MULTIPLE TAIL MODELS INCLUDING INVERSE MEASURES FOR STRUCTURAL DESIGN UNDER UNCERTAINTIES

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY UNIVERSITY OF FLORIDA 2007
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Dedicated to my parents Meenakshi and Ramu
ACKNOWLEDGMENTS

This dissertation would not have been possible if not for the help, support and motivation by my teachers, family and friends. I am extremely thankful to my advisors Dr. Nam Ho Kim and Dr. Raphael T. Haftka for, providing me this wonderful opportunity to pursue a Ph.D, their constant encouragement, patience and excellent guidance. I learnt a lot both in academics and personal life from them. They have been more than mere advisors – friends, philosophers and mentors. I have always admired their in-depth understanding and knowledge and feel that I am fortunate to have worked under their guidance.

I would like to thank Dr. Peter Ifju, Dr. Tony Schmitz and Dr. Stanislav Uryasev for agreeing to serve on my committee, reviewing my dissertation and for providing constructive criticism that helped to enhance the quality of this work. I would like to thank Dr. Choi, Dr. Missoum, Dr. Qu, and Dr. Youn, for collaborating with me. I also thank the staff of the Mechanical and Aerospace Engineering department for their help with administrative support. Thanks are due to my former and present colleagues in the Structural and Multidisciplinary Optimization group. I thank all my friends, especially Gators for Asha group members, for making my life outside work wonderful and enjoyable. I greatly appreciate the financial support that I received from the Institute for Future Space Transport

Last, but not the least, I would not have completed this work if not for the unconditional love, emotional understanding and support of my family. To you, I dedicate this work!
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\( w \) Width

\( X \) Load in x direction

\( Y \) Load in Y direction

\( \beta \) Reliability Index

\( \Phi \) Standard Normal Cumulative Density Function

\( \sigma \) Scale parameter

\( \xi \) Shape parameter

\( z \) Exceedance data

\( F_u(z) \) Conditional CDF

\( \hat{F}_{\xi,\sigma}(z) \) Conditional CDF approximated by GPD

\( P_{\text{target}} \) Target probability of failure
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By

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December 2007

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Major: Aerospace Engineering

Sampling-based reliability estimation with expensive computer models may be computationally prohibitive due to a large number of required simulations. One way to alleviate the computational expense is to extrapolate reliability estimates from observed levels to unobserved levels. Classical tail modeling techniques provide a class of models to enable this extrapolation using asymptotic theory by approximating the tail region of the cumulative distribution function (CDF). This work proposes three alternate tail extrapolation techniques including inverse measures that can complement classical tail modeling. The proposed approach, multiple tail models, applies the two classical and three alternate extrapolation techniques simultaneously to estimate inverse measures at the extrapolation regions and use the median as the best estimate. It is observed that the range of the five estimates can be used as a good approximation of the error associated with the median estimate. Accuracy and computational efficiency are competing factors in selecting sample size. Yet, as our numerical studies reveal, the accuracy lost to the reduction of computational power is very small in the proposed method. The method is demonstrated on standard statistical distributions and complex engineering examples.
CHAPTER 1
INTRODUCTION

Uncertainty is an acknowledged phenomenon in the process of structural design. In an optimization framework design under uncertainties refers to a safe design that not only needs to be optimal but should also warrant survivability against uncertainties. Traditionally safety factors were used to account for the uncertainties. However, use of safety factors does not usually lead to minimum cost designs for a given level of safety because different structural members or different failure modes require different safety factors. Alternately, probabilistic approaches offer techniques to characterize the uncertainties using a statistical method and have the potential to provide safer designs at a given cost. However, the probabilistic approaches require solving an expensive, complex optimization problem that needs robust formulations and efficient computational techniques for stable and accelerated convergence. In addition, they also require statistical information which may be expensive to obtain.

Structural reliability analysis requires the assessment of the performance function which dictates the behavior of the structure. The performance function is called the limit state function which is typically expressed as the difference between the capacity (e.g., yield strength, allowable vibration level) and the response of the system (e.g., stress, actual vibration). Approaches available for reliability assessment and analysis can be widely classified as analytical and simulation approaches. Analytical approaches are simple to implement but are mostly limited to single failure modes, whereas simulation methods like Monte Carlo simulation (MCS) are computationally intensive but can handle multiple failure modes. Moreover, they can handle any type of limit state functions unlike analytical approaches which are mostly appropriate for linear limit state functions. Most real life applications exhibit multiple failure modes and the limit state function is not available explicitly in closed form. Since there is no information on nonlinearity
of the limit state function, MCS is the obvious choice in such situations. In reliability-based
design optimization, a widely used probabilistic optimization approach (Rackwitz, 2000) in
structural engineering, reliability analysis is an iterative process and using crude MCS is
computationally prohibitive. Researchers develop variants of MCS or other approximation
methods like response surface that replaces the reliability analysis and obviate the need to
repeatedly access the expensive computer models.

In reliability based design context, high reliability, typical of aerospace applications
translates to small probability in the tails of the statistical distributions. Reliability analysis when
dealing with high reliability (or low failure probability) designs is mostly dependent on how the
tails of the random variables behave. In few cases, the safety levels can vary by an order of
magnitude with slight modifications in the tails of the response variables (Caers and Maes,
1998). Therefore, the tails need to be modeled accurately. Limitations in computational power
prevent us in employing direct simulation methods to model the tails. One way to alleviate the
computational expense is to extrapolate into high reliability levels with limited data at lower
reliability levels. Statistical techniques from extreme value theory referred to as classical tail
modeling techniques here are available to perform this extrapolation. They are based on the
concept of approximating the tail portion of the cumulative distribution function (CDF) with a
generalized pareto distribution (GPD).

In structural engineering, reliability is measured by quantities like probability of failure or
reliability index. Recently, alternate safety measures like the inverse measures in performance
space have cornered enough interest because of their multifaceted advantages (Ramu et al,
2006). The inverse measures in performance space ties the concepts of safety factor and target
failure probability. The inverse measures translate the deficiency in failure probability to
deficiency in performance measure and hence provide a more quantitative measure of the resources needed to satisfy safety requirements. Among the several advantages they exhibit, inverse measures like probabilistic sufficiency factor (PSF) help in stable and accelerated convergence in optimization, better response surface approximations compared to surfaces fit to other reliability measures. Since inverse measure exhibit several advantages, its usefulness in tail modeling is explored in this work. Here we develop 3 alternate tail extrapolation schemes including inverse measures that can complement the classical tail modeling techniques. The alternate extrapolation schemes are based on variable transformations and approximating the relationship between the inverse measure and transformed variable.

The primary motivation of this work is to develop an efficient reliability estimation procedure employing tail modeling techniques including inverse measures that can estimate reliability measures corresponding to high reliability with samples that are only sufficient to assess reliability measure corresponding to low reliability levels. This is a tradeoff between accuracy and computational power. Yet, as the studies reveal, the accuracy lost to the reduction of computational power is very reasonable. The reduction in computational cost is about a minimum of 3 orders of magnitude for the same level of accuracy.

Goel et al., (2006) developed a method to extend the utility of an ensemble of surrogates. When faced with multiple surrogates, they explored the possibility of using the best surrogate or a weighted average surrogate model instead of individual surrogate models. In a similar fashion, in order to take advantage of both the classical tail modeling techniques and alternate extrapolation schemes and still come up with the best prediction, we propose to apply all the techniques simultaneously and use the median of the five estimates as the best estimate. Here, we call using all the techniques simultaneously as multiple tail models (MTM). In addition to
arriving at the estimate, the range of the five techniques can be used to approximate the error associated with the median estimate. Moreover, the MTM approach can be used to replace the reliability analysis in reliability based design framework.
CHAPTER 2
RELIABILITY BASED DESIGN OPTIMIZATION

Introduction

Optimization is the process of minimizing a cost which is a function of design variables (and other variables) subject to constraints that prescribe the design requirement. Often times, the variables involved are uncertain and probabilistic approaches are called for to account for the uncertainty as an alternate to the traditional safety factor approach, to obtain better designs. Several probabilistic approaches has been proposed in the last decade like Robust Design Optimization (Phadke, 1989, Gu, 2000), Reliability-based Design Optimization (Frangopol, 1985, Tu, 1999), Fuzzy optimization (Sakawa 1993, Maglaras et al, 1997), Reliability design using evidence theory (Soundappan et al., 2004). Reliability-based Design Optimization (RBDO) is widely used because it allows the designer to prescribe the level of reliability required. This section presents a review of the standard reliability-based design formulations and reliability estimation methods.

Standard Formulations

Primarily, reliability-based design consists of minimizing a cost function while satisfying reliability constraints. The reliability constraints are based on the failure probability corresponding to each failure mode or a single failure mode describing the system failure. The estimation of failure probability is usually performed by reliability analysis.

In structural engineering, the system performance criterion is described by the limit state function which is typically expressed as the difference between the capacity of the system and the response of the system, which is expressed as:

\[ G(x) = G_c(x) - G_r(x) \]  \hspace{1cm} (2-1)

where, \( G \) is the limit state function and \( G_c, G_r \) are the capacity and the response respectively. All the three quantities are functions of a vector \( x \) which consists of design (and random) variables.
and other parameters. The system is considered failed if \( G < 0 \) and safe if \( G > 0 \). If \( x \) is random, the design is considered a failure if the probability of failure is unacceptably high.

Often times, RBDO is formulated as a double loop (nested loop) problem. The outer loop performs the optimization with respect to design variables, while the inner loop performs the reliability analysis estimating the failure probability. This is mathematically expressed as:

\[
\min \text{ cost}(x) \\
\text{subject to } P_f^j = P(G^j < 0) \leq P_{ftarget}^j \quad j = 1, 2, \ldots nm \quad (2-2)
\]

where the cost is a function of design variables, \( nm \) is the number of failure modes, \( G \) is the limit state function. \( P_{ftarget} \) is the prescribed failure probability levels to which the designer intends to design the structure. Here, estimation of probabilistic constraint requires reliability analysis.

Often times the magnitude of failure probability is low and hence engineers prefer to work with reliability index as the alternate measure of reliability. The reliability index and failure probability are related as:

\[
P_f = \Phi(-\beta) \quad (2-3)
\]

where, \( \Phi \) is the standard normal cumulative density function (CDF). Since the inverse of the standard normal CDF is required to estimate the reliability index, it is an inverse measure. Probabilistic constraints in Eq.(2-2) can be written as:

\[
\beta^j \geq \beta_{target}^j \quad (2-4)
\]

Employing Eq. (2-4) for prescribing the probabilistic constraint is called the Reliability Index Approach (RIA). In RIA, the failure reliability measure is usually calculated via FORM which is an iterative process and so computationally expensive and sometimes features convergence problems (Tu et al., 1999) In order to reduce the computational cost of double loop, various techniques has been proposed which can be classified into two categories as: (i) techniques that improve the efficiency of uncertainty analysis like fast probability integration (Wu 1994) and two-point adaptive non linear approximations (Grandhi and Wang 1998) (ii) techniques that
modify the formulation of the probabilistic constraints, for instance, using inverse reliability measures in the probabilistic constraint. Sometimes, researchers formulate a single loop RBDO that avoids nested loops. The idea of single loop formulations rests on the basis of formulating the probabilistic constraint as deterministic constraints by either approximating the Karush-Kuhn-Tucker conditions at the MPP or approximating the relationship between probabilistic design and deterministic design safety factors. A detailed survey of both single and double loop methods in RBDO is presented in the literature survey of Chapter 3.

Kharmanda et al (2002, 2004, and 2007) developed RBDO solution procedures relative to two views points: reliability and optimization. From an optimization view point, Kharmanda et al (2002) developed a hybrid method based on simultaneous application of the reliability and the optimization problem that reduced the computational time. The hybrid method, compared to classical RBDO, minimizes a new form of the objective function which is expressed as the product of the original objective function and the reliability index in the hybrid space. Since the minimization of the objective function is carried out in both the deterministic variables and random variable space, it is referred to as hybrid design space. The reliability index in the hybrid space is the distance between the optimum point and the design point in the hybrid space. However, the hybrid RBDO problem was more complex than that of the deterministic design and may not lead to local optima In order to address both the challenges, Kharmanda et al (2004) propose an optimum safety factor approach that computes safety factors satisfying a reliability level without demanding additional computing cost for the reliability evaluation. Kharmanda et al (2007) developed the hybrid design spaces and optimal safety factor equations for three distributions namely, the normal, lognormal and uniform. The optimal safety factors are computed by sensitivity analysis of the limit state function with respect to the random variables.
They report that the optimal safety factor approach has several advantages like smaller number of optimization variables, good convergence stability, lower computing time and satisfaction of required reliability levels. In order to estimate the reliability measure for reliability analysis, one need to use analytical or simulation approaches which are described below.

**Moment-Based Methods**

In standard reliability techniques, an arbitrary random vector $\mathbf{x}$ is mapped to an independent standard normal vector $\mathbf{U}$ (variables with normal distribution of zero mean and unit variance). This transformation is known as Rosenblatt transformation (Rosenblatt, 1952, Melchers, 1999, pp.118-120). The limit state function in the standard normal space can be obtained as $G(\mathbf{U}) = G(T(\mathbf{X}))$ where $T$ is the transformation. If the limit state function in the standard normal space is affine, i.e., if $G(\mathbf{U}) = \alpha^T \mathbf{U} + \psi$, then failure probability can be exactly calculated as $P_f = \Phi\left(-\frac{\psi}{\|\alpha\|}\right)$. This is the basis for moment based methods.

Moment-based methods provide for less expensive calculation of the probability of failure compared to simulation methods, although they are limited to a single failure mode. The First Order Reliability Method (FORM) is the most widely used moment-based technique. FORM is based on the idea of the linear approximation of the limit state function and is accurate as long as the curvature of the limit state function is not too high. In the standard normal space the point on the first order limit state function at which the distance from the origin is minimum is the MPP.

When the limit state has a significant curvature, second order methods can be used. The Second Order Reliability Method (SORM) approximates the measure of reliability more accurately by considering the effect of the curvature of the limit state function (Melchers, 1999, pp 127-130).
Figure 2-1 illustrates the concept of the reliability index and MPP search for a two variable case in the standard normal space. In reliability analysis, concerns are first focused on the $G(U) = 0$ curve. Next, the minimum distance to the origin is sought. The corresponding point is the MPP because it contributes most to failure. This process can be mathematically expressed as:

To find $\beta = \| U^* \|

Minimize $\sqrt{U^T U}$

Subject to $G(U) = 0$ 

(2-5)

Where $U^*$ is the MPP. The calculation of the failure probability is based on linearizing the limit function at the MPP. It is to be noted that reliability index is the distance to the limit state function from the origin in the standard normal space. That is, it is an inverse measure in the input space (standard normal).

**Monte Carlo Simulation**

Variants of Monte Carlo methods were originally practiced under more generic names such as statistical sampling. The name and the method were made famous by early pioneers such as Stanislaw Ulam and John von Neumann and is a reference to the famous casino in Monaco. The methods use of randomness and repetitive nature is similar to the casino’s activities. It is discussed in Metropolis (1987) that the method earns its name from the fact that Stanislaw Ulam had an uncle who would borrow money from relative just to go to Monte Carlo.

Monte Carlo integration is essentially numerical quadrature using random numbers. Monte Carlo integration methods are algorithms for the approximate evaluation of definite integrals, usually multidimensional ones. The usual algorithms evaluate the integrand at a regular grid. Monte Carlo methods, however, randomly choose the points at which the integrand is evaluated. Monte Carlo Methods are based on the analogy between volume and probability. Monte Carlo calculates the volume of a set by interpreting the volume as probability. Simply put, it means,
sampling from all the possible outcomes and considering the fraction of random draws that in a
given set as an estimate of the set’s volume. The law of large numbers ensures that this estimate
converges to the correct value as the number of draws increases. The central limit theorem
provides information about the likely magnitude of the error in the estimate after finite number
of draws.

For example, consider estimating the integral of a function $f$ over a unit interval. The
$$
\alpha = \int_0^1 f(x)dx
$$
(2-6)
integral can be expressed as an expectation $E[f(U)]$, with $U$ uniformly distributed between 0
and 1. The Monte Carlo estimate for the integral can be obtained as:
$$
\hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} f(U_i)
$$
(2-7)
If $f$ is continuous and integrable over $[0, 1]$, then, by the strong law of large numbers, the
estimate in Eq. (2-7) converges to the actual value with probability 1 as $n \to \infty$. In fact, if $f$ is
square integrable and we set
$$
\sigma_f^2 = \int_0^1 (f(x) - \alpha)^2 dx
$$
(2-8)
then the error $\hat{\alpha} - \alpha$ in the Monte Carlo estimate is approximately normally distributed with zero
mean and standard deviation of $\sigma_f / \sqrt{n}$, the quality of this approximation improving as $n$
increases. The parameter $\sigma_f$ in Eq.(2-8) will not be known in a real setting but can be
approximated by the sample standard deviation. Thus, we not only obtain the estimate but also
the error contained in the estimate. The form of the standard error is the main feature of the
MCS. Reducing this error to half require about increasing the number of points by four;
increasing the precision by one decimal point requires 100 times as many points. Yet, the
advantage of MCS lies in the fact that its $O(n^{-1/2})$ convergence rate is not restricted to integrals
over the unit interval and can be extended to any dimension. The standard error will still have the same form irrespective of dimension. This advantage is unlike other numerical methods. Thus, Monte Carlo simulation is attractive to solve integrals in higher dimension.

In structural reliability estimation, MCS involves sampling each random variable to give a sample value. Then, the limit state function is computed at every realization. If the limit state is violated, the structure has failed. The experiment is repeated many times with a randomly chosen vector. If \( N \) trials are conducted, the estimate \( \hat{P}_f \) of the probability of failure is given approximately as:

\[
\hat{P}_f = \frac{n(G(\hat{x}) \leq 0)}{N}
\]

(2-9)

Where \( n(G(x) \leq 0) \) denotes the number of trials \( n \) for which the structure failed. The number of trials \( N \) is related to the desired accuracy of failure probability. MCS need at least \( 100*N \) samples when the failure probability is of the order of \( 1/\sqrt{N} \). One can find the standard deviation of failure probability in Eq. (2-9) using the following expression:

\[
\sigma = \sqrt{\frac{\hat{P}_f (1 - \hat{P}_f)}{N}}
\]

(2-10)

There are many variants to the abovementioned crude MCS. Researchers use several variance reduction techniques like importance sampling (Glasserman, 2004) to make better use of MCS and extract more information from the same level of simulation. Importance sampling requires one to find a sampling PDF which uses the information of greatest probability density in the failure zone. This is conceptually similar to MPP which is essentially a point that contributes most to failure. An alternate formulation of the MCS is used by Smarslok et al (2006) to exploit the advantage of separable Monte Carlo simulations. This integration method takes advantage of a special form of the limit state function in which the limit state is composed of difference between capacity and response.
Figure 2-1. Reliability analysis and MPP. Subscript refers to iteration number.
In probabilistic approaches the difference between the computed probability of failure or reliability index and their target values does not provide the designer with easy estimates of the change in the design cost needed to achieve these target values. Alternately, inverse reliability measures are capable of providing this information. Several inverse reliability measures (e.g., probabilistic performance measure and probabilistic sufficiency factor) that are essentially equivalent have been introduced in recent years. The different names for essentially the same measure reflect the fact that different researchers focused on different advantages of inverse measures. This chapter reviews the various inverse measures and describes their advantages compared to the direct measures of safety such as probability of failure and reliability index. Methods to compute the inverse measures are also described. Reliability-based design optimization with inverse measure is demonstrated with a beam design example.

**Literature Review**

In deterministic approaches, a safety factor is defined as the ratio of the capacity to resistance of a system, typically calculated at the mean values. However, in probabilistic approaches, as both capacity and response are random, the safety factor is also a random number. One safety measure that combines the advantages of the safety factor and probability of failure was proposed by Birger (1970, as reported by Elishakoff, 2001). Birger relates the safety factor and the fractile of the safety factor distribution corresponding to a target probability of failure. It belongs to a class of inverse reliability measures, which carry that name because they require use of the inverse of the cumulative distribution function. Several researchers developed equivalent inverse reliability methods (Lee and Kwak 1987; Kiureghian et al. 1994; Li and Foschi 1998; Tu et al. 1999; Lee et al. 2002; Qu and Haftka 2003; Du et al. 2003) that are closely related to the
Birger measure. These measures quantify the level of safety in terms of the change in structural response needed to meet the target probability of failure.

Lee and Kwak (1987) used the inverse formulation in RBDO and showed that it is preferable for design when the probability of failure is very low in some region of the design space so that the safety index approaches infinity. Kiureghian et al. (1994) addressed the inverse reliability problem of seeking to determine one unknown parameter in the limit state function such that a prescribed first order reliability index is attained. To solve the inverse reliability problem, they proposed an iterative algorithm based on the Hasofer-Lind-Rackwitz-Fiessler algorithm. Li and Foschi (1998) employed the inverse reliability strategy in earthquake and offshore applications to solve for multiple design parameters. They show that it is an efficient method to estimate design parameters corresponding to target reliabilities. Kirjner-Neto et al. (1998) reformulated the standard RBDO formulation similar to Lee and Kwak (1987) except that they use an inequality constraint. They developed a semi-infinite optimization algorithm to solve the reformulated problem. This formulation does not require second order derivatives of the limit state functions and obviates the need for repeated reliability indices computation. However, they found that the approach can result in conservative design. Royset et al. (2001) extended the reformulation technique discussed in Kirjner-Neto et al. (1998) for reliability-based design of series structural systems. The required reliability and optimization calculations are completely decoupled in this approach. Tu et al. (1999) dubbed the inverse measure approach the Performance Measure Approach (PMA) and called the inverse measure probability performance measure (PPM). Lee et al. (2002) adopted the same procedure as Tu et al. (1999) and named it target performance based approach, calling the inverse measure the target performance. They compared the reliability index approach and inverse measure based approach and found that the latter was
superior in both computational efficiency and numerical stability. Youn et al. (2003) showed that PMA allows faster and more stable RBDO compared to the traditional Reliability Index Approach (RIA).

When accurate statistical information is not available for input data, it is not appropriate to use probabilistic method for stochastic structural analysis and design optimization and researchers resort to possibility based design optimization (PBDO) methods in such situations. Mourelatos and Zhou, 2005, discuss a PMA based PBDO. They formulate the inverse possibility analysis problem for the possibilistic constraint. Unlike FORM based RBDO which is based on linear approximation, the PBDO is exact. To perform the inverse possibility analysis, Du et al (2006b) proposes a maximal possibility search (MPS) method with interpolation to address the challenges of accuracy and computational expense exhibited by traditional methods like the multilevel cut and alpha level optimization method respectively. They report that the MPS evaluates possibility constraint efficiently and accurately for nonlinear structural applications. Often times industry design problems deal with uncertainties with sufficient data and uncertainties with insufficient data simultaneously. To address such situations Du et al., (2006a) extend the possibility-based design optimization (PBDO) method. They propose a two step approach to generate membership function from the available data. Initially, they develop a temporary PDF using the available data and then generate a membership function from the temporary PDF. They report that PMA based PBDO combined with MPS method can adequately address design problems with mixed input variables and demonstrates that it evaluates possibility constraint efficiently, stably and accurately for nonlinear structural applications.

Qu and Haftka (2003, 2004) called the inverse measure probability sufficiency factor (PSF) and explored its use for RBDO with multiple failure modes through Monte Carlo
Simulation (MCS) and response surface approximation (RSA). They showed that PSF leads to more accurate RSA compared to RSA fitted to failure probability, provides more effective RBDO, and that it permits estimation of the necessary change in the design cost needed to meet the target reliability. Moreover, PSF enables performing RBDO in variable-fidelity fashion and sequential deterministic optimization fashion to reduce the computational cost (Qu and Haftka, 2004a and 2004b). An initial study of using PSF to convert RBDO to sequential deterministic optimization was performed to solve problems with reliability constraints on individual failure modes (Qu and Haftka, 2003). An improved version for system reliability problems with multiple failure modes was developed for reliability-based global optimization of stiffened panels (Qu and Haftka, 2004b).

Du et al. (2003, 2004) employed PMA to formulate RBDO, but used percentile levels of reliability (1 minus failure probability) in the probabilistic constraint and called the inverse measure as percentile performance. Traditionally, design for robustness involves minimizing the mean and standard deviation of the performance. Here, Du et al. (2003) proposed to replace the standard deviation by percentile performance difference, which is the difference between the percentile performance corresponding to the left tail of a CDF and the right tail of that CDF. They demonstrated increased computational efficiency and more accurate evaluation of the variation of the objective performance. In an effort to address reliability-based designs when both random and interval variables are present, Du and Sudijianto (2003) proposed the use of percentile performance with worst-case combination of the interval variables for efficient RBDO solutions.

Du and Chen (2004) developed the sequential optimization and reliability assessment (SORA) to improve the efficiency of the probabilistic optimization. The method is a serial single
loop strategy, which employs percentile performance and the key is to establish equivalent
deterministic constraints from probabilistic constraints. This method is based on evaluating the
constraint at the most probable point of the inverse measure in Section IV below) based on the
reliability information from the previous cycle. This is referred to as “design shift”
(Chiralaksanakul and Mahadevan 2004; Youn et al. 2004). They show that the design quickly
improves in each cycle and is computationally efficient. The sequential optimization and
reliability assessment, however, is not guaranteed to lead to an optimal design. Single-level (or
unilevel) techniques that are equivalent to the standard RBDO formulation are based on
replacing the RIA or PMA inner loop by the corresponding Karush-Kuhn-Tucker conditions.
Here again, Agarwal et al. (2004) showed that the PMA approach is more efficient than the
unilevel RIA approach due to Kuschel and Rackwitz (2000). Since most commercial optimizers
are numerically unreliable when applied to problems accompanied by many equality constraints,
Agarwal et al (2007) use homotopy methods for constraint relaxation and to obtain a relaxed
feasible design. They solve a series of optimization problem as the relaxed optimization problem
is transformed via a homotopy to the original problem. They show that it is easier to solve the
relaxed problem and make a gradual progress towards the solution than solve the original
problem directly.

The several inverse measures discussed above are all based on the common idea of using
the inverse of the cumulative distribution function. The numerous names for the inverse
measures contemplate that they were developed by different researchers for different
applications. Since these inverse measures come under various names, it is easy to fail to notice
the commonality among them.
Birger Safety Factor

The safety factor, $S$ is defined as the ratio of the capacity of the system $G_c$ (e.g., allowable strength) to the response $G_r$ with a safe design satisfying $G_r \leq G_c$. To account for uncertainties, the design safety factor is greater than one. For example, a load safety factor of 1.5 is mandated by FAA in aircraft applications. To address the probabilistic interpretation of the safety factor, Birger (1970) proposed to consider its cumulative distribution function (CDF) $F_S$:

$$F_S(s) = \text{Prob}(\frac{G_c}{G_r} \leq s) \quad (3-1)$$

Note that unlike the deterministic safety factor, which is normally calculated for the mean value of the random variables, $G_c/G_r$ in Eq.(3-1) is a random function. Given a target probability, $P_{\text{target}}$, Birger suggested a safety factor $s^*$ (which we call here the Birger safety factor) defined in the following equation

$$F_S(s^*) = \text{Prob}(\frac{G_c}{G_r} \leq s^*) = \text{Prob}(S \leq s^*) = P_{\text{target}} \quad (3-2)$$

That is, the Birger safety factor is found by setting the value of cumulative distribution function (CDF) of the safety factor equal to the target probability. That is, we seek to find the value of the safety factor that makes the CDF of the safety factor equal to the target failure probability. This requires the inverse transformation of the CDF, hence the terminology of inverse measure.

Probabilistic Sufficiency Factor

Qu and Haftka (2003, 2004) developed a similar measure to the Birger safety factor, calling it first the probabilistic safety factor and then the probabilistic sufficiency factor (PSF). They obtained the PSF by Monte Carlo simulation and found that the response surface for PSF was more accurate than the response surface fitted to failure probability. Later, they found the
reference to Birger’s work in Elishakoff’s review (2001) of safety factors and their relations to probabilities. It is desirable to avoid the term safety factor for this entity because the common use of the term is mostly deterministic and independent of the target safety level. Therefore, while noting the identity of the Birger safety factor and the probabilistic sufficiency factor, we will use the latter term in the following.

Failure happens when the actual safety factor $S$ is less than one. The basic design condition that the probability of failure should be smaller than the target probability for a safe design may then be written as:

$$ P_f = \text{Prob}(S \leq 1) = F_S(1) \leq P_{\text{ftarget}} \quad (3-3) $$

Using inverse transformation, Eq.(3-3) can be expressed as:

$$ 1 \leq F_S^{-1}(P_{\text{ftarget}}) = s^* \quad (3-4) $$

The concept of PSF is illustrated in Figure. 3-1. The design requirement $P_{\text{ftarget}}$ is known and the corresponding area under the probability density function of the safety factor is the shaded region in Figure.3-1. The upper bound of the abscissa $s^*$ is the value of the PSF. The region to the left of the vertical line $S = 1$ represents failure. To satisfy the basic design condition $s^*$ should be greater than or equal to one. In order to achieve this, it is possible to either increase $G_c$ or decrease $G_r$. The PSF $s^*$, represents the factor that has to multiply the response $G_r$ or divide the capacity $G_c$, so that the safety factor be raised to 1.

For example, a PSF of 0.8 means that $G_r$ has to be multiplied by 0.8 or $G_c$ be divided by 0.8 so that the safety factor ratio increases to one. In other words, this means that $G_r$ has to be decreased by 20 % (1-0.8) or $G_c$ has to be increased by 25% ((1/0.8)-1) in order to achieve the target failure probability. It can be observed that PSF is a safety factor with respect to the target failure probability and is automatically normalized in the course of its formulation.
PSF is useful in estimating the resources needed to achieve the required target probability of failure. For example, in a stress-dominated linear problem, if the target probability of failure is $10^{-5}$ and a current design yields a probability of failure of $10^{-3}$, one cannot easily estimate the change in the weight required to achieve the target failure probability. Instead, if the failure probability corresponds to a PSF of 0.8, this indicates that maximum stresses must be lowered by 20% to meet the target. This permits the designers to readily estimate the weight required to reduce stresses to a given level.

**Probabilistic Performance Measure**

In probabilistic approaches, instead of the safety factor it is customary to use a performance function or a limit state function to define the failure (or success) of a system. For example, the limit state function is expressed as in Eq(2-1). In terms of safety factor $S$, another form of the limit state function is:

$$
\tilde{G}(X) = \frac{G_c(X)}{G_r(X)} - 1 = S - 1 \geq 0
$$

(3-5)

Here, $G(X)$ and $\tilde{G}(X)$ are the ordinary and normalized limit state functions, respectively. Failure happens when $G(X)$ or $\tilde{G}(X)$ is less than zero, so the probability of failure $P_f$ is:

$$
P_f = \text{Prob}(G(X) \leq 0)
$$

(3-6)

$$
P_f = \text{Prob}(\tilde{G}(X) \leq 0)
$$

(3-7)

Using Eq. (3-6), Eq. (3-7) can be rewritten as:

$$
P_f = \text{Prob}(G(X) \leq 0) = F_G(0) \leq P_{ftarget}
$$

(3-8)

$$
P_f = \text{Prob}(\tilde{G}(X) \leq 0) = F_{\tilde{G}}(0) \leq P_{ftarget}
$$

(3-9)

where $F_G$ and $F_{\tilde{G}}$ are the CDF of $G(X)$ and $\tilde{G}(X)$, respectively. Inverse transformation allows us to write Eqs. (3-8) and (3-9) as:
Here, $g^*$ and $\tilde{g}^*$ are the ordinary and normalized Probabilistic Performance Measure (PPM, Tu et al, 1999), respectively. PPM can be defined as the solution to Eqs. (3-8) and (3-9). That is, the value in (•) (instead of zero) which makes the inequality an equality. Hence PPM is the value of the limit state function that makes its CDF equals the target failure probability. Figure 3-2 illustrates the concept of PPM. The shaded area corresponds to the target failure probability. The area to the left of the line $G = 0$ indicates failure. $g^*$ is the factor that has to be subtracted from Eq. (3-6) in order to make the vertical line $g^*$ move further to the right of the $G = 0$ line and hence facilitate a safe design.

For example, a PPM of -0.8 means that the design is not safe enough, and -0.8 has to be subtracted from $G(X)$ in order to achieve the target probability of failure. A PPM value of 0.3 means that we have a safety margin of 0.3 to reduce while improving the cost function to meet the target failure probability.

Considering $\tilde{g}^*$ as the solution for Eq.(3-11), it can be rewritten in terms of the safety factor as:

$$\text{Prob}(\tilde{G}(X) = S-1 \leq \tilde{g}^*) = P_{\text{target}}$$

Comparing Eqs. (15), (23) and (24), we can observe a relationship between $s^*$ and $\tilde{g}^*$. PSF ($s^*$) is related to the normalized PPM ($\tilde{g}^*$) as:

$$s^* = \tilde{g}^* + 1$$

This simple relationship between PPM and PSF shows that they are closely related, and the difference is only in the way the limit state function is written. If the limit state function is expressed as the difference between capacity and response as in Eq.(3-6), failure probability
formulation allows us to define PPM. Alternatively, if the limit state function is expressed in terms of the safety factor as in Eq. (3-7), the corresponding failure probability formulation allows us to define PSF. PSF can be viewed as PPM derived from the normalized form of Eq. (3-6). The performance-measure approach (PMA) notation may appeal because of its generality, while the PSF notation has the advantage of being automatically scaled and being expressed in terms that are familiar to designers who use safety factors.

The PPM can be viewed as the distance from the $G = 0$ to target failure probability line, in the performance function space. This is analogical to reliability index being a measure of distance in the input variable space. The major difference is the measurement of distance in different spaces, the performance function (or output) space and the input space.

**Inverse Measure Calculation**

**Simulation Approach- Monte Carlo Simulation**

Conceptually, the simplest approach to evaluate PSF or PPM is by Monte Carlo simulation (MCS), which involves the generation of random sample points according to the statistical distribution of the variables. The sample points that violate the safety criteria in Eq. (3-6) are considered failed. Figure 3-3 illustrates the concept of MCS. A two-variable problem with a linear limit state function is considered. The straight lines are the contour lines of the limit state function and sample points generated by MCS are represented by small circles, with the numbered circles representing failed samples. The zero value of the limit state function divides the distribution space into a safe region and a failure region. The dashed lines represent failed conditions and the continuous lines represent safe conditions.

The failure probability can be estimated using Eq.(2-9) In Figure. 3-3, the number of sample points that lie in the failure region above the $G = 0$ curve is 12. If the total number of
samples is 100,000, the failure probability is estimated at $1.2 \times 10^{-4}$. For a fixed number of samples, the accuracy of MCS deteriorates with the decrease in failure probability.

For example, with only 12 failure points out of the 100,000 samples, the standard deviation of the probability estimate is $3.5 \times 10^{-5}$, more than a quarter of the estimate. When the probability of failure is significantly smaller than one over the number of sample points, its calculated value by MCS is likely to be zero.

PPM is estimated by MCS as the $n^{th}$ smallest limit state function among the $N$ sampled functions, where $n = N \times P_{\text{target}}$. For example, considering the example illustrated in Figure 3-3, if the target failure probability is $10^{-4}$, to satisfy the target probability of failure, no more than 10 samples out of the 100,000 should fail. So, the focus is on the two extra samples that failed. PPM is equal to the value of the highest limit state function among the $n$ (in this case, $n = 10$) lowest limit state functions. The numbered small circles are the sample points that failed. Of these, the three highest limit states are shown by the dashed lines. The tenth smallest limit state corresponds to the sample numbered 8 and has a limit state value of -0.4, which is the value of PPM. Mathematically this is expressed as:

$$g^* = n^{th} \min_{i=1}^{N} (G(X_i))$$

(3-14)

where, $n^{th} \min$ is the $n^{th}$ smallest limit state function. So, the calculation of PPM in MCS requires only sorting the lowest limit state function in the MCS sample. Similarly, PSF can be computed as the $n$th smallest factor among the $N$ sampled safety factors and is mathematically expressed as **:

$$s^* = n^{th} \min_{i=1}^{N} (S(X_i))$$

(3-15)

** A more accurate estimate of PPM or PSF will be obtained from the average of the $n^{th}$ and $(n+1)^{th}$ smallest values. So in the case of Figure 4, PPM is more accurately estimated as 0.35.
Finally, probabilities calculated through Monte Carlo Simulation (MCS) using a small sample size are computed as zero in the region where the probability of failure is lower than one over the number of samples used in MCS. In that region, no useful gradient information is available to the optimization routine. On the other hand, PSF value varies in this region and thus provides guidance to optimization.

MCS generates numerical noise due to limited sample size. Noise in failure probability may cause RBDO to converge to a spurious minimum. In order to filter out the noise, response surface approximations are fitted to failure probability to create a so-called design response surface. It is difficult to construct a highly accurate design response surface because of the huge variation and uneven accuracy of failure probability. To overcome these difficulties, Qu and Haftka (2003, 2004) discuss the usage of PSF to improve the accuracy of the design response surface. They showed that design response surface based on PSF is more accurate compared to design response surface based on failure probability and this accelerates the convergence of RBDO. For complex problems, response surface approximations (RSA) can also be used to approximate the structural response in order to reduce the computational cost. Qu and Haftka (2004b) employ PSF with MCS based on RSA to design stiffened panels under system reliability constraint.

**Analytical Approach- Moment-based Methods**

Inverse reliability measures can be estimated using moment based methods like FORM. The FORM estimate is good if the limit state is linear but when the limit state has a significant curvature, second order methods can be used. The Second Order Reliability Method (SORM) approximates the measure of reliability more accurately by considering the effect of the curvature of the limit state function (Melchers, 1999, pp 127-130).
Figure 3-4. illustrates the concept of inverse reliability analysis and MPP search. The circles represent the \( \beta \) curves with the target \( \beta \) curve represented by a dashed circle. Here, among the different values of limit state functions that pass through the \( \beta_{\text{target}} \) curve, the one with minimum value is sought. The value of this minimal limit state function is the MPP as shown by Tu et al. 1999. The point on the target circle with the minimal limit state function is sought. This point is also an MPP and in order to avoid confusion between the usual MPP and MPP in inverse reliability analysis, Du et al. (2003) coined the term most probable point of inverse reliability (MPPIR) and Lee et al. (2002) called it the minimum performance target point (MPTP). Du et al. (2003) developed the sequential optimization and reliability analysis method in which they show that evaluating the probabilistic constraint at the design point is equivalent to evaluating the deterministic constraint at the most probable point of the inverse reliability. This facilitates in converting the probabilistic constraint to an equivalent deterministic constraint. That is, the deterministic optimization is performed using a constraint limit which is determined based on the inverse MPP obtained in the previous iteration. Kiureghian et al. (1994) proposed an extension of the Hasofer-Lind-Rackwitz-Fiessler algorithm that uses a merit function and search direction to find the MPTP. In Figure 3-4, the value of the minimal limit state function or the PPM is -0.2. This process can be expressed as:

\[
\begin{align*}
\text{Minimize} & \quad : G(U) \\
\text{Subject to} & \quad : \| U \| = \sqrt{U^T U} = \beta_{\text{target}} \\
\end{align*}
\] (3-16)

In reliability analysis the MPP is on the \( G(U) = 0 \) failure surface. In inverse reliability analysis, the MPP search is on the \( \beta_{\text{target}} \) curve. One of the main advantages when inverse measures are used in the FORM perspective is that, the formulation of the optimization problem is as expressed in Eq. (3-16) where the constraint has a simple form (circle) compared to the cost function. The cost function has in general a complicated expression due to the nonlinear
transformation between physical random variable X to the standard random variable U. It is well known that it is easy to satisfy a simple constraint compared to a complicated constraint irrespective of the cost function. On comparing the formulation in Eq.(2-5) with the formulation in Eq.(3-16), it is evident that it is easier to solve the latter because of the simpler constraint.

Reliability – Based Design Optimization with Inverse Measures

The probabilistic constraint in RBDO can be prescribed by several methods like the Reliability Index Approach (RIA), the Performance Measure Approach (PMA), the Probability Sufficiency Factor approach (PSF), see Table 3-1.

<table>
<thead>
<tr>
<th>Method</th>
<th>RIA</th>
<th>PMA</th>
<th>PSF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probabilistic constraint</td>
<td>$\beta \geq \beta_{target}$</td>
<td>$g^* \geq 0$</td>
<td>$s^* \geq 1$</td>
</tr>
<tr>
<td>Quantity to be computed</td>
<td>Reliability index ($\beta$)</td>
<td>PPM ($g^*$)</td>
<td>PSF ($s^*$)</td>
</tr>
</tbody>
</table>

In RIA, $\beta$ can be computed as the product of reliability analysis as discussed in the previous section. The PPM or PSF can be computed through inverse reliability analysis or as a byproduct of reliability analysis using MCS.

To date, most researchers have used RIA to prescribe the probabilistic constraint. However, the advantages of the inverse measures, illustrated in the next section, have led to its growing popularity.

Beam Design Example

The cantilever beam shown in Figure. 3-5, taken from Wu et al. (2001), is a commonly used demonstration example for RBDO methods. The length $L$ of the beam is 100". The width and thickness is represented by $w$ and $t$. It is subjected to end transverse loads $X$ and $Y$ in orthogonal directions as shown in the Figure 3-5. The objective of the design is to minimize the

---

1 For this example, the random variables are shown in bold face.
weight or equivalently the cross sectional area: \( A = wt \) subject to two reliability constraints, which require the safety indices for strength and deflection constraints to be larger than three. The first two failure modes are expressed as two limit state functions:

\[
\text{Stress limit: } G_s = R - \sigma = R - \left( \frac{600}{wt^2} Y + \frac{600}{w^2t} X \right) \quad (3-17)
\]

\[
\text{Tip displacement limit: } G_d = D_0 - D = D_0 - \frac{4L}{Ewt} \sqrt{\left( \frac{Y}{r^2} \right)^2 + \left( \frac{X}{w^2} \right)^2} \quad (3-18)
\]

where \( R \) is the yield strength, \( E \) is the elastic modulus, \( D_0 \) the displacement limit, and \( w \) and \( t \) are the design parameters. \( R, X, Y \) and \( E \) are uncorrelated random variables and their means and standard deviations are defined in Table 3-2.

<table>
<thead>
<tr>
<th>Random Variables</th>
<th>( X ) (lb)</th>
<th>( Y ) (lb)</th>
<th>( R ) (psi)</th>
<th>( E ) (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distribution ( (\mu, \sigma) )</td>
<td>Normal (500,100)</td>
<td>Normal (1000,100)</td>
<td>Normal (40000,2000)</td>
<td>Normal (29×10^6,1.45×10^6)</td>
</tr>
</tbody>
</table>

### Design for Stress Constraint

The design with strength reliability constraint is solved first, followed by the design with a system reliability constraint. The results for the strength constraint are presented in Table 3-3. For the strength constraint, the limit state is a linear combination of normally distributed random variables, and FORM gives accurate results for this case. The MCS is performed with 100,000 samples. The standard deviation in the estimated failure probability is calculated by MCS as:

\[
\sigma_p = \sqrt{\frac{P_f (1-P_f)}{N}} \quad (3-19)
\]

In this case, the failure probability of 0.0013 calculated from 100,000 samples has a standard deviation of 1.14×10^-4. It is seen from Table 3-3 that the designs obtained from RIA, PMA and PSF match well. Since the stress in Eq. (3-17) is a linear function of random variables,
the RIA and PMA are exact. The more conservative design from PSF is due to limited sampling of MCS.

Table 3-3. Comparison of optimum designs for the stress constraint
Minimize objective function $A = wt$ such that $\beta \geq 3$

<table>
<thead>
<tr>
<th>Method</th>
<th>Reliability Analysis</th>
<th>Inverse Reliability Analysis</th>
<th>Exact Optimum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FORM (RIA)</td>
<td>FORM (PMA)</td>
<td>(Qu and Haftka,2003)</td>
</tr>
<tr>
<td>Optima $w$</td>
<td>2.4460</td>
<td>2.4460</td>
<td>2.4526</td>
</tr>
<tr>
<td></td>
<td>3.8922</td>
<td>3.8920</td>
<td>3.8884</td>
</tr>
<tr>
<td>Reliability Index</td>
<td>3.00</td>
<td>3.00</td>
<td>3.0162</td>
</tr>
<tr>
<td>Failure Probability</td>
<td>0.00135</td>
<td>0.00135</td>
<td>0.00128</td>
</tr>
</tbody>
</table>

Comparison of Inverse Measures

The relation between PSF and PPM in Eq. (3-13) is only approximate when PPM is calculated by FORM and PSF by MCS. PSF suffers from sampling error and PPM from error due to linearization. For the linear stress constraint, and with a large Monte Carlo sample, the difference is small, as seen in Table 3-4. It may be expected that the MPTP (minimum performance target point) should also be close to the point used to calculate PSF. This result may be useful, because when a response surface is used for an approximation of the response, it is useful to center it near the MPTP. In order to check the accuracy of the MPTP estimation, the MPP of PPM and the point corresponding to the PSF are compared and the results are tabulated in Table 3-5. The coordinates corresponding to the PSF, computed by a million-sample MCS, deviate considerably from the MPTP. Since the accuracy of the points computed by MCS depends on the number of samples used, an increased number of samples lead to more accurate results, albeit at increased computational cost. Alternative approaches to obtain the points with better accuracy without increasing the number of samples are to average the co-ordinates computed by repeated MCS simulations with fewer numbers of samples. Alternatively, we can average a number of points
that are nearly as critical as the PSF point. That is, instead of using only the \( X_i \) corresponding to the \( S_n \) in Eq (3-15), we use also the points corresponding to \( S_{n-p}, S_{n-p+1}, \ldots, S_{n+p} \) in computing the PSF, where \( 2p \) is the total number of points that is averaged around the PSF. It can be observed from Table 3-5 that averaging 11 points around the PSF matches well with the MPTP, reducing a Euclidean distance of about 0.831 for the raw PSF to 0.277 with 11-point average. The values of \( X, Y \) and \( R \) presented in Table 3-5 are in the standard normal space.

Table 3-4. Comparison of inverse measures for \( w=2.4526 \) and \( t=3.8884 \) (stress constraint)

<table>
<thead>
<tr>
<th>Method</th>
<th>FORM</th>
<th>MCS (1×10^6 samples)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_f )</td>
<td>0.001238</td>
<td>0.001241</td>
</tr>
<tr>
<td>Inverse Measure</td>
<td>PPM: 0.00258</td>
<td>PSF: 1.002619</td>
</tr>
</tbody>
</table>

Table 3-5. Comparison of MPP obtained from calculation of PSF for \( w=2.4526 \) and \( t=3.8884 \) (stress constraint)

<table>
<thead>
<tr>
<th>Co-ordinates</th>
<th>( X )</th>
<th>( Y )</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>MPTP</td>
<td>2.1147</td>
<td>1.3370</td>
<td>-1.6480</td>
</tr>
<tr>
<td>PSF</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1)10^6 samples</td>
<td>2.6641</td>
<td>1.2146</td>
<td>-1.0355</td>
</tr>
<tr>
<td>2)Average:10 runs of each 10^5 samples</td>
<td>2.1888</td>
<td>1.8867</td>
<td>-1.1097</td>
</tr>
<tr>
<td>3)Average:11 points around PSF of 10^6 Samples</td>
<td>2.0666</td>
<td>1.5734</td>
<td>-1.5128</td>
</tr>
</tbody>
</table>

**Use of PSF in Estimating the Required Change in Weight to Achieve a Safe Design**

The relation between the stresses, displacement and weight for this problem is presented to demonstrate the utility of PSF in estimating the required resources to achieve a safe design.

Consider a design with the dimensions of \( A_0 = w_0 t_0 \) with a PSF of \( s_0^* \) less than one. The structure can be made safer by scaling both \( w \) and \( t \) by the same factor \( c \). This will change the stress and displacement expressed in Eq.(3-17) and Eq. (3-18) by a factor of \( c^3 \) and \( c^4 \), respectively, and the area by a factor of \( c^2 \). If stress is most critical, it will scale as \( c^3 \) and the PSF will vary with the area, \( A \), as:
Equation (3-20) indicates that a one percent increase in area will increase the PSF by 1.5 percent. Since non-uniform scaling of width and thickness may be more efficient than uniform scaling, this is a conservative estimate. Thus, for example, considering a design with a PSF of 0.97, the safety factor deficiency is 3% and the structure can be made safer with a weight increase of less than two percent, as shown by Qu and Haftka (2003). For a critical displacement state, $s^*$ will be proportional to $A^2$ and a 3% deficit in PSF can be corrected in under 1.5% weight increase. While for more complex structures we do not have analytical expressions for the dependence of the displacements or the stresses on the design variables, designers can usually estimate the weight needed to reduce stresses or displacements by a given amount.

**System Reliability Estimation Using PSF and MCS**

System reliability arises when the failure of a system is defined by multiple failure modes. Here, the discussion is limited to series systems. The estimation of system failure probability involves estimating the failure due to individual modes and failure due to the interaction between the modes. This is mathematically expressed for a system with n failure modes as:

$$P_{sys} = P(F_1) + P(F_2) + ... + P(F_n)$$
$$- P(F_1 \cap F_2) - P(F_1 \cap F_3) - ... - P(F_{n-1} \cap F_n)$$
$$+ P(F_1 \cap F_2 \cap F_3) + ... + P(F_{n-2} \cap F_{n-1} \cap F_n) - ...$$

Higher order terms

It is easy to employ MCS to estimate system failure, as the methodology is a direct extension of failure estimation for single modes. In the context of employing analytical methods like FORM, estimation of failure regions bounded by single modes are easy to estimate but estimating the probability content of odd shaped domains created because of the intersection of two or more modes is challenging. Techniques are available to estimate the probability content at the intersection regions but require evaluating multiple integrals (Melchers, pp: 386-400). Since the
number of multiple integrals that has to be evaluated depends on the number of variables, the
task becomes arduous when more variables are involved. Instead, an alternate technique is to
develop upper and lower bounds on the failure probability of the structural system. Considering
the terms in the first and the third rows (positive contributions) in Eq. (3-21) permits us to estimate the upper bound. When the terms in first and second rows (negative contributions) are considered, the lower bounds can be estimated. Improved bounds for the system reliability are available in the literature.

Owing to these challenges, MCS is mostly used when system reliability is to be estimated. As discussed earlier, MCS is computationally prohibitive when the failure probability to be estimated is low, because of the number of samples required. PSF can be used to estimate system reliability. The method is an extension of the PSF estimation for single failure mode. When \( M \) failure modes are considered, the most critical safety factor is calculated for each sample. Then the sorting of the \( n^{th} \) minimum safety factor can proceed as in Eq.(3-15). This process is explained in the flow chart in Figure 3-6. The estimation of PSF for a system reliability case is demonstrated with an example in Appendix A.

**Design for System Reliability by MCS and PSF**

Monte Carlo simulation is a good method to use for system reliability analysis with multiple failure modes. For the cantilever beam discussed in the earlier section, the allowable deflection was chosen to be 2.25" in order to have competing constraints (Wu *et al.*, 2001). The results are presented in Table 3-6. It can be observed that the contribution of the stress mode to failure probability dominates the contribution due to displacement and the interaction between the modes. The details of the design process are provided in Qu and Haftka (2004). They demonstrated the advantages of using PSF as an inverse safety measure over probability of failure or the safety index as a normal safety measure. They showed that the design response
surface (DRS) of the PSF was much more accurate than the DRS for the probability of failure. For a set of test points the error in the probability of failure was 39.11% from the DRS to the PSF, 96.49% for the DRS to the safety index, and 334.78% for the DRS to the probability of failure.

Table 3-6. Design for System Reliability by PSF, Qu and Haftka (2004)**

<table>
<thead>
<tr>
<th>Optima</th>
<th>Objective Function</th>
<th>$P_f$</th>
<th>Safety Index</th>
<th>$P_{f1}$</th>
<th>$P_{f2}$</th>
<th>$P_{f1 \cap P_{f2}}$</th>
<th>PSF $s^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w = 2.6041$</td>
<td>9.5691</td>
<td>0.001289</td>
<td>3.01379</td>
<td>0.001133</td>
<td>0.000208</td>
<td>0.000052</td>
<td>1.0006</td>
</tr>
<tr>
<td>$t = 3.6746$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**100,000 samples, $P_{f1}, P_{f2}, P_{f1 \cap P_{f2}}$ - Failure probabilities due to stress constraint, displacement constraint, and intersection between the modes, respectively.
Figure 3-1. Schematic probability density of the safety factor $S$. The PSF is the value of the safety factor corresponding to the target probability of failure.

Figure 3-2. Schematic probability density of the limit state function. The Probabilistic Performance Measure is the value of the limit state function corresponding to the target probability of failure.
Figure 3-3. Illustration of the calculation of PPM with Monte Carlo Simulation for the linear performance function

Figure 3-4. Inverse reliability analysis and MPP for target failure probability 0.00135 \( (\beta = 3) \)

Figure 3-5. Cantilever beam subjected to horizontal and vertical loads
Figure 3-6. Flowchart for estimating system PSF

\[ x_i \quad i = 1...N \]

\[ S^j(x_i) \quad j = 1...M (\text{#of modes}) \]

\[ S_{cr} = \text{sort}(\text{min}(S(x^j))) \]

\[ S^*_{sys} = \min_{i=1}^{N} n^{th} \quad n = N \times P_{\text{target}} \]
CHAPTER 4
TAIL MODELLING AND RELIABILITY ANALYSIS

Low failure probability problems (extreme value) require one to have sufficient data in the tails of the distribution which represent the extremes. But this is seldom possible and instead researchers use tail modeling techniques based on extreme value theory to predict the probability of extreme events. The theory comprises a principle for model extrapolation based on the implementation of mathematical limits as finite level approximations. Since several advantages are reported by working with inverse measures, it is logical to justify an attempt to perform tail modeling in the performance space along with inverse measures to estimate quantities at unobserved levels. This section discusses the tail modeling technique and how to apply it to find inverse measures.

**Tail Equivalence and Generalized Extreme Value Theory**

Tail modeling is based on the concept of approximating the tail distribution of the variable of interest by associating it with the tail of already known distribution and using the parameters and characteristics therein for further analysis. This criterion is adequately defined by tail equivalence. That is, two CDFs $F(x)$ and $Q(x)$ can be called tail equivalent (Maes and Breitung, 1993) at the right tails when:

$$\lim_{{x \to \infty}} \frac{1 - F(x)}{1 - Q(x)} = 1 \quad (4-1)$$

This reveals that the error in approximating the small probability of $1 - F(x)$ by $1 - Q(x)$ tends to zero as $x \to \infty$.

The extremal theorem by Fisher and Tippet (1928) states that, if $M_n$ is the maximum of a sequence of ‘n’ independent random variables having a common distribution function and if there exists sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ such that
\[ \Pr \left\{ \frac{(M_n - b_n)}{a_n} \leq z \right\} \rightarrow H(z) \text{ as } n \rightarrow \infty \] 

(4-2)

for a non-degenerate distribution function \( H \), then \( H \) belongs to one of the extreme value distribution families namely, the Gumbel, Frechet and Weibull families. It is to be noted that regardless of the distribution for the population, the three types of the extreme value distributions are the only possible limits for the distribution of the normalized maxima. In this respect, this theorem is to extreme value theory what central limit theorem is to central statistics. The central limit theorem states that the approximating distribution of the sample mean is normal regardless of the parent distribution.

The three families of distributions can be combined into a single family of models having distribution function of the form

\[
H(z) = \exp \left\{ - \left[ 1 + \xi \left( \frac{z - \mu}{\sigma} \right) \right] \right\}^{-1/\xi} 
\]

(4-3)

such that \( \{ z : 1 + \xi (z - \mu) / \sigma > 0 \} \), where the parameters satisfy \(-\infty < \mu < \infty, \sigma > 0 \) and \(-\infty < \xi < \infty \). This is the generalized extreme value (GEV) family of distributions. Here, \( \mu \) is the location parameter, \( \sigma \) is the scale parameter, \( \xi \) is the shape parameter. A more detailed discussion of the generalized extreme value distributions is presented in Appendix B

**Generalized Pareto Distribution**

In engineering applications, rather than maxima, the interest is to address the excesses over threshold. In these situations, the generalized pareto distribution (GPD) arises as the limiting distribution. The concept of GPD is presented in Figure 4-1. Let \( y \) be a model output which is random and \( u \) be a large threshold of \( y \). The observations of \( y \) that exceed \( u \) are called exceedance. The conditional distribution \( F_u(z) \) of the exceedance given that the data \( y \) is greater than the threshold \( u \), is modeled fairly well by the GPD. Here, \( z = y - u \). Let approximation of
For a large enough $u$, the distribution function of $(y-u)$, conditional on $y > u$, is approximately written as (Coles, 2001):

$$
\hat{F}_{\xi,\psi}(z) = \begin{cases} 
1-\left(1+\frac{\xi}{\psi} z\right)^{-\frac{1}{\xi}} & \text{if } \xi \neq 0 \\
1-\exp\left(-\frac{z}{\psi}\right) & \text{if } \xi = 0
\end{cases}
$$

(4-4)

In Eq (4-4), $\langle A \rangle_s = \max(0, A)$ and $z > 0$. $\xi$ plays a key role in assessing the weight of the tail. Eq (4-4) can be justified as a limiting distribution as $u$ increases (Coles, 2001, pp:75-76).

Tails can be classified based on $\xi$ as:

- $\xi > 0$, heavy tail (pareto type tail)
- $\xi = 0$, medium tail (exponential type tail)
- $\xi < 0$, light tail (Beta-type tails)

There are several parameter estimation methods to estimate the parameters $\xi$ and $\psi$. It is to be noted that conditional excess CDF $F_u(z)$ is related to the CDF of interest $F(y)$ through the following expression:

$$
F_u(z) = \frac{F(y) - F(u)}{1 - F(u)}
$$

(4-5)

From Eq (4-5), the CDF of $y$ can be expressed as:

$$
F(y) = (1 - F(u))F_u(z) + F(u)
$$

(4-6)

When Eq (4-4) is substituted for $F_u(z)$ in the above expression, Eq (4-6) becomes:

$$
F(y) = 1 - \left(1 - F(u)\right)\left(1+\frac{\xi}{\psi} (y-u)\right)^{-\frac{1}{\xi}}
$$

(4-7)

For simplicity of presentation, only the case of $\xi \neq 0$ is considered here. Once we obtain estimates of the parameters as $\xi$ and $\psi$ using some parameter estimation method like maximum
likelihood estimation method, least square regression that are discussed later in the chapter, it is possible to estimate the pth quantile by inverting Eq.(4-7):

$$\hat{y}_p = F^{-1}(p) = u + \frac{\psi}{\xi} \left( \frac{1 - F(p)}{1 - F(u)} \right)^{-\frac{1}{\xi}} - 1 \quad (4-8)$$

In order to extend the concept to structural applications, if the output $y$ is replaced by the limit state in Eq (3-5) then the interest is to model the lower tail rather than the upper tail. However, modeling the upper tail of $y$ is equivalent to modeling the lower tail of $-y$. Therefore, an equivalent way of expressing Eq.(3-5) is:

$$G_i(x) = 1 - S_i(x) \quad (4-9)$$

where $S_i$ is the reciprocal of the safety factor. Failure occurs when $G_i > 0$. $S_r$ at any required target reliability index can be estimated using Eq(4-8). If $P_f$ refers to the target failure probability that we wish to design the structure, PSF can be obtained as:

$$S_r = F^{-1} \left( 1 - P_f \right) = u + \frac{\psi}{\xi} \left[ \left( \frac{P_f}{1 - F(u)} \right)^{-\frac{1}{\xi}} - 1 \right] \quad (4-10)$$

Coles (2001) opines that it is easy to object to the procedure of tail modeling on the grounds that even with the support of asymptotic argument, there is an implicit assumption that the underlying stochastic mechanism of the process being modeled is sufficiently smooth to enable extrapolation. However no credible alternative has been proposed to date.

**Threshold Selection**

The theory of excess over threshold is referred to as peaks over threshold method also. Performance of this approach is based on the choice of the threshold value $u$. The basic theory identified by Pickands (1975) states that the pareto distribution will be a practical family for statistical estimation of the tails, provided that the threshold is taken sufficiently high. The CDF is composed of 3 portions, the lower tail, upper tail and the central portion. In theory, the
threshold should be selected where the actual upper tail starts. But there is no straightforward globally accepted method to select threshold.

Selection of threshold is a tradeoff between bias and variance. If the threshold selected is low, then some data points belong to the central part of the distribution and do not provide a good approximation to the tails. On the other hand, if the threshold selected is too high, the number of data available for the tail approximation is much less and this might lead to excessive scatter in the final estimate. The proper selection of threshold is very important because it has important repercussions on the estimated value of the shape factor (Caers and Maes, 1998, McNeil and Saladin, 1997) and hence on the final estimates such as the quantile, extreme values etc.

There are many exploratory techniques like the mean excess plot which help in selecting the threshold. But in a simulation study, it is impractical to perform interactive data analysis required by the exploratory techniques to choose the threshold. The mean excess plot is the sample mean excess function defined as the sum of the excesses over the threshold divided by the number of the data points which exceed the threshold plotted with respect to the threshold. A reasonable choice of the optimal threshold is where this plot becomes linear. For further reading the reader is referred to Coles (2001).

Boos (1984) suggests that the ratio of $N_{ex}$ (number of tail data) over $N$ (total number data) should be 0.02 ($50 < N < 500$) and the ratio should be 0.1 for $500 < N < 1000$. Hasofer (1996) suggests using $N_{ex} = 1.5 \sqrt{N}$. Caers and Maes (1998) propose to use a finite sample mean square error (MSE) as a criterion for estimating the threshold. They use the threshold value that minimizes the MSE. In a similar fashion Beirlant et al (1996) find an optimal threshold by minimizing an approximate expression for asymptotic mean square error. The other methods
include plotting the quantile, shape or scale factor or any quantity of interest with respect to different thresholds and look for a stability in the curve (Bassi et al, Coles pp:84-86)

Parameter Estimation

Generally one is interested to generate values (PDF) from a particular distribution given a set of parameter values \( \theta \) which is expressed as:

\[
f(x / \theta)
\]  

(4-11)

where, \( x \) is a data vector \([x_1...x_m]\) of frequency. However, here we are faced with an inverse problem. Given the set of observed data and a model of interest, one is required to find one PDF that among all probability distribution that is possible for the model is most likely to have produced the data. The parameters can be estimated by several methods. The methods used in the literature include, maximum likelihood estimation (MLE), method of moments, probability weighted moments, elemental percentile method, the least square regression method. Generally, maximum likelihood estimation method is widely used and accepted by researchers though it has limitations. That is as long as \( \xi > -0.5 \), the asymptotic properties (Appendix B) of the maximum likelihood estimators are preserved. When \(-1 < \xi < -0.5\) MLE are obtainable but do not have the standard asymptotic properties. MLE is not obtainable when \( \xi < -1 \) (Coles, 2001, pp54-55). The likelihood estimation approach for the GPD parameters is presented in the Appendix B. However, accurate estimation of the parameters is a huge research area in itself. Beirlant et al (2004), Castillo et al (2005) discuss these methods in detail. Hasking and Wallis (1987) report a comparison between the maximum likelihood estimation method, method of moments and the probability weighted moments. They conclude that method of moments and the probability weighted moments are more reliable compared to the MLE method unless the sample size is greater than 500. This section discusses the maximum likelihood method and least square regression approach in detail
**Maximum Likelihood Estimation**

MLE is a popular statistical method that is used to make inferences about the parameters of the underlying probability distribution of a given data set. There is nothing visual about the MLE method but it is very powerful and is precise for large samples. The likelihood of a set of data is the probability of obtaining that particular set of data, given the chosen probability distribution model. ML estimation starts with writing the likelihood function which contains the unknown distribution parameters. The values of these parameters that maximize the likelihood function are called the maximum likelihood estimators.

In order to estimate the parameters, the likelihood function has to be constructed by reversing the roles of the data and parameter in Eq (4-11), i.e.,

\[ L(\theta / x) = f(x / \theta) \]  

Thus \( L \) represent the likelihood of the parameter \( \theta \) given the observed data \( x \) and is a function of \( \theta \). For a model with \( k \) parameters, the likelihood function takes the shape of \( k \)-dim geometrical surface sitting above a \( k \)-dim hyperplane spanned by the parameter vector (Myung, 2003).

Both \( f \) and \( L \) are PDFs but the major difference lies in the fact that they are defined in two different axes and are not directly comparable to each other. PDF \( f \) is a function of data given a particular set of parameter values defined on the data scale. The likelihood function \( L \) is the function of parameter given a particular set of observed data defined on the parameter scale.

ML approach is totally analytical in concept. MLE's generally have very desirable large sample properties (Appendix C) However, MLE cannot be adopted in all situations. There are some limitations when small numbers of failures (less than 5, and sometimes less than 10 is small), MLE's can be heavily biased and the large sample asymptotic properties do not apply
Least Square Regression

The method of least squares assumes that the best-fit curve of a given type is the curve that has the minimal sum of the deviations squared (least square error) from a given set of data. The parameters are obtained by solving the following minimization problem

\[
\text{Min} \sum_{i=N_u}^{N} \left( \hat{F}_{\xi,\mu}(z_i) - P_i \right)^2
\]

The GPD CDF can be obtained by using Eq. (4-4). The empirical CDF are plotting positions which are computed as:

\[
P_i = \frac{i}{N + 1}, i=1...N
\]

where \(N\) is the total number of samples, \(N_u\) is the index corresponding to the threshold \(u\) and \(P\) is the plotting position. Since it is a minimum finding, it can be seen as an optimization technique similar to maximum likelihood estimate method where we try to maximize the likelihood function. Least square regression requires no or minimal distributional assumptions. Unlike MLE, there is no basis for testing hypotheses or constructing confidence intervals.

Accuracy and Convergence Studies for the Quantile Estimation

In order to understand the capability of GPD to model the tails of the distribution, an error analysis in terms of modeling and sampling error was performed on the probability content estimation using tail modeling approach. A detailed discussion of this is presented in Appendix D. Based on the results, it can be concluded that the sampling error is high compared to the modeling error. If the sampling error is accounted for, tail modeling has a fairly good capacity to estimate extreme probabilities with good accuracy. In structural reliability studies, one is required to model the tails of the response which might follow some standard statistical distributions. Normal and Lognormal distributions are widely used to model physical quantities in structural reliability studies. Therefore, an investigation was done to study the accuracy and
convergence capabilities of tail modeling approach to model the tails of normal and lognormal
distribution which is discussed in detail in Appendix E. Then, the approach is demonstrated on a
cantilever beam example and a tuned mass damper example to estimate the failure probability.

The simulation studies (Appendix E) help us to conclude that based on the target quantile
value we wish to estimate, the threshold and number of exceedance data need to be chosen. In
the context of structural analysis, the interest is to address the response function. The failure
probability is based on the response. If we wish to design a system with a low failure probability
of the order of 1e-4 or higher, then we are required to estimate quantiles of 0.9999 or higher. So,
if the response follows a distribution close to normal, and we wish to estimate a quantile of
0.9999, a quantile of 0.95 and 100 exceedance data would enable us to estimate the quantile
under 3% error. On the other hand, if the response follows more of a lognormal distribution
(heavy tailed), then one is required to choose a higher threshold of the order of 0.99 or 0.995
with 100 or more exceedance data to be guaranteed to have a bias error of less than one percent
and a RMSE error of less than 5 percent. However, it is not possible to come up with a table that
explains the minimum exceedance data required for a level of quantile and required accuracy
level because, in practical situations the distribution of the response is not known. Moreover, the
challenge is to develop methodologies that can estimate the inverse measure corresponding to
high reliability, with limited samples.

**Cantilever Beam Example**

The beam example that was considered in Chapter 3 is treated here also. Though
simulating the beam example is not very expensive, we pretend that we have limited computer
resources and are restricted to use a maximum of 500 simulations. We are interested in
estimating the PSF for low failure probabilities with only 500 response data. Three different
cases are discussed. Case 1 discusses the estimation of PSF for the strength failure mode using
the tail modeling approach at different threshold values. Similarly the PSF estimation for the
deflection failure mode is investigated in Case 2 and the system failure case where both the
failure modes are considered simultaneously is discussed in Case 3. Case 1 and Case 2 are
presented in Appendix F. For all the cases, the convergence of PSF with respect to different
thresholds and accuracy of PSF at different number of samples are investigated. Two methods,
namely the maximum likelihood method and the least square regression methods are used to
estimate the parameters. The results from these methods are compared.

In the system failure mode, both the failure modes are considered simultaneously. The safety
factor for each failure mode is evaluated. For each realization, the critical of the two safety factor
is considered the system safety factor. Once, the system safety factor is obtained for 100
simulations, the tail modeling approach is carried out and the corresponding system PSF is
estimated. Here the design variables are $w=2.6041; t=3.6746$ and the allowable deflection
$D_0=2.145$. This combination of the variables allows equal contribution of the modes to the total
failure. The contribution of the modes are: $P_{f_1} = 0.00099; P_{f_2} = 0.00117; P_{f_1} \cap P_{f_2} = 0.00016$.
The convergence plots of PSF for the system failure case is presented in Figure 4-2. Figure 4-2
shows that the PSF estimated through parameters estimated by the regression method is unstable
in contrast to the PSF estimated through parameters from the ML method. This behavior was
observed in the other two cases too. The PSF estimate at a threshold of 0.9 is 0.99 with a
standard deviation of 0.003. The corresponding failure probability estimate is 0.0017 with a
standard deviation of 0.0002 which is about 10% of the estimate. The number of samples used in
the above studies is 500. The PSF is expected to converge to the actual value when the number
of samples is increased.
The tail modeling approach to estimate the failure probability is demonstrated here with the help of a tuned mass-damper example. The tuned mass-damper problem in Figure 4-3 is taken from Chen et al (1999). The problem involves the failure probability estimation of a damped single degree of freedom system with dynamic vibration absorber. Figure 4-3 illustrates the tuned damper system consisting of the single degree of freedom system and a dynamic vibration absorber to reduce the vibrations. The original system is externally excited by a harmonic force. The absorber serves to reduce the vibration. The amplitude of vibration depends on

\[ R = \frac{m}{M}, \text{ the mass ratio of the absorber to the original system} \]

\[ \zeta, \text{ the damping ratio of the original system} \]

\[ r_1 = \frac{\omega_1}{\omega}, \text{ ratio of the natural frequency of the original system to the excitation frequency} \]

\[ r_2 = \frac{\omega_2}{\omega}, \text{ ratio of the natural frequency of the absorber to the excitation frequency} \]

The amplitude of the original system \( \hat{y} \) is normalized by the amplitude of its quasi static response and is a function of four variables expressed as:

\[
y = \sqrt{1 - R \left( \frac{1}{r_1} \right)^2 - \left( \frac{1}{r_2} \right)^2 + \frac{1}{r_1^2 r_2^2}} + 4 \zeta^2 \left[ \left( \frac{1}{r_1} \right) - \frac{1}{r_1 r_2^2} \right]^2 \]

This example treats \( r_1 \) and \( r_2 \) as random variables. They follow a normal distribution \( N(1,0.025) \) and \( R = 0.01, \zeta = 0.01 \). The normalized amplitude of the original system is plotted in
Figure 4-4. There are two peaks where the normalized amplitude reached undesirable vibration levels. The corresponding contour plot is presented in Figure 4-5. The failure region is denoted by the red band. This failure region is an island. That is, the failure region has safe regions on either side of it. This introduces additional challenges of not being able to use analytical approaches like FORM because the failure region is not a half plane. The objective of the problem is to reduce the risk of the normalized amplitude being larger than a certain value. The limit state for this case is expressed as:

\[ g(r_1, r_2) = y(r_1, r_2) - y_0 \]  

(4-16)

where \( y_0 \) can be considered as the allowable level of vibration. When the limit state in Eq. (4-16) is greater than 0, failure is said to occur. Increasing or decreasing \( y_0 \) will help in decreasing or increasing the failure probability respectively. \( y_0 = 27 \) is considered for the discussion. The corresponding failure probability with 1e7 sample MCS is estimated to be 0.01031. The tail modeling approach with 500 samples and 100 repetitions are used to study the convergence and accuracy estimates of PSF. The plot for PSF at various thresholds is presented in Figure 4-6. From the plots, it seems that regression behaves better compared to the ML method. There is a discrepancy in the plot corresponding to the ML method. The PSF estimated at 0.96 threshold is more accurate than the PSF estimate at a threshold of 0.98. In order to understand the behavior of the tail of the tuned mass damper, a log plot of the CDF with 1e5 samples is examined. The plot is presented in Figure 4-7. It is evident from the plot that there are two curvatures in the tail which are difficult to model. The GPD has to capture this shape with less exceedance data. This is the reason for the discrepancy in the ML plot, the shape of the tail at the area of interest (\( S_r = 1 \)) modeled by ML with a threshold of 0.96 was better than the tail model with a threshold of 0.98.
In order to further explore the tail models from each method, the tail model from each method is superimposed on each other and the comparative plots are presented in Figure 4-7. It is clearly observed that the tail modeled based on ML approach denoted by the dashed line attempts to capture the second curvature and in the process introduces error in the PSF value corresponding to a failure probability level of 0.01. On the other hand, the model obtained based on regression parameters represented by the solid line, approximates the tail in a linear fashion and is accurate compared to the tail based on ML method for a failure probability of 0.01. However, the ML method is expected to perform better when the failure probability to be estimated is low. When the failure probability is low, it becomes necessary to model the second curvature adequately to estimate the PSF with reasonable accuracy. The ML method can perform better than the regression approach in modeling the second curvature. Such types of CDF with double curvature might not be encountered in structural applications often and solving this problem with less number of samples is very challenging. With more insight on the problem, it is possible to select the suitable parameter estimation method depending on the requirements. Here, if one needs to estimate PSF corresponding to failure probabilities lesser than 1e-3, then the second curvature has to be modeled well and hence ML method is a good choice. Else, for higher failure probabilities, regression can perform better.

**Alternate Tail Extrapolation Schemes**

In this section, simple tail extrapolation techniques are developed to estimate the PSF for low target failure probability using MCS that is sufficient only to estimate the PSF for substantially higher failure probability (lower target reliability index). Failure probability can be transformed to reliability index by using Eq. (2-3). The same transformation is applied here to the CDF of the reciprocal of inverse measure. Since we are modeling the upper tail, reciprocal of the inverse measure is of interest here. The tail portion of the resulting transformed CDF
(relationship between inverse measure and reliability index) is approximated by a linear polynomial in order to take advantage of fact that normally distributed inverse measure is linearly related to the reliability index. Since this approximation will not suffice if the inverse measure follows distributions other than normal, a logarithmic transformation is then applied to the reliability index which tends to linearize the tail of the transformed CDF. This tail is approximated using a quadratic polynomial. Since both these techniques use only data from the tail, another technique that approximates the relationship between inverse measure and reliability index from the mean to the maximum data using a quadratic polynomial is developed. The three extrapolation techniques discussed are described with the help of a figure. A data set of size 500 with a mean of 10 and variance 9 following a lognormal distribution is used to illustrate the three techniques.

Figure 4-8, A) shows the general relationship between PSF reciprocal and 1-$P_f$. Applying the transformation modifies the CDF as in B). If the PSF reciprocal follows a normal distribution, this curve will be linear. Therefore, the first technique approximates the tail of the curve in B) by a linear polynomial. In order to take advantage of the remaining data, half of the curve in B) is approximated by a quadratic polynomial. It is sufficient to consider only half of the data because of the symmetry in reliability index. The third technique applies a logarithmic transformation to the reliability index and approximates the tail of the curve shown in C) by a quadratic polynomial. In all the above three techniques, once the coefficients of the polynomial are obtained, the PSF reciprocal corresponding to any higher reliability index can be found. It is to be noted that with 500 samples, it is possible to estimate only PSF reciprocal at reliability levels less than 3. Classical tail models discussed in the previous chapter and the alternate
extrapolation techniques discussed in this chapter allow predicting PSF reciprocal at much higher reliability indices without additional reliability analysis.

The alternative extrapolation techniques and classical tail modeling techniques are conceptually the same. The major difference in perceiving the two classes is that the classical tail modeling techniques model the CDF of PSF reciprocal, whereas the extrapolation schemes approximates the trend of PSF reciprocal in terms of reliability index.
Figure 4-1. Tail modeling of $F(u)$ using the threshold $u$. The region of $y>0$ is failure.
Figure 4-2  Convergence of PSF at different thresholds. A) MLE B) Regression. Cantilever beam system failure mode. 500 Samples. 100 repetitions.
Figure 4-3. Tuned vibration absorber

Figure 4-4. Normalized amplitude vs $r_1$ and $r_2$
Figure 4-5. Contour of the normalized amplitude
Figure 4-6. Convergence of PSF at different thresholds. Tuned Mass Damper. A) MLE. B) Regression. 500 samples. 100 repetitions. LCI – Lower Confidence Interval. UCI – Upper Confidence Interval
Figure 4-7 Tuned Mass Damper. Comparison of tail models using regression and ML method
Figure 4-8. Transformation of the CDF of PSF reciprocal ($S_r$). A) CDF of $S_r$. B) Inverse Standard normal cumulative distribution function applied to the CDF. C) Logarithmic transformation applied to the reliability index.
CHAPTER 5
MULTIPLE TAIL MODELS

Multiple Tail Model
A total of five methods to model the tails of data were discussed in the previous two sections. However, in practical applications where there is no information on the form of the output distribution, there is no way to explore which technique performs better. A single technique that performs well for a particular example might not perform the best for another example. In fact, a single method that performs the best at a particular reliability index might not perform the best at a different reliability index for the same example, as will be shown in the numerical examples. In addition, there are no known straightforward ways to quantify the error in the estimates.

In order to take advantage of all the techniques and still come up with the best prediction, we propose to apply the five techniques simultaneously and use the median of the five estimates. It is observed that the median is a robust estimate than the mean. This method is demonstrated first on seven different statistical distributions followed by two engineering examples. In all examples, it is observed that the median of the five estimates is at least the second best compared to estimates from individual techniques. Moreover, the range of the five estimates closely approximates the error in the median. Thus using multiple tail models not only ensures to choose a good estimate and buys insurance against bad predictions but also provides an error of the estimate. Figures 5.1 to 5.5 show the various tail modeling techniques for a lognormal distribution with a data mean of 10 and standard deviation of 3. In each case, the 1e5 sample curve and 500 sample curve are superimposed to highlight the extrapolation. The $S_r$ is computed as discussed in the implementation in next section. Figure 5.4 shows the multiple tail model approach (MTM). In order to understand to clearly see the range of the five methods, the region beyond the threshold is zoomed and presented in Figure 5.5. The range of the five methods at
three different reliability indices is marked by double headed arrows. Mean of these ranges computed over the entire extrapolation zone are capable of representing the error associated with the median estimate from MTM. A representative actual error is presented in the plot. The QH is the median and overlaps with the MTM curve.

**Numerical Examples**

**Standard Statistical Distributions**

The standard statistical distributions that are discussed in this section are presented in Table 5-1. A data set with mean 10 and standard deviation 3 is considered. In order to compare the performance of the proposed method on different distributions, a common target failure probability is used to find the capacity which is in turn is used to normalize the parameters.

Implementation of this exercise is as follows:

\[
P_f \text{target} = 0.00135 (\beta_{\text{target}} = 3), \bar{x} = 10, COV = 0.3, \sigma = COV \times \bar{x}
\]

1. Use \(\bar{x}\) and \(COV\) to estimate parameters \((a_0\) and \(b_0)\) for each distribution from Table 5-1.
   These parameters can be used to generate random numbers.
2. Find capacity \(C\) as \((1 - P_{\text{target}})^a\) quantile. Inverse cumulative distribution functions (icdf) can be used for this. In the case of single parameter distributions, add shift factor to the \(C\).
3. Find normalized mean using \(\bar{x} = \frac{\bar{x}}{C}\). We are interested in the safety factor reciprocal, \(S_r\), following different statistical distributions. The normalized mean is the mean of the \(S_r\).
4. Use \(\hat{x}\) and \(COV\) to find new parameters \((a\) and \(b)\) for all the distributions.
5. Generate \(N = 500\) Latin hypercube samples in \([0,1]\)
6. Find \(S_r\) at each sample using \((a, b)\) and inverse icdf
7. Use Eq (4-14) to estimate plotting positions
8. Plotting \(S_r\) and \(P\) provides the empirical CDF of \(S_r\). It is to be noted that the ordinate is \((1-P_f)\). That is, we are interested in the upper tail. As \(P_f\) decreases, \(1-P_f\) increases, reliability index increases, \(S\) decreases and \(S_r\) increase.
9. A threshold of 0.9 is considered for classical tail modeling techniques and the parameters of GPD that approximates the conditional CDF beyond the threshold are estimated using Maximum Likelihood and least square regression approaches.
10. The transformations discussed in the previous section can be applied to the CDF of \(S_r\) and \(S_r\) at higher reliability indices can be estimated directly from the approximated relationships.
11. There are totally five estimates (steps 9 and 10) of $r_s$ denoted by $\hat{r}_{\text{ind}}$ in the extrapolated regions. Record the median and range of $\hat{r}_{\text{ind}}$ as $5m$ and $5Ra$ respectively.
12. Exact values of $r_s$ can be obtained using icdf functions.
13. Compute error in the estimate of individual methods $e_{\text{ind}}$ as $\text{abs}(r_s - \hat{r}_{\text{ind}})$.
14. Compute error in multiple tail model (MTM) estimate $e_{\text{MTM}}$ as $\text{abs}(r_s - 5m)$
15. Compute the ratio of mean of $e_{\text{MTM}}$ to mean of $5Ra$ over the entire extrapolation zone and denote it as $\eta$
16. Repeat steps 5 to 15, 1000 times for each distribution.
17. Record the median, mean of $e_{\text{ind}}$ and $e_{\text{MTM}}$ over 1000 repetition as $1000m_{\text{ind}}$, $1000\mu_{\text{ind}}$ and $1000m_{\text{MTM}}$, $1000\mu_{\text{MTM}}$
18. Record the median and mean of the ratio in step 15 over 1000 repetition as $1000\eta$

For each distribution, boxplots of $e_{\text{ind}}$ and $e_{\text{MTM}}$ are presented. $1000m_{\text{ind}}$ and $1000m_{\text{MTM}}$ compared at three reliability indices between 3 and 4.2. The results for the normal distribution are presented in Figure 5-6. The objective of this plot is to see how well the $5m$ estimate compares to the $\hat{r}_{\text{ind}}$. For 1000 repetitions, only the median and the mean can be compared because the range is volatile. It was observed that the median was a robust estimate compared to the mean. It is evident that the median is less sensitive to the outliers compared to the mean. At $\beta=3$, the regression (Rg) performs the best followed by the linear tail (LT). The MTM performs as good as the second best. At $\beta=3.6$, the LT performs the best followed by the regression (Rg) and MTM performs as well as LT. At $\beta=4.2$, the LT performs the best followed by the quadratic tail (QT) and MTM’s performance is as good as LT. It is to be noted that the individual techniques that provided the best estimate differed at different reliability indices. However, the MTM estimate was consistent and it performed at least as good as the second best estimate from the individual techniques. This approach serves as an insurance against bad predictions, if we were to use a single technique.
Box plots of $\eta$ ratio are presented to see how well the range of the methods compares to the mean of the $e_{\text{MTM}}$ over the entire extrapolation region. The results for the normal distribution are presented in Figure 5-7. It is clearly observed that the ratio of the mean of the error to the mean of the range in the entire extrapolation region is around 1. If the ratio is 1, then the range captures the error in the median perfectly. Here, the range provides an estimate of $e_{\text{MTM}}$ that might slight over or under estimate the actual error. The boxplots for the remaining distributions are presented in the Appendix G. Table 5-2 provides a summary of the performance of the individual methods and the MTM for all the distributions. Table 5-2 shows that not a particular technique was the best for all the distributions and the technique that provided the best estimate for a particular distribution varied as the reliability indices changed. However, the MTM estimate was mostly close to the 2nd best available estimate. The summary of $\eta$ ratio is presented in Table 5-3. It is observed that other than uniform distribution, the ratio varies between 0.15 to Table 5-1. 7 standard statistical distributions and their parameters

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>$a = \bar{x}$, $b = \sigma$</td>
</tr>
<tr>
<td>LogNormal</td>
<td>$\ln(\bar{x}) - 0.5(b^2)$, $\sqrt{\ln\left[1 + \left(\frac{\sigma}{\bar{x}}\right)^2\right]}$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\left(\frac{\bar{x}}{\sigma}\right)^2$, $\frac{\sigma}{\sqrt{a}}$</td>
</tr>
<tr>
<td>Extreme Type 1</td>
<td>$\bar{x} = \frac{0.577}{b}$, $\frac{\pi}{\sigma\sqrt{6}}$</td>
</tr>
<tr>
<td>Uniform</td>
<td>$\bar{x} = \frac{\sqrt{12} - \sigma}{2}$, $\bar{x} + \frac{\sqrt{12} - \sigma}{2}$</td>
</tr>
<tr>
<td>Single parameter</td>
<td></td>
</tr>
<tr>
<td>Exponential</td>
<td>$\bar{x} - \sigma$, $\bar{x}$</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>$\bar{x} = \sqrt{\frac{\pi}{2}}(b)$, $\frac{\sigma}{\sqrt{2 - \frac{\pi}{2}}}$</td>
</tr>
</tbody>
</table>
0.35. This means that half of the range over estimates the error by a factor of 1.5 to 3. In order to provide an insight what number should one expect for the $\eta$ ratio, a simple exercise is performed. Standard normal random numbers of size $5 \times 7$ (represents 5 estimates at 7 different reliability indices) are generated and $\eta$ is computed over $1 \times 5$ repetition. The resultant number is 0.18.

Table 5-2. Summary of the performance of individual methods and MTM for different distributions

<table>
<thead>
<tr>
<th>Rel Index</th>
<th>3</th>
<th>3.6</th>
<th>4.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Performance</td>
<td>1st best</td>
<td>2nd best</td>
<td>MTM</td>
</tr>
<tr>
<td>Distributions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normal</td>
<td>Rg</td>
<td>LT</td>
<td>2nd</td>
</tr>
<tr>
<td>Lognormal</td>
<td>Rg</td>
<td>LT</td>
<td>1st</td>
</tr>
<tr>
<td>Gamma</td>
<td>Rg</td>
<td>LT</td>
<td>2nd</td>
</tr>
<tr>
<td>Ev</td>
<td>QH</td>
<td>Rg</td>
<td>1st</td>
</tr>
<tr>
<td>Uniform</td>
<td>QT</td>
<td>Rg</td>
<td>1st</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>Rg</td>
<td>LT</td>
<td>2nd</td>
</tr>
<tr>
<td>Exponential</td>
<td>LT</td>
<td>Rg</td>
<td>2nd</td>
</tr>
</tbody>
</table>

Table 5-3. Summary of the Error to range ratio for different distributions

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$1000 \eta_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>0.279</td>
</tr>
<tr>
<td>Lognormal</td>
<td>0.349</td>
</tr>
<tr>
<td>Gamma</td>
<td>0.181</td>
</tr>
<tr>
<td>Ev</td>
<td>0.183</td>
</tr>
<tr>
<td>Uniform</td>
<td>0.003</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>0.148</td>
</tr>
<tr>
<td>Exponential</td>
<td>0.241</td>
</tr>
</tbody>
</table>

Since the concept of regression is used to estimate the parameters in all methods other than the ML approach, the residuals are monitored for each method to see whether they follow the normal distribution as per classical statistical theory. It is expected that the type of distribution of the residuals can provide a good idea about the quality of the fit. A similar study was made for the normal distribution case and the normal QQ plots for the residuals are presented in Figure 5-8. It is observed that the residuals of none of the methods follow a normal distribution. Investigating the corresponding histograms reveal that they follow a log normal
distribution. In these situations, other regression schemes like weighted approaches can be adopted depending on the knowledge of the histograms and QQ plots of the residual which is scope for future work.

Engineering Examples

Application of multiple tail models for reliability estimation of a cantilever beam

The cantilevered beam example treated in Chapter 3 is considered here. The objective of the original problem is to minimize the weight or, equivalently, the cross sectional area, \( A = w \cdot t \) subject to two reliability constraints, which require the reliability indices for strength and deflection constraints to be larger than three. Eqs (3-17) and (3-18) are rewritten as

\[
\tilde{G}_s = 1 - \frac{S}{R} = 1 - \frac{\left( \frac{600}{wt^2} Y + \frac{600}{w^2 t} X \right)}{R} 
\]

(5-1)

\[
\tilde{G}_d = 1 - \frac{D}{D_o} = 1 - \frac{4L^3}{Ewt} \sqrt{\left( \frac{Y}{t^2} \right)^2 + \left( \frac{X}{w^2} \right)^2} \frac{D_o}{D_o}
\]

(5-2)

It is to be noted that the performance functions are expressed in a fashion such that failure occurs when \( \tilde{G}_s \) or \( \tilde{G}_d \) is greater than 1. Here, we consider a system failure case. That is, both the failure modes are considered simultaneously. The optimal design variables \( w = 2.6041, \ t = 3.6746 \) is taken from Qu and Haftka (2003) for a system reliability case. The corresponding reliability index is 3. The allowable deflection \( D_o \) is taken to be 2.145. For each sample the critical safety ratio reciprocal (maximum of the two) is computed. Since we are dealing with \( S_r \), it is to be noted that \( n = 1 - P_{target} \). The conditional CDF of \( S_r \) can be approximated by classical techniques and the relationship between \( S_r \) and reliability index be approximated by three different alternate techniques. Finally, using the MTM approach will give a better prediction compared to any individual method. This procedure is repeated 1000 times and the results are presented in Figure
It is observed that the LT performs the best at all three reliability indices followed by the QH as the second best. MTM consistently performed close to the second best estimate.

Figure 5-10 presents the plot of $\eta$ ratio. The ratio is 0.23 and thus strengthens our conclusion that the half of the range approximates the error in the median by a factor between 1.5 and 3, which is close to 2 here. The main advantages of the techniques discussed are that no statistical information is required and huge reduction in computational power. In order to understand the sacrifice in accuracy for the reduction in computational power a MCS study is performed.

100 repetitions of $S_r$ estimates with $5E5$ samples and the corresponding standard deviation are computed and presented in Table 5-4. At a reliability index of 4.2, the standard deviation in $S_r$ estimate is 0.04 which is the same error from MTM using 500 samples. Therefore, for a same level of accuracy, the reduction in computational power is about 3 orders of magnitude ($5E5$ to $5E2$).

<table>
<thead>
<tr>
<th>Rel Index</th>
<th>3</th>
<th>3.2</th>
<th>3.4</th>
<th>3.6</th>
<th>3.8</th>
<th>4</th>
<th>4.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_r$</td>
<td>1.012</td>
<td>1.032</td>
<td>1.05</td>
<td>1.07</td>
<td>1.09</td>
<td>1.12</td>
<td>1.13</td>
</tr>
<tr>
<td>SD</td>
<td>0.003</td>
<td>0.004</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.02</td>
<td>0.04</td>
</tr>
</tbody>
</table>

**Application of multiple tail models for reliability estimation of a composite panel**

The design of the wall of a hydrogen tank which is a composite laminate operating in cryogenic temperatures addressed by Qu et al., (2003) is considered as a second engineering example. In order to reduce matrix cracking and hence hydrogen leakage in cryogenic environments, the laminates need to have smaller angle between the plies. However, these types of laminates do not carry the biaxial hoop and axial stresses due to pressure. Qu et al., (2003) investigated options for minimizing the increase in thickness required to carry both mechanical
and thermal residual strains. The geometry and loading conditions of the problem is presented in Figure 5-11. The laminate is subject to mechanical (\(N_x\) and \(N_y\)) load and thermal loading due to operating temperature -423°F where the stress free temperature is 300°F. The objective of the actual problem was to optimize the weight of laminate with two ply angles \([\pm \theta_1, \pm \theta_2]\). The ply angles and ply thickness \((t_1\) and \(t_2\)) are the design variables. The material used is IM600/133 graphite epoxy of ply thickness 0.005 inch. Qu et al., (2003) performed the deterministic design optimization using continuous design variables. In order to account for the uncertainties and make the deterministic optimal design comparable to probabilistic optimal design, they used a safety factor of 1.4. For further details, the reader is referred to Qu et al., (2003). The mechanical properties are presented in Table 5-5.

The deterministic optimization problem is formulated as

\[
\text{Minimize} \quad h = 4(t_1 + t_2) \\
\text{Such that} \quad t_1, t_2 \geq 0.005 \\
\varepsilon^U_i \leq S_F \varepsilon_i \leq \varepsilon^U_i, \quad \varepsilon^L_i \leq \varepsilon^L_i \leq \varepsilon^U_i, \quad S_F |\gamma_{12}| \leq \gamma_{12}^U \tag{5-3}
\]

All material properties are assumed to be random, uncorrelated and follow normal distributions. The coefficient of variation is presented in Table 5-6. Here, thickness is in inches and \(h\) is the laminate thickness. Superscripts \(U\) and \(L\) represent upper and lower limits of the associated quantities. \(\varepsilon_i\), \(\varepsilon^L_i\), and \(\gamma_{12}\) are the ply strains along fiber direction, transverse to fiber direction, and shear strain, respectively. \(E_2, G_{12}, \alpha_1\) and \(\alpha_2\) are functions of temperature. Qu et al (2001) show that at 20.2K (-423 °F) the thickness of the optimum laminates obtained by using temperature dependent material properties is 80% less than that using constant material properties at 298 K (77° F). Qu et al (2003) observed that \(\varepsilon^U_2\) is the active constraint. For a feasible design, 21 different temperatures uniformly distributed between 20°K and 300°K is
considered and the strain constraints are applied to these temperatures. The mean is computed at a particular temperature and random numbers generated based on coefficient of variation. The mean for other parameters are presented in Table 5-7. The deterministic optimal design obtained by Qu et al (2003) is presented in Table 5-8.

The transverse strain on the first ply (direction 2) is the critical strain. The limit state is defined as the difference between the critical strain and the upper bound of the allowable strain.

\[ G = \varepsilon_2 - \varepsilon_1^U \]  

(5-4)

Failure is said to occur if Eq. (5-4) > 0. In order to extend this example for demonstrating the multiple tail models approach, the second optimal design in Table 5-8 is considered. The loads are increased by a factor of 1.1 (Nx -5280 lb/inch and Ny -2640 lb/inch) so that the failure probability is of the order of 1e-3.

500 random critical strain values are generated. Similar to the previous examples, 5 different techniques are used to fit the tail of the response distribution. The results are presented in Figures 5-12 and 5-13. As observed in other examples the MTM compared at least close to the second best in estimates from individual methods. However, the ratio of mean error to mean range is 0.22 which means that half of the range overestimates the error by a factor of around 2 which falls within the range of 1.5 and 3 that was observed earlier in section. The 1e7 sample based PSF reciprocal estimation and the corresponding standard deviation is presented in Table 5-9. At reliability index of 4.2, it is seen that the standard deviation is 0.0063 and the multiple tail model provides an estimate with an error of 0.05 which is off by a factor of 8. The number of samples used for this estimation is 500, which is about 4 orders of magnitude less than 1e7. Due to the prohibitive computational power required, an exercise similar to the previous example is
not performed here. Instead the estimate from 500 samples and 1e7 samples are compared directly.

Table 5-5. Mechanical properties of the composite laminates

<table>
<thead>
<tr>
<th>Elastic properties</th>
<th>$E_1, E_2, G_{12}$ and $v_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficients of thermal expansion</td>
<td>$\alpha_1$ and $\alpha_2$</td>
</tr>
<tr>
<td>Stress-free temperature</td>
<td>$T_{\text{ref}}$</td>
</tr>
<tr>
<td>Failure strains</td>
<td>$\varepsilon_1^L, \varepsilon_1^V, \varepsilon_2^L, \varepsilon_2^V$ and $\gamma_{12}^V$</td>
</tr>
<tr>
<td>Safety factor</td>
<td>$S_p$</td>
</tr>
</tbody>
</table>

Table 5-6: Coefficient of variation for random material properties

<table>
<thead>
<tr>
<th>$E_1, E_2, G_{12}, v_{12}$</th>
<th>$\alpha_1, \alpha_2$</th>
<th>$T_{\text{ref}}$</th>
<th>$\varepsilon_1^L$</th>
<th>$\varepsilon_1^V$</th>
<th>$\varepsilon_2^L$</th>
<th>$\varepsilon_2^V$</th>
<th>$\gamma_{12}^V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.035</td>
<td>0.035</td>
<td>0.03</td>
<td>0.06</td>
<td>0.09</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5-7: Mean of random variables

<table>
<thead>
<tr>
<th>$E_1$</th>
<th>$v_{12}$</th>
<th>$T_{\text{ref}}$</th>
<th>$\varepsilon_1^L$</th>
<th>$\varepsilon_1^V$</th>
<th>$\varepsilon_2^L$</th>
<th>$\varepsilon_2^V$</th>
<th>$\gamma_{12}^V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>21.5x10^6</td>
<td>0.359</td>
<td>300</td>
<td>-0.0109</td>
<td>0.0103</td>
<td>-0.013</td>
<td>0.0154</td>
<td>0.0138</td>
</tr>
</tbody>
</table>

Table 5-8: Deterministic optima found by Qu et al (2003)

<table>
<thead>
<tr>
<th>$\theta_1$ (deg)</th>
<th>$\theta_2$ (deg)</th>
<th>$t_1$ (in)</th>
<th>$t_2$ (in)</th>
<th>$h$ (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>28.16</td>
<td>0.005</td>
<td>0.02</td>
<td>0.1</td>
</tr>
<tr>
<td>27.04</td>
<td>27.04</td>
<td>0.01</td>
<td>0.015</td>
<td>0.1</td>
</tr>
<tr>
<td>25.16</td>
<td>27.31</td>
<td>0.005</td>
<td>0.020</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 5-9: Composite laminate. $S_r$, reciprocal estimates and standard deviation at different reliability indices

<table>
<thead>
<tr>
<th>Rel Index</th>
<th>3</th>
<th>3.2</th>
<th>3.4</th>
<th>3.6</th>
<th>3.8</th>
<th>4</th>
<th>4.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1E+07</td>
<td>1.0043</td>
<td>1.0304</td>
<td>1.0579</td>
<td>1.0862</td>
<td>1.1157</td>
<td>1.1480</td>
<td>1.1824</td>
</tr>
<tr>
<td>SD</td>
<td>0.0007</td>
<td>0.0009</td>
<td>0.0015</td>
<td>0.0021</td>
<td>0.0025</td>
<td>0.0046</td>
<td>0.0063</td>
</tr>
</tbody>
</table>
Figure 5-1. Lognormal distribution. Classical tail modeling techniques – MLE and Regression
Figure 5-2. Lognormal distribution. Linear fit to tail (LT) and Quadratic fit to half of the data (QH)
Figure 5-3. Lognormal distribution. Quadratic fit to the tail (QT)
Figure 5-4. Lognormal distribution. Multiple tail models
Figure 5-5 Lognormal distribution. Multiple tail models-extrapolation region
Med= Median. ML= Maximum Likelihood, Rg- Regression, QT – Quadratic fit to the Tail data between $S_r$ and ln($\beta$), LT – Linear fit to the Tail data between $S_r$ and $\beta$, QH – Quadratic fit to Half of the data between $S_r$ and $\beta$. Dashed line inside the box depicts the mean and the solid line, the median.

Figure 5-6. Normal distribution. 1000 repetitions. Boxplot comparison of $e_{\text{ind}}$ and $e_{\text{MTM}}$ at different reliability indices
Figure 5-7: Normal distribution. Boxplot of $\eta$ ratio.
Figure 5-8. Cantilever Beam. 1000 repetitions. Boxplot comparison of $e_{\text{ind}}$ and $e_{\text{MTM}}$ at different reliability indices.

Med = Median. ML = maximum likelihood, Rg = Regression, QT = Quadratic fit to the Tail data between $S_r$ and $\ln(\beta)$, LT = Linear fit to the Tail data between $S_r$ and $\beta$, QH = Quadratic fit to Half of the data between $S_r$ and $\beta$. Dashed line inside the box depicts the mean and the solid line, the median.
Figure 5-9. Normal QQ plot of the residuals for the normal distribution
Figure 5-10. Cantilever beam. Boxplot of $\eta$ ratio.

Figure 5-11. Geometry and loading for the composite laminate. X-hoop direction, Y-axial direction
ML = Maximum Likelihood, Rg = Regression, QT = Quadratic fit to the Tail data between $r_S$ and $\ln(\beta)$, LT = Linear fit to the Tail data between $r_S$ and $\beta$, QH = Quadratic fit to half of the data between $r_S$ and $\beta$. Dashed line inside the box depicts the mean and the solid line, the median.

Figure 5-12. Composite laminate of cryogenic tank. 1000 repetitions. Boxplot comparison of $e_{\text{ind}}$ and $e_{\text{MTM}}$ at different reliability indices...
Figure 5-13. Composite laminate of cryogenic tank. Boxplot of $\eta$ ratio.
High reliability estimations with expensive computer models pose a challenge because of the high computational expense. Extrapolating into higher reliability indices with information from lower reliability indices can be one solution to this problem. In order to do this, the tails of the distribution are to be modeled well. Inverse reliability measures feature several advantages and is widely used by researchers in structural reliability. This work explores the usefulness of inverse measures in tail modeling. Two classical tail modeling techniques were discussed. Three extrapolation techniques that can complement the classical tail modeling techniques were developed. This work proposes to use a multiple tail model approach in which all the five techniques are applied simultaneously to estimate the inverse measure, and the median is taken to be the best estimate. It is shown that the range of the five methods can be utilized to approximate the error associated with the estimate. The proposed method was demonstrated on seven statistical distributions and engineering examples. It was shown that the sacrifice in accuracy compared to the crude MCS is relatively small compared to the immense reduction in computational power.
APPENDIX A
PSF ESTIMATION FOR A SYSTEM RELIABILITY

The estimation of PSF for a system reliability case is demonstrated with an example. This example is artificial and is an illustration of a case with multiple limit state functions. The example is presented in Figure A1. The limit state functions are:

\[ G_1 = y - \left( \frac{x^3 - x^2 + 15}{40} \right) \] (A1)

\[ G_2 = \left( \frac{(x + y - 5)^2}{30} + \frac{(x - y - 12)^2}{140} - 0.8 \right) \] (A2)

\[ G_3 = y - \left( \frac{80}{x} \right) - 20 \] (A3)

The limit states are functions of two random variables \( x \) and \( y \). The distributions of the random variables are presented in Table A-1. The example has been created in such a fashion that each limit state and its intersection with other limit states contributes significantly to the total failure. It should be noted that non-linear functions are used to emphasize the advantage of using MCS and a PSF-based approach to estimate failure probability. Using FORM in these circumstances might introduce huge error due to linearizing the limit state function at the most probable point of failure. The contributions of failure modes to the total failure are listed in Table A-2. The system PSF can be estimated as described in the flowchart in Figure A-1. The limit states are functions of two random variables \( x \) and \( y \). The PSF estimated for different target failure probabilities are listed in Table A-3. For a target failure probability of 0.557 which is the actual failure probability as listed in Table A-3, the PSF is estimated as 1. When a lower target failure probability of 0.5 is used, the PSF is estimated to be 0.844 indicating that the system is unsafe and when a larger target failure probability of 0.6 is used, the PSF is found to be 1.12 showing that the system is over safe. This shows that PSF based approach can adequately address system reliability case accounting for the interactions between the modes.
Table A-1. Distribution of random variables. Multiple failure modes example

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distribution</td>
<td>Normal (4,0.5)</td>
<td>Normal (2,3)</td>
</tr>
</tbody>
</table>

Table A-2. Contribution of failure modes to the system failure

<table>
<thead>
<tr>
<th>Limit states</th>
<th>Failure probability</th>
<th>% Contribution to total failure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.432</td>
<td>77.55</td>
</tr>
<tr>
<td>2</td>
<td>0.468</td>
<td>84.02</td>
</tr>
<tr>
<td>3</td>
<td>0.336</td>
<td>60.32</td>
</tr>
<tr>
<td>1 &amp; 2</td>
<td>0.386</td>
<td>69.29</td>
</tr>
<tr>
<td>1 &amp; 3</td>
<td>0.253</td>
<td>45.42</td>
</tr>
<tr>
<td>2 &amp; 3</td>
<td>0.249</td>
<td>44.71</td>
</tr>
<tr>
<td>1,2 &amp; 3</td>
<td>0.209</td>
<td>37.52</td>
</tr>
<tr>
<td>Total</td>
<td>0.557</td>
<td>100</td>
</tr>
</tbody>
</table>

Table A-3. PSF at different target failure probabilities

<table>
<thead>
<tr>
<th>Target Pf</th>
<th>PSF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.557</td>
<td>0.9996</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8444</td>
</tr>
<tr>
<td>0.6</td>
<td>1.1191</td>
</tr>
</tbody>
</table>

Figure A-1. Two variable system. Algebraic multiple limit state functions example
APPENDIX B
GENERALIZED EXTREME VALUE THEORY

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with a common distribution function $F$, and let

$$M_n = \max \{X_1, \ldots, X_n\}$$

$$\Pr\{M_n \leq z\} = \Pr\{X_1 \leq z, \ldots, X_n \leq z\} = \Pr\{X_1 \leq z\} \times \cdots \times \Pr\{X_n \leq z\} = \left(F(z)\right)^n$$

Generally, the distribution function of $M_n$ can be derived exactly for all values of $n$. But here $F$ is not known, so it does not help. Here we look for approximate families of models for $F^n$ which can be estimated based on extreme data only. This is done as follows:

$$M_n^* = \frac{M_n - b_n}{a_n}$$

for sequences of constants $\{a_n > 0\}$ and $\{b_n\}$. Appropriate choices of constants stabilize the location and scale of $M_n^*$ as $n$ increases. Essentially we seek limit distributions for $M_n^*$ with appropriate choices of constants. The all possible range of limit distributions for $M_n^*$ is given by the following theorem, the extremal type theorem, a product of Fisher-Tippet (1928) work.

If there exists sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ such that

$$\Pr\{(M_n - b_n)/a_n \leq z\} \to H(z) \text{ as } n \to \infty$$

for a non degenerate distribution function $H$, then $H$ belong to one of the following families:

1. $$H(z) = \exp\left\{-\exp\left[-\left(\frac{z-b}{a}\right)\right]\right\}, \quad -\infty < z < \infty$$

2. $$H(z) = \begin{cases} 0 & z \leq b \\ \exp\left\{-\left(\frac{z-b}{a}\right)^{-\alpha}\right\}, & z > b \end{cases}$$


3. \[
H(z) = \begin{cases} 
\exp\left\{-\left[-\left(\frac{z-b}{a}\right)^\alpha\right]\right\}, & z < b, \\
1 & z \geq b,
\end{cases}
\] (B6)

Here, \(a\) and \(b\) are the scale and location parameters. \(\alpha\) is the shape parameter. These three classes of distributions are called the extreme value distributions with types I, II and III, widely known as Gumbel, Frechet and Weibull families respectively. The above theorem implies that the normalized maxima have a limiting distribution that must be one of the three types of extreme value distribution. The three families of distributions can be combined into a single family of models having distribution function of the form

\[
H(z) = \exp\left\{-\left[1 + \xi\left(\frac{z-\mu}{\sigma}\right)^{-1/\xi}\right]\right\}
\] (B7)

such that \(\{z : 1 + \xi(z-\mu)/\sigma > 0\}\), where the parameters satisfy \(-\infty < \mu < \infty\), \(\sigma > 0\) and \(-\infty < \xi < \infty\). This is the generalized extreme value (GEV) family of distributions. Here, \(\mu\) is the location parameter, \(\sigma\) is the scale parameter, \(\xi\) is the shape parameter. Type II and III families correspond to \(\xi > 0\) and \(\xi < 0\) cases in (9). \(\xi = 0\) can be interpreted as the Gumbel family with distribution function:

\[
H(z) = \exp\left[-\exp\left\{-\left(\frac{z-\mu}{\sigma}\right)\right\}\right]
\] (B8)
APPENDIX C
MAXIMUM LIKELIHOOD ESTIMATION

The PDF of the Pareto Distribution is given by:

\[
f(x) = \begin{cases} 
\frac{1}{\sigma} \left(1 + \frac{\xi z_i}{\sigma}\right)^{-\left(1 + \frac{1}{\xi}\right)} & \xi \neq 0 \\ 
\frac{1}{\sigma} e^{-\frac{z_i}{\sigma}} & \xi = 0
\end{cases}
\]  
(C1)

Here, the \( z \) represents the exceedances. \( \xi \) and \( \sigma \) are the shape and scale parameters respectively.

The likelihood function for a sample of \( z_1, \ldots, z_n \) of iid GP random variables is given by:

\[
L(z_i, \xi, \sigma) = \begin{cases} 
\frac{1}{\sigma^n} \prod_{i=1}^{n} \left(1 + \frac{\xi z_i}{\sigma}\right)^{-\left(1 + \frac{1}{\xi}\right)} & \xi \neq 0 \\ 
\frac{1}{\sigma^n} \prod_{i=1}^{n} e^{-\frac{1}{\sigma}} & \xi = 0
\end{cases}
\]  
(C2)

The log likelihood function is expressed as:

\[
\log L(\xi, \sigma) = \begin{cases} 
-n \log \sigma - \left(1 + \frac{1}{\xi}\right) \sum_{i=1}^{n} \log \left(1 + \frac{\xi z_i}{\sigma}\right) & 1 + \frac{\xi z_i}{\sigma} > 0 \\ 
-n \log \sigma - \frac{1}{\sigma} \sum_{i=1}^{n} z_i & \xi = 0
\end{cases}
\]  
(C3)

The maximization of (C3) is best made by reparametrization:

\((\sigma, \xi) \to (\tau, \tilde{\xi}) \text{ with } \tau = \frac{\xi}{\sigma}\)

yielding

\[
h = \log L(\tau, \tilde{\xi}) = -n \log \left(\frac{\xi}{\tau}\right) - \left(1 + \frac{1}{\tilde{\xi}}\right) \sum_{i=1}^{n} \log \left(1 + \tau z_i\right)
\]  
(C4)

\[
h = -n \log \tilde{\xi} + n \log \tau - \left(1 + \frac{1}{\tilde{\xi}}\right) \sum_{i=1}^{n} \log (1 + \tau z_i)
\]  
(C5)

The max of (C5) is found as the minimum of the negative of (C5). The MLE are given by differentiating (C5) and equating to zero.
\[
\frac{\partial h}{\partial \xi} = -n \frac{\hat{\xi}}{\xi^2} + \frac{1}{\hat{\xi}^2} \sum_{i=1}^{n} \log(1 + \hat{\xi}z_i) = 0 \quad \text{(C6)}
\]
\[
\frac{\partial h}{\partial \tau} = n \frac{1}{\hat{\xi} + 1} \sum_{i=1}^{n} \frac{1}{1 + \hat{\xi}z_i} = 0 \quad \text{(C7)}
\]

(C6) yields \( \hat{\xi} = \frac{1}{n} \sum_{i=1}^{n} \log(1 + \hat{\xi}z_i) \) and (C7) can be solved for \( \hat{\tau} \). Numerical techniques are used for the maximization of \( h \) in (C5) using algorithms that are available in standard software packages.

In most situations estimation is done using iid samples. In such cases, the interest is to determine the behavior of the estimator as a sample size \( \to \infty \). This is referred to as asymptotic behavior. Under certain regularity conditions like: the first and second derivatives of the log likelihood function must be defined, the fisher information matrix must not be equal to zero, MLEs exhibit following properties

- Asymptotically unbiased (i.e.,) its bias tends to zero as the number of samples increases to infinity.
- MLE is asymptotically guassian. That is, as sample increases the distribution of MLE tends to normal distribution. Here, the covariance matrix can be obtained as the inverse of the fisher information matrix.
APPENDIX D
ERROR ANALYSIS FOR GPD

This appendix describes the error analysis for the probability content estimated using tail modeling. When the distribution of exceedance is approximated using GPD, there are two sources of error:

1. Modelling error

2. Sampling component of the total error

The modelling error refers to the capacity of the GPD model to accurately represent the distribution of exceedance data or the tail of the CDF. If the exact CDF is available, the CDF estimated through GPD can be compared and the resulting error is the modeling error. Depending on the empirical CDF, the GP fit will vary. This introduces sampling error. The sampling error comes from the fact that the GPD parameters vary depending on the exceedance. The input variables generally come from sampling distributions like the LHS or any other suitable DOE. Depending on these sampling points, the threshold differs. Hence, the exceedance is affected and therefore the estimated parameters and eventually the fitted GPD.

If we have the privilege to generate large number of samples and use a fraction of it as exceedance data to fit a GPD, the estimated parameters in the process can be considered as ‘optimal’ parameters since they are based on a large sample size. These parameters can be used to estimate probability contents at certain levels and can be compared with the probability content estimated by a GPD with small sample size. This difference will dictate the sampling error contribution to the total error.

In an attempt to study the modeling and sampling error, an example problem is treated here. The example under investigation is \( Y = \exp(x) \) and \( x \sim \mathcal{N}(0,1) \). The modeling error is studied comparing the estimated probability content using a GPD with a large sample size and the exact
probability content generated by analytical expression of the lognormal distribution. The parameters for the GPD are obtained by MLE method.

Let the conditional CDF $P\{Y > u + z / Y > u\}$ be represented by $F_u(z)$. In this example, it is possible to estimate this quantity exactly. Using extreme value theory for exceedances, $F_u(z)$ can be approximated by a GP distribution denoted by $\hat{F}_{\hat{\xi},\hat{\sigma}}(z)$. Let the estimated CDF be denoted by $\hat{F}_{\hat{\xi},\hat{\sigma}}(z)^\diamond$. Let the GPD which is based on the optimal parameters be denoted by $\hat{F}_{\hat{\xi},\hat{\sigma}}(z)^\star$.

$1e6$ samples based on LHS design are generated. The $Y$'s are sorted as: $Y_{(1)} \leq Y_{(2)} \leq \ldots \leq Y_{(1e6)}$. This is called the order statistics of $Y$. Choosing the appropriate $u$, the threshold value is very critical. $u$ defines the beginning of the upper tail. The $F_u$ is chosen to be $0.90$ and the corresponding value of $Y$ is chosen as $u$. Once $u$ is chosen, the exceedance data is formed and the ML estimates are obtained for the parameters. These estimates are used to estimate the probability content at different levels of $Y$. The results are presented in Table D-1. Five repetitions of $1e6$ samples are used for each level. The mean and standard deviation of the CDF estimates based on GPD are presented. The error refers to the total error in percentage. The sampling error component of the total error is extracted using estimated optimal parameters. The optimal parameters were obtained based on a million samples. Here, $3$ percent of the data was considered as exceedance. The results are presented in Table D-2. From Tables D-1 and D-2, it can be clearly seen that the sampling error is high compared to the modeling error. So, if we can account for the sampling errors, tail modeling has a fairly good capacity to model the tails of the distribution to estimate probability contents.
TableD-1. Modelling error \(( F_g(z) - \hat{F}_{\xi,\sigma}(z)^* ) \). The standard deviation is very low that it is not presented.

<table>
<thead>
<tr>
<th>Y</th>
<th>Exact ( F_g(z) )</th>
<th>GPD ( \hat{F}_{\xi,\sigma}(z)^A )</th>
<th>Error%</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.08283</td>
<td>0.07988</td>
<td>3.56</td>
</tr>
<tr>
<td>5</td>
<td>0.05376</td>
<td>0.05299</td>
<td>1.43</td>
</tr>
<tr>
<td>6</td>
<td>0.03659</td>
<td>0.03647</td>
<td>0.33</td>
</tr>
<tr>
<td>7</td>
<td>0.02583</td>
<td>0.02587</td>
<td>-0.16</td>
</tr>
<tr>
<td>8</td>
<td>0.01879</td>
<td>0.01884</td>
<td>-0.28</td>
</tr>
<tr>
<td>9</td>
<td>0.01400</td>
<td>0.01403</td>
<td>-0.19</td>
</tr>
<tr>
<td>10</td>
<td>0.01065</td>
<td>0.01065</td>
<td>0.00</td>
</tr>
<tr>
<td>11</td>
<td>0.00824</td>
<td>0.00822</td>
<td>0.24</td>
</tr>
<tr>
<td>12</td>
<td>0.00648</td>
<td>0.00645</td>
<td>0.49</td>
</tr>
</tbody>
</table>

TableD-2: Sampling error \(( \hat{F}_{\xi,\sigma}(z)^A - \hat{F}_{\xi,\sigma}(z)^S ) \)

<table>
<thead>
<tr>
<th>Y</th>
<th>GPD- 1e6 samples ( \hat{F}_{\xi,\sigma}(z)^A )</th>
<th>GPD- 1e2 samples ( \hat{F}_{\xi,\sigma}(z)^S )</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.07988</td>
<td>0.08710</td>
<td>-9.04</td>
</tr>
<tr>
<td>5</td>
<td>0.05299</td>
<td>0.05797</td>
<td>-9.39</td>
</tr>
<tr>
<td>6</td>
<td>0.03647</td>
<td>0.03997</td>
<td>-9.61</td>
</tr>
<tr>
<td>7</td>
<td>0.02587</td>
<td>0.02824</td>
<td>-9.15</td>
</tr>
<tr>
<td>8</td>
<td>0.01884</td>
<td>0.02039</td>
<td>-8.20</td>
</tr>
<tr>
<td>9</td>
<td>0.01403</td>
<td>0.01503</td>
<td>-7.17</td>
</tr>
<tr>
<td>10</td>
<td>0.01065</td>
<td>0.01135</td>
<td>-6.53</td>
</tr>
<tr>
<td>11</td>
<td>0.00822</td>
<td>0.00877</td>
<td>-6.69</td>
</tr>
<tr>
<td>12</td>
<td>0.00645</td>
<td>0.00696</td>
<td>-7.95</td>
</tr>
</tbody>
</table>
APPENDIX E
ACCURACY AND CONVERGENCE STUDIES FOR NORMAL AND LOGNORMAL DISTRIBUTIONS

The quantile of a probability distribution can be estimated using Eq.(4-8). The quantile\(^2\) can be related to the inverse measure in the structural reliability context where the probability content is expressed as the target failure probability. Due to the various advantages of using inverse measures in RBDO, we are interested in estimating them using tail modeling approach. Here, we study the accuracy of the tail model to capture the quantile in the tails of the normal distribution. In the following simulation study a \( y=x \) example where \( x \sim N(0, 1) \) is treated. The error metrics that are monitored are:

a) Bias error:

\[
\text{Bias}(\hat{x}_p) = \mathbb{E}\left[\hat{x}_p - x_p\right]
\]

(bias)

\[
= \frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} \hat{x}_p - x_p
\]

(E2)

where,

\( \hat{x}_p \): Estimated quantile

\( x_p \): Actual quantile

\( N_{sim} \): Number of simulations

b) RMSE error:

\[
\sqrt{\left(\hat{x}_p - x_p\right)^2} = \sqrt{\frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} (\hat{x}_p - x_p)^2}
\]

(E3)

Here, we set out to study the effects of the threshold value, the number of exceedance data on the accuracy of the quantile estimate. For each threshold value, four different exceedances are tested. Since the exceedances vary, the number of samples used in each case varies. The results are presented in Table E-1. The overall capacity of the tail modeling approach to capture the quantiles in the tails of the normal distribution seems to be pretty good. For all the four threshold

\(^2\) A \( p \) quantile of a random variable \( X \) is any value \( x \) such that \( \Pr(X \leq x_p) = p \)
values, the % Bias error for the 25 exceedance data seem to be very less compared to higher
exceedance data. But, it is to be noted that the corresponding %RMSE errors are high
emphasizing the fact that the estimates from low number of exceedance data could be highly
unreliable. 25 exceedance data is considered low to estimate quantiles. 50 to 100 data is
considered realistic and 200 ideal. It can be clearly observed that the accuracy of the % Bias
increases as the threshold increases for the

Table E-1. Accuracy of the tail modeling approach to estimate quantiles. Normal distribution
\( y = x \), \( x \sim N(0,1) \). 500 simulations. ML estimated parameters

<table>
<thead>
<tr>
<th>Fu</th>
<th>Nex</th>
<th>NS</th>
<th>% Bias</th>
<th>% Bias</th>
<th>% Bias</th>
<th>% RMSE</th>
<th>% RMSE</th>
<th>% RMSE</th>
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<td>0.99999</td>
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<td>-4.5</td>
<td>2.8</td>
<td>5.5</td>
<td>8.5</td>
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</tbody>
</table>

Fu – Threshold in the CDF. Nex – Number of exceedance data. NS- Number of total samples used. 0.999, 0.9999, 0.99999 → Quantiles required to be estimated.

For a threshold of 0.85 and Nex=100, the -2,-6,-10 % Bias error in quantile translates to -27.6, -139.9 and -524.12% error in probability content

same number of exceedance data. The threshold values, number of exceedance data is dependent

on the level of target quantile we wish to estimate. It can be observed that for a quantile of 0.999,
reasonably low quantiles such as 0.8 perform well enough with 50 to 100 exceedance data, if 5% bias error is allowable. For a 0.9999 quantile, a threshold of 0.9 with 100 exceedance data
performs fairly well. A 0.95 threshold is capable of estimating a 0.99999 quantile well with about 100 data. But it can be observed that there is oscillation in the error values as the 100 exceedance data seem to perform better than the 200 exceedance data in % bias error.

A higher threshold value is expected to rectify this oscillation. Since normal distribution is the most encountered distribution in structural analysis, it is good news that the tail modeling approach can capture the quantiles in the tail of the normal distribution fairly well. The next distribution that is treated here is a log normal distribution which has a heavy tail and is generally used to model loads in structural analysis. The simulation procedure is same as above. The results are provided in Table E-2. Comparison of Tables E-1 and E-2 allows us to conclude

Table E-2. Accuracy of the tail modeling approach to estimate quantiles. LogNormal distribution $y = x \cdot x \sim \logN(0,1)$. 500 simulations. ML estimated parameters

<table>
<thead>
<tr>
<th>Fu</th>
<th>Nex</th>
<th>NS</th>
<th>% Bias</th>
<th>% RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
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<td>17.4</td>
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<td>166</td>
<td>18</td>
<td>170</td>
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<td>-0.1</td>
<td>5.6</td>
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</tbody>
</table>

Fu – Threshold in the CDF. Nex – Number of exceedance data. NS- Number of total samples used. 0.999, 0.9999, 0.99999 $\Rightarrow$ Quantiles required to be estimated.

For a 0.95 threshold and Nex=100, the 1.4, 12.4 and 39% Bias error in quantile translates to 4.5, 37.5 and 78.3% error in probability content
that estimating quantiles in lognormal distribution is more challenging than normal distribution. For a 0.999 quantile, 0.8 and 0.85 thresholds perform fairly well with 200 samples. Above a 0.9 threshold, 100 samples seem to render very good results. There is considerable difficulty in estimating 0.9999 quantile with even a 100 exceedance data and a 0.95 threshold value. Needless to mention, trying to estimate 0.99999 quantile deteriorates the accuracy of the estimation only. In order to test whether higher threshold values enable us to estimate the extreme quantiles accurately, simulation is conducted for threshold values ranging from 0.96 to 0.99 and results are presented in Table E-3.

Table E-3. Accuracy of the tail modeling approach to estimate quantiles. Higher thresholds.

<table>
<thead>
<tr>
<th>Fu</th>
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<th>NS</th>
<th>% Bias</th>
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<td>19</td>
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<td>10000</td>
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<td>10</td>
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<tr>
<td></td>
<td>200</td>
<td>20000</td>
<td>0.2</td>
<td>6</td>
</tr>
</tbody>
</table>

Fu – Threshold in the CDF. Nex – Number of exceedance data. NS- Number of total samples used. 0.9999, 0.99999 Æ Quantiles required to be estimated

Results from Table E-3 clearly show that as the threshold is increased, the extreme quantiles are also captured fairly well. With a 0.99 threshold and 100 exceedance data, we can achieve less
than one percent error in bias for estimating 0.9999 quantile. This requires 10000 samples. Similarly the accuracy in estimating 0.99999 quantile is also better with 0.99 threshold and 200 exceedance data. A more accurate estimate can be expected if we raise the threshold to 0.999 or higher.
APPENDIX F
ACCURACY AND CONVERGENCE STUDIES FOR THE CANTILEVER BEAM

Stress Failure Mode

The stress failure mode for the cantilever beam discussed in Chapter 3 is considered here. The optimal design variables $w=2.4526; \ t=3.8884$ for a target failure probability of 0.00125 ($1e7$ Samples) are adopted from (Qu and Haftka, 2004). The corresponding value of PSF for this design is 1. For the optimal combination of the design variables, 500 samples based on the distribution of the random variables are generated and the PSF is estimated for different thresholds as discussed in Chapter 3. This procedure is repeated 100 times and the mean, 5%, 95% confidence intervals are estimated. The results are presented in Figure E-1. It can be observed from the figure that the PSF estimated by regression approach is very unstable whereas the median of PSF estimated by the ML approach is relatively stable. There is instability towards higher threshold like 0.98 but the corresponding exceedance data are very less, which explains the instability. The estimated mean of the PSF in the ML approach is close to 0.99 which is about 1% bias error in estimating the actual value of the PSF. The standard deviation of the PSF estimate at a threshold of 0.9 with ML approach is about 0.003 which is 0.3% of the estimate. The corresponding estimate of failure probability is 0.0011 with a standard deviation of 0.0001 which is about 10% of the estimate.

Deflection Failure Mode

The allowable deflection is chosen to be $D_0=2.25$. Similar to the stress failure case, the PSF is estimated at different thresholds for the optimal combination of the design variables. The results are presented in Figure E-2. At a threshold of 0.9 and ML approach, the PSF is estimated to be 0.99 with a standard deviation of 0.004. The corresponding failure probability estimate is 0.0013 with a standard deviation of 0.0001 which is about 8% of the estimate itself.
Figure F-1. Convergence of PSF at different thresholds. A) MLE B) Regression. Cantilever beam stress failure mode. 500 Samples. 100 repetitions
Figure F-2. Convergence of PSF at different thresholds. A) MLE B) Regression. Cantilever beam deflection failure mode. 500 Samples. 100 repetition
APPENDIX G
STATISTICAL DISTRIBUTIONS ERROR PLOTS

Boxplot comparison of $e_{\text{ind}}$ and $e_{\text{MTM}}$ at different reliability indices. ML=Maximum Likelihood, Rg- Regression, QT – Quadratic fit to the Tail data between $S_r$ and $\ln(\beta)$, LT – Linear fit to the Tail data between $S_r$ and $\beta$, QH – Quadratic fit to half of the data between $S_r$ and $\beta$. Dashed line inside the box depicts the mean and the solid line, the median. 1000 repetition.
Figure G-1 Lognormal distribution
Figure G-2: Gamma Distribution
Figure G-3. Extreme Value type 1 Distribution
Figure G-4. Uniform Distribution.
Figure G-5. Rayleigh Distribution.
Figure G-6. Exponential Distribution.
Boxplot of $\eta$ ratio. Different distributions

Figure G-7. Lognormal Distribution.

Figure G-8. Gamma Distribution.
Figure G-9. Extreme Value-Type 1 Distribution

Figure G-10. Uniform Distribution.
Figure G-11. Rayleigh Distribution.

Figure G-12. Exponential Distribution.
REFERENCES

design optimization, Proceedings of the 45th AIAA/ASME/ASCE/AHS/ASC Structures, Structural

Constraint Relaxation in Unilevel Reliability-based Design Optimization, Submitted to the ASME
Journal of Mechanical Design

Practical Guide to Heavy Tails, Edited by Adler, R. J., et al., Boston, Birkhaeuser, pp. 111-130

Applications, John Wiley & Sons

Beirlant, J.; Vynckier, P.; Teugels, J.L. 1996: Excess Functions and Estimation of Extreme Value
Index. Bernoulli, 2, pp:293-318

Boos, D.D. 1984: Using Extreme Value Theory to Estimate Large Percentiles, Technometrics,
26, pp:33-39

Structural Safety, Vol 20, pp 1-23

Models with Applications in Engineering and Science, Hoboken, New Jersey:Wiley Interscience

Structures, Structural dynamics, and Materials Conference and Exhibit, 2860-2874.

Chiralaksanakul, A.; Mahadevan, S. 2005: Multidisciplinary Design Optimization Under
Uncertainty, in press, Optimization and Engineering.

Springer-Verlag


Uncertainty Using Inverse Reliability Strategy, ASME Journal of Mechanical Design, Vol.126,
No.4, pp. 561-764.

Elishakoff, I. 2001: Interrelation between Safety Factors and Reliability, NASA report CR-2001-
211309


Glasserman, P. 2004: Monte Carlo Methods in Financial Engineering. Springer Verlag


Sakawa, M. 1993: *Fuzzy sets and Interactive Multiobjective Optimization*. Plenum press


BIOGRAPHICAL SKETCH

Palani Ramu was born in Chennai, India in 1978. He earned his BEng in Mechanical from Madurai Kamaraj University, India in 1999. He worked at the Indian Institute of Technology, Bombay and InfoTech - Pratt & Whitney Center of Excellence as Research Assistant and Design Engineer respectively till 2002. In 2003, he started his graduate studies at the University of Florida, Gainesville and worked towards a Ph.D in Aerospace Engineering. During his graduate studies, he was a visiting researcher at the Leonardo Da Vinci University, Paris, France and was awarded a summer fellowship at the Center for Space Nuclear Research – Idaho National Labs, Idaho. His areas of interest include application of probability and statistical methods to design aerospace structures, specifically highly reliable designs.