1-\alpha)^2} = \frac{-2 + 2\alpha + 1 - 2\alpha}{(1-\alpha)^2} = \frac{-1}{(1-\alpha)^2} < 0
\]

Since \( k_\alpha < 0 \),

\[
\frac{\partial (k/\varepsilon^*)}{\partial \alpha} = \frac{\partial ((P - c)/P)}{\partial \alpha} < 0
\]
(2) \( \alpha = \frac{1}{2} \Rightarrow P(q) = c \) (i.e., maximize total surplus)

Proof: \( k|_{x=1} = \frac{1-2(\frac{1}{2})}{1-\frac{1}{2}} = 0 = \frac{0}{1} = 0 \) \( \therefore \frac{P(q) - c}{P(q)} = 0 \Rightarrow P(q) = c \)

(3) \( \alpha = 0 \Rightarrow MR = MC \) (monopoly pricing... inverse elasticity rule)

Proof: \( k|_{x=0} = \frac{1-2(0)}{1-0} = 1 = \frac{1}{1} \) \( \therefore \frac{P(q) - c}{P(q)} = \frac{1}{|\varepsilon|} \)

(4) Largest output increment is priced at MC

Proof: \( \theta^*(q) = 1 \) (maximum value of \( \theta \) in this scenario)

\( \varepsilon|_{\theta=1} = \frac{\partial (1-\varepsilon) P(q)}{\partial p} \frac{1}{1-1} \rightarrow \infty \) \( \therefore \frac{P(q) - c}{P(q)} \rightarrow \frac{k}{\infty} = 0 \Rightarrow P(q) = c \)

Note: this is only on the last item purchased by the \( \theta_{\text{max}} \) type; he still buys all previous units of output some \( P > MC \)

Revisit Constraint - ignored the \( \frac{d\theta^*(q)}{dq} \geq 0 \) constraint

English - the constraint says \( \theta^*(q) \) is weakly increasing in \( q \)

Single Crossing - the constraint is satisfied with single crossing (i.e., \( P(q) \) is everywhere flatter than \( p(q, \theta) \forall \theta \))

Example - from top graph:

\[
\begin{align*}
\theta^*(q_0) &= \theta_0 \\
\theta^*(q_1) &= \theta_1 \\
\theta^*(q_2) &= \theta_2 \\
\theta^*(q_3) &= \theta_3
\end{align*}
\]

\( q \uparrow \Rightarrow \theta^*(q) \uparrow \)

Second Graph:

\[
\begin{align*}
\theta^*(q_0) &= \theta_0 \\
\theta^*(q_1) &= \theta_1 \\
\theta^*(q_2) &= \theta_2 \\
\theta^*(q_3) &= \theta_3
\end{align*}
\]

Violates \( \frac{d\theta^*(q)}{dq} \geq 0 \)

Solution - in the range that violates the constraint, the solution (with binding constraint) is to use the willingness to pay schedule (demand) of the last \( \theta_i \) type where \( P(q) \) is flatter than \( p(q, \theta) \); assume each customer buys the maximum amount if he's indifferent (as shown in third graph)

Problem Set 5 Hints

Given \( U(q, \theta) \)

Use \( U_q(\cdot) = P(q) \) to solve for \( q^*(\theta) \), which you can solve for

\( \theta^*(q) = P(q) + \text{something} \)

Objective: \( \max_{P(q)} \int_0^1 \int_{\theta^*(q)} \left[ U_q(\cdot) - P(\cdot) \right] d\theta + (1 - \alpha) \ldots \)
**Peak Load Pricing** (not on test)

Sometimes can distinguish between demand for product at different points in time (e.g., electricity used more during day than at night)

Want to manage demand to shift high demand to different time to lower MC

**Tricky** - $Q_D$ & $Q_N$ are interdependent; could have such a good incentive to shift usage that regulator (or firm) creates new high peak at another time (e.g., phone company’s had busy circuits at 11pm when they first started charging lower prices at 11pm)