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INVERSE MEASURE ESTIMATION FOR HIGH RELIABILITY USING LOW-NUMBER MONTE CARLO SIMULATIONS

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Inverse Measure Estimation for High Reliability Using Low-Number Monte Carlo Simulations

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Monte Carlo simulations (MCS) are more accurate than moment-based methods with multiple failure modes, highly nonlinear limit functions, and large departure from normality. However, they may require very large number of simulations when a system needs to be designed for very low probability of failure. We propose a method based on an inverse probability measure called the probabilistic sufficiency factor (PSF). It measures the limit state margin from the desired probability of failure. We show that a relative low-number MCS can be used to extrapolate the PSF with respect to a much lower probability of failure. This would permit reliability estimation with moderate size MCS, which will be demonstrated in the proposed paper.

1. Introduction

Traditionally, deterministic approaches have been successfully employed in designing structures with safety factors accounting for uncertainties in loading, material properties, engineering simulations, and manufacturing processes. The quest for more efficient methods to incorporate uncertainties leads to probabilistic approaches like Reliability-based Design Optimization (RBDO). RBDO involves minimizing a cost function subject to the constraint that the reliability of the structure should meet a target (or target reliability index). The value of the target reliability index is application dependent and is usually high for aerospace applications (i.e., corresponding to low probability of failure)

Researchers widely use Monte Carlo Simulation (MCS) and moment–based techniques to solve RBDO problems. Recently, several researchers (e.g., Lee and Kwak., 1987, Tu et al., 1999, Lee et al., 2002, Qu and Haftka., 2003, Du et al., 2003 and review paper by Ramu et al., 2006) have demonstrated that inverse reliability measures enhance the efficiency of RBDO with both MCS and moment–based techniques. MCS is simple to implement, robust and suitable for solving problems with multiple failure modes. When the target reliability index is high (low failure probability), the number of samples must be large leading to high computational cost. Moment based techniques are approximate methods and are capable of efficiently solving high reliability index problems, but are restricted to single failure mode.

The probabilistic sufficiency factor (PSF, Qu and Haftka, 2003)) is an inverse measure that provides a traditional safety factor with respect to the target probability of failure. For example, in a stress-constrained problem, a PSF of 0.9 indicates that stresses need to be lowered by 10% in order to meet the target reliability of failure. PSF may be calculated by moment-based methods such as stylesFORM. However, with multiple modes of failure, MCS may be called for, and a high target reliability index would require a large Monte Carlo sample.

Here, we suggest that even a relatively small-sample MCS contains information on the derivative of the PSF with respect to the target reliability index, and this derivative may be used to estimate the PSF corresponding to a higher target reliability index. The advantage of the method is that it uses only MCS to estimate the PSF corresponding to low failure probability and hence can be easily extended to address

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system reliability. Section 2 of this paper describes the probabilistic sufficiency factor and reliability estimation using it. Estimation of system reliability using PSF and PSF based RBDO is also discussed. The proposed method is discussed in Section 3. Section 4 discusses numerical examples. Firstly, the method is demonstrated on single failure modes of a cantilever beam example. Then, a circular limit state function is used to demonstrate the method. Finally, a system reliability case is discussed. Section 5 presents a discussion on how to use the extrapolated PSF value to estimate the required resources to achieve a safe design, followed by concluding remarks in Section 6.

2. Reliability Estimation Using Probabilistic Sufficiency Factor

The inverse measure used here is the probabilistic sufficiency factor (PSF) introduced by Qu and Haftka (2001, 2003). PSF is a safety factor with respect to the target probability of failure and hence combines the concepts of safety factor and the probability of failure. Let the capacity of the system be g_c (e.g., allowable strength) and the response be g_r. For the given vector $\mathbf{x}$ of input variables, the traditional safety factor is defined as the ratio of the capacity to the response, as

$$S(\mathbf{x}) = \frac{g_c(\mathbf{x})}{g_r(\mathbf{x})} \tag{1}$$

The system is considered to be failed when $S \le 1$ and safe when $S > 1$.

In probabilistic approaches, it is customary to use a performance function or a limit state function instead of the safety factor to define failure (or success) of a system. For example, the limit state function can be expressed as

$$G(\mathbf{x}) = S(\mathbf{x}) - 1 . \tag{2}$$

The failure of the system is defined as $G(\mathbf{x}) \le 0$, while the system is considered to be safe when $G(\mathbf{x}) > 0$. A performance function is often defined as the difference between capacity and response. However, the role of safety factor is clear in the definition in Eq. (2).

When the vector $\mathbf{x}$ of input variables is random, $g_c(\mathbf{x})$ and $g_r(\mathbf{x})$ are random in nature, resulting in the safety factor being a random function. In such instances, the safety of the system can be enforced by using the following reliability constraint:

$$P_f := \Pr\left(G(\mathbf{x}) \le 0\right) = \Pr\left(S(\mathbf{x}) \le 1\right) \le P_{f\ \text{target}} , \tag{3}$$

where P_f is the failure probability of the system and $P_{f\ \text{target}}$ is the target failure probability, which is the design requirement.

Reliability analysis calculates P_f with given random input $\mathbf{x}$, and reliability–based design optimization (RBDO) imposes Eq. (3) as a constraint. Since the magnitude of the probabilities in Eq. (3) tends to be small, the notion of reliability index is often employed. From the observation that the cumulative distribution is monotonic, the inverse transformation of the probability constraint in Eq. (3) is taken in the standard normal random space, to obtain

$$\beta := \left[-\Phi^{-1}(P_f)\right] \ge \left[-\Phi^{-1}(P_{f\ \text{target}})\right] := \beta_{\text{target}} , \tag{4}$$

where $\Phi(\bullet)$ is the cumulative distribution function (CDF) of the standard normal random variable, β the reliability index, and β_{target} the target reliability index. The reliability index is the value of standard normal random variable that has the same probability with P_f. The RBDO using Eq. (4) is called the Reliability Index Approach (RIA) (Enevoldsen, 1994; Tu *et al*., 1999)

The last inequality in Eq. (3) can be converted into equality, if the upper bound of the safety factor is relaxed (in this case it is one). Let the relaxed upper bound be s^*. Then, the last part of the reliability constraint in Eq. (3) can be rewritten, as

$$\Pr(S(\mathbf{x}) \le s^*) = P_{f\text{ target}} . \quad (5)$$

The relaxed upper bound s^* is called the Probabilistic Sufficient Factor (PSF). Using PSF, the goal is to find the value of PSF that makes the CDF of the safety factor equals to the target failure probability. Finding s^* requires inverse mapping of CDF, from which the terminology of inverse measure comes.

The concept of PSF is illustrated in Figure 1. The shaded region represents the target failure probability. Since the region to the left of $S = 1$ denotes failure, s^* should be larger than one in order to satisfy the basic design condition that the failure probability should be less than target failure probability. This can be achieved by either increasing the capacity g_c or decreasing the response g_r, which is similar to the conventional notion of safety factor, but now it is extended to probabilistic problems using PSF.

A unique advantage of PSF is that design engineers, who are familiar to the deterministic design using the safety factor, can apply the similar notion to the probabilistic design.

The PSF s^* is the factor that has to be multiplied by the response or divided by the capacity so that the safety factor be raised to one. For example, a PSF of 0.8 means that g_r has to be multiplied by 0.8 or g_c be divided by 0.8 so that the safety factor increases to one. In other words, it means that g_r has to be decreased by 20% or g_c has to be increased by 25% in order to achieve the target failure probability.

The PSF can be computed using either Monte Carlo Simulation (MCS) or moment–based methods. If MCS with N samples is used to calculate PSF, the location n is first determines as the smallest integer larger than $N \times P_{f\text{ target}}$. Then, the PSF is the n-th smallest safety factor, which is mathematically expressed as:

$$s^* = n^{th} \min_{i=1}^{N}(S(x_i)) . \quad (6)$$

The calculation of PSF requires sorting the safety factors from the MCS samples and choosing the n-th smallest one.

2.1 System Reliability Using PSF

System reliability arises when the failure of a system is defined by multiple failure modes. Here, the discussion is limited to series systems. The estimation of system failure probability involves estimating the failure due to individual modes and failure due to the interaction between the modes. This is mathematically expressed for a system with n failure modes as:

$$\begin{aligned} P_{sys} &= P(F_1) + P(F_2) + \ldots + P(F_n) \\ &- P(F_1 \cap F_2) - P(F_1 \cap F_3) - \ldots - P(F_{n-1} \cap F_n) \\ &+ P(F_1 \cap F_2 \cap F_3) + \ldots + P(F_{n-2} \cap F_{n-1} \cap F_n) - \ldots \text{Higher order terms} \end{aligned} \quad (7)$$

where, F_i is the failure of the ith mode.

It is easy to employ MCS to estimate system failure, as the methodology is a direct extension of failure estimation for single modes. In the context of employing analytical methods like FORM, estimation

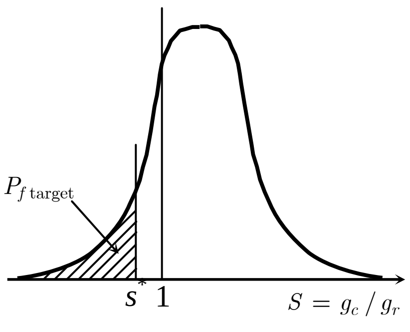


Figure 1: Probabilistic distribution of safety factor S. PSF is the value of the safety factor whose CDF corresponds to the target probability of failure.

of failure regions bounded by single modes are easy to estimate but estimating the probability content of odd shaped domains created because of the intersection of two or more modes is challenging. Techniques are available to estimate the probability content at the intersection regions but require evaluating multiple integrals (Melchers, pp: 386-400). Since the number of multiple integrals that has to be evaluated depends on the number of variables, the task becomes arduous when more variables are involved. Instead, an alternate technique is to develop upper and lower bounds on the failure probability of the structural system. Considering the terms in the first and the third rows (positive contributions) in Eq. (7) permits us to estimate the upper bound. When the terms in first and second rows (negative contributions) are considered, the lower bounds can be estimated. Improved bounds for the system reliability are available in the literature.

Owing to these challenges, MCS is mostly used when system reliability is to be estimated. As discussed earlier, MCS is computationally prohibitive when the failure probability to be estimated is low, because of the number of samples required. In order to estimate a failure probability of the order of 1/*N*, MCS requires at least 100**N* samples. PSF can be used to estimate system reliability. The method is an extension of the PSF estimation for single failure mode. When *M* failure modes are considered, the most critical safety factor is calculated for each sample. Then the sorting of the n^{th}-minimum safety factor can proceed as in Eq. (6). This process is explained in the flow chart in Figure 2.

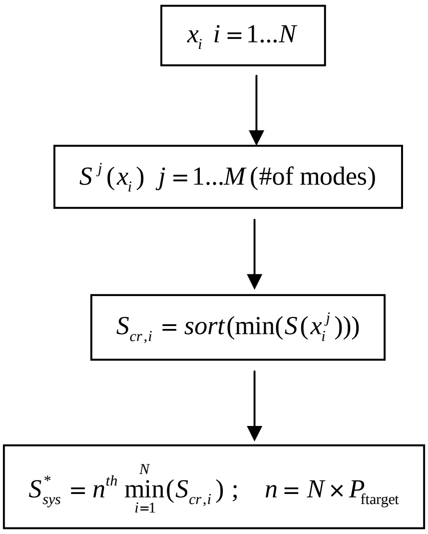


Figure 2: Flowchart for estimating system PSF

The estimation of PSF for a system reliability case is demonstrated with an example. This example is artificial and is an illustration of a case with multiple limit state functions. The example is presented in Figure 3.The limit state functions are:

$$G_1 = y - \left(\frac{x^3 - x^2 + 15}{40} \right) \tag{8}$$

$$G_2 = \left(\frac{(x + y - 5)^2}{30} + \frac{(x - y - 12)^2}{140} - 0.8 \right) \tag{9}$$

$$G_3 = y - \left(\frac{80}{x} \right) - 20 \tag{10}$$

The limit states are functions of two random variables *x* and *y*. The distributions of the random variables are presented in Table 1. The example has been created in such a fashion that each limit and its intersection with other limit states contributes significantly to the total failure.

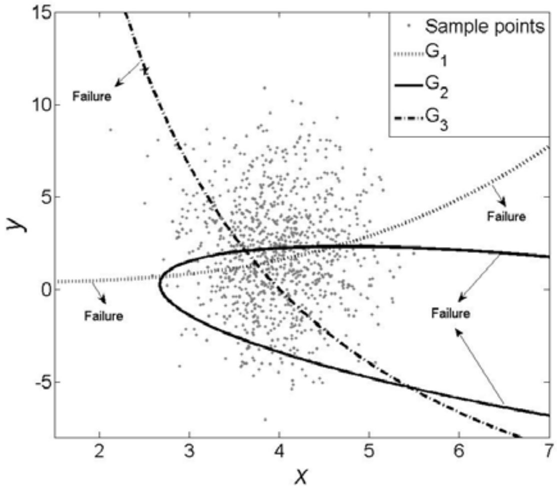


Figure 3: 2 Variable system. Algebraic multiple limit state functions example

Table 1: Distribution of random variables. Algebraic multiple failure modes example

Random Variable	*x*	*y*
Distribution	Normal (4,0.5)	Normal (2,3)

Table 2: Contribution of failure modes to the system failure for algebraic example

Limit states	Failure probability	% Contribution to total failure
1	0.432	77.55
2	0.468	84.02
3	0.336	60.32
1 & 2	0.386	69.29
1 & 3	0.253	45.42
2 & 3	0.249	44.71
1,2 & 3	0.209	37.52
Total	0.557	100

Table 3: PSF at different target failure probabilities for algebraic example

P_{ftarget}	PSF
0.557	0.9996
0.5	0.8444
0.6	1.1191

It should be noted that nonlinear functions are used to emphasize the advantage of using MCS and a PSF based approach to estimate failure probability. Using FORM in these circumstances might introduce huge error due to linearizing the limit state function at the most probable point of failure. The contributions of failure modes to the total failure are listed in Table 2. The system PSF can be estimated as described in the flowchart in Figure 2. The limit states are functions of two random variables *x* and *y*. The PSF estimated for different target failure probabilities are listed in Table 3. For a target failure probability of 0.557 which is the actual failure probability as listed in Table 2, the PSF is estimated as 1. When a lower target failure probability of 0.5 is used, the PSF is estimated to be 0.844 indicating that the system is unsafe and when a larger target failure probability of 0.6 is used, the PSF is found to be 1.12 showing that the system is over safe. This shows that PSF based approach can adequately address system reliability case accounting for the interactions between the modes.

If moment–based techniques are used to calculate PSF, the inverse measure can be computed as the value of the minimal limit state function that passes through the target reliability index curve. Mathematically, this is expressed as:

$$\begin{array}{ll} \text{Minimize} & G(\mathbf{u}) := s^* - 1 \\ \text{subject to} & |\mathbf{u}| = \sqrt{\mathbf{u}^T\mathbf{u}} = \beta_{\text{target}} \end{array}, \quad (11)$$

where $\mathbf{u}$ is the vector of standard normal random variables, $G(\mathbf{u})$ is the normalized limit state function in the standard normal space, and $\beta_{\text{target}} := \Phi^{-1}(P_{f\,\text{target}})$ is the target reliability index. A more detailed discussion about the computation of inverse measure can be found in Ramu *et al*. (2006).

2.2 RBDO using PSF

RBDO using inverse measures are being developed mainly because of the advantages involved in working with inverse measures. The various advantages include better computational efficiency, numerical stability (Tu *et al*., 1999), faster and stable RBDO (Youn *et al*., 2003) compared to RIA. Qu and Haftka (2002) showed that usage of inverse measure leads to more accurate response surface approximation compared to response surfaces fitted to reliability index or failure probability. Table 4 compares the manner in which the probabilistic constraint is prescribed in the direct reliability index approach (RIA) and the inverse approach (PSF). RBDO using PSF involves a two–level optimization shown in Figure 4, for a single mode of failure. When the MCS is employed in the inverse reliability analysis, the lower–level optimization problem can be replaced by Eq. (6).

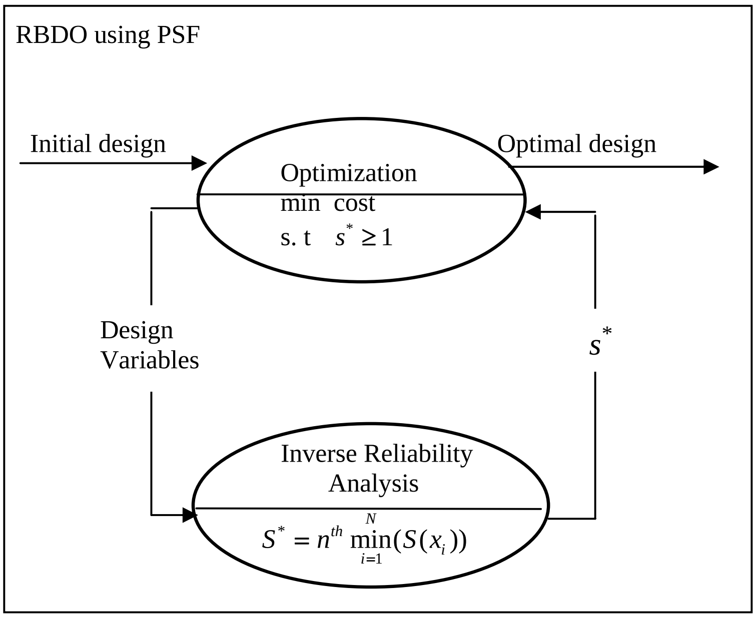


Figure 4: The double-loop RBDO

Table 4: Different approaches to prescribe probabilistic constraint

Method	RIA	PSF
Probabilistic Constraint	$\beta \geq \beta_{\text{target}}$	$s^* \geq 1$
Quantity to be computed	Reliability Index (β)	PSF (s^*)

3. Estimation of PSF for Low Target Failure Probability

In general, reliability analysis using MCS is more accurate than the moment–based method when the performance function shows nonlinearity. In addition, MCS is useful with multiple failure modes. However, the required number of samples in MCS increases significantly when the target failure probability is relatively small. In this paper, an extrapolation scheme is proposed to estimate the PSF for low target failure probability using an MCS which is sufficient only to estimate the PSF for substantially higher failure probability (lower target reliability index). This is based on approximating the relationship between the PSF and the reliability index by a quadratic polynomial. The PSF for each reliability index in a range of small reliability indices is obtainable using smaller sample size MCS. A quadratic polynomial is fit to the PSF in terms of the natural logarithm of the reliability index in this range. Once the polynomial is obtained, the PSF corresponding to any higher reliability index can be estimated using it. Hence, once the PSF for each reliability index in a range of low reliability indices is obtained, the problem reduces to a data fitting problem.

A FORM perspective of the problem definition is illustrated in Figure 5 when two random variables are involved. The PSF s^*_{hp} is calculated with a low target reliability index $\beta_{\text{target}} = \beta_{hp}$, using the value of limit state function that passes through MPP1, which is the MPP in the inverse reliability analysis. (MPP is the acronym of the Most Probable Point and as the name suggests it is a point which contributes most to failure). In the standard normal space the point at which the minimum first order limit state function forms a tangent to the target reliability curve is the MPP. The goal is to estimate s^*_{lp} without performing additional reliability analysis. From the perspective of moment–based techniques, PSF is the minimal value of the limit state function that passes through the target reliability index curve. Recalling the

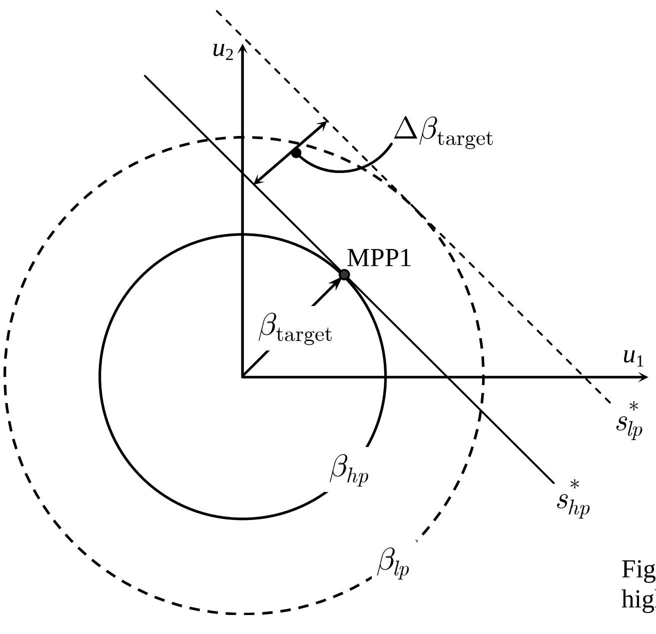


Figure 5: Problem definition: Find the PSF at higher target reliability index

optimization problem in Eq. (11), *the sensitivity of PSF with respect to the target reliability index is essentially the effect on cost function due to changes in the constraint limits*. The Lagrange multiplier from the optimization problem of the inverse reliability loop provides information for this sensitivity analysis. The variation of solution sensitivity with respect to constraint variation is equal to the negative of the Lagrange multiplier of the constraint in Eq. (11) (Haftka and Gürdal, 1993). Thus, the sensitivity of s^* with respect to β_{target} is the negative of the Lagrange multiplier at MPP1.

The proposed method employs the MCS to compute PSF corresponding to the higher failure probability (small sample size is suitable). Based on the approximating the relationship between the PSF and a range of low reliability indices using a polynomial, it is possible to estimate PSF corresponding to the low failure probability. This method works well for small perturbations of the target reliability index as shown in the next section. It is to be noted that small perturbations in reliability index as 3 to 4 corresponds to 2 orders magnitude in the failure probability space.

4. Numerical Examples

This section demonstrates the proposed method on several examples. First, the method is demonstrated for limit state functions of a cantilever beam design example. The second example treats a circular limit state example and finally the system reliability case is discussed for the cantilever beam example.

Single-mode beam example:

The cantilever beam shown in Figure 6 is taken from Wu *et al.* (2001). The objective of the design is to minimize the weight or, equivalently, the cross-sectional area $A = wt$, subject to two reliability constraints, which require the reliability indices for strength and deflection constraints to be larger than three. The two failure modes are expressed as:

Strength:
$$g_s = \frac{R}{\sigma} - 1 = \frac{R}{\left(\frac{600}{wt^2}Y + \frac{600}{w^2 t}X\right)} - 1 \tag{12}$$

and

Tip Displacement:
$$g_d = \frac{D_O}{D} - 1 = \frac{D_O}{\frac{4L^3}{Ewt}\sqrt{\left(\left(\frac{Y}{t^2}\right)^2 + \left(\frac{X}{w^2}\right)^2\right)}} - 1 \tag{13}$$

where R is the yield strength; X and Y are the horizontal and vertical loads; w and t are the deterministic design parameters; L is the length; and E is the elastic modulus. R, X, Y and E are random variables, whose distributions are defined in Table 5. D_0 is the allowable deflection. To demonstrate the proposed extrapolation technique, the two single failure modes of stress and displacement are discussed

Design for the strength case is discussed in detail in Qu and Haftka (2001, 2003), and Ramu *et al.* (2004). The optimization is initially performed for the target reliability index of three (target failure probability is 0.00135). During the optimization process, PSF is calculated by using MCS with 100,000 samples. The PSF is estimated as an average of PSF at 11 points around the critical point. In order to verify the accuracy of the inverse measure at the optimal design, MCS with 10 million samples is performed after the optimal design is obtained. The accuracy of the inverse measure computed by different approaches is discussed in Ramu *et al.* (2004).

The values of PSF corresponding to target reliability indices in the range of 2 to 3 at an increment of 0.1 are evaluated and employed to fit a quadratic polynomial to the PSF as function of the natural logarithm of the target reliability index. The polynomial can be used to extrapolate the inverse measure at target reliability index levels higher than 3.0 Table 6 shows the results for the stress failure mode. It is seen that the extrapolated inverse measure matches well with the inverse measure estimated based on 1E7 samples at various higher target reliability indices. As can be seen in Table 6 the error in PSF for target reliability

Table 5. Random variables for cantilever beam problem

Random Variables	X	Y	R	E
Distribution	Normal (500,100)lb	Normal (1000,100)lb	Normal (40000,2000) psi	Normal (29E6,1.45E6) psi

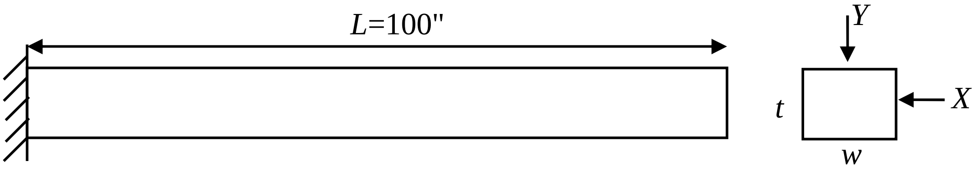


Figure 6: Cantilever beam subjected to horizontal and vertical random loads

Table 6: Cantilever beam stress constraint. Comparison of PSF estimated based on 1E7 samples and PSF estimated based on quadratic fit
w = 2.4526, t = 3.8884 (Optimum design obtained by 100,000 sample MCS)

		PSF	
β_{target}	$P_{ftarget}$	1E7 samples	Quadratic Fit* (1E5 samples)
2	2.27E-02	1.1090	1.1088
2.2	1.39E-02	1.0864	1.0869
2.4	1.89E-03	1.0646	1.0649
2.6	4.66E-03	1.0434	1.0435
2.8	2.55E-03	1.0227	1.0221
3	1.35E-03	1.0024	1.0015
			Extrapolated
3.2	6.87E-04	0.9825	0.9824
3.4	3.37E-04	0.9632	0.9634
3.6	1.59E-04	0.9447	0.9453
3.8	7.23E-05	0.9267	0.9276
4	3.17E-05	0.9110	0.9102
4.2	1.33E-05	0.8939	0.8932

* Median of 10 simulations

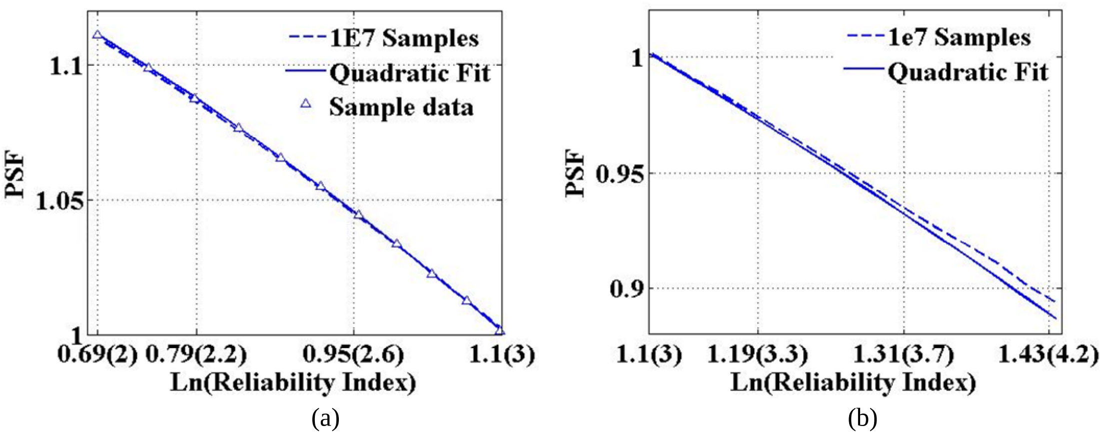


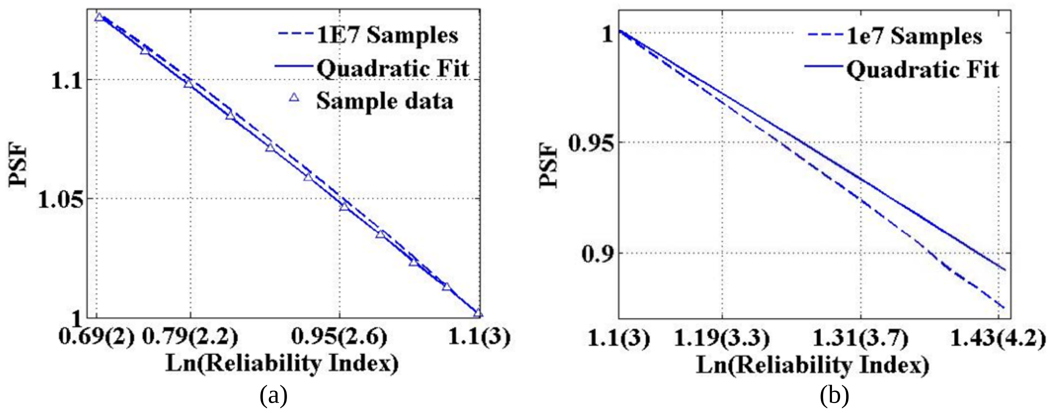
Figure 7: Cantilever beam stress failure mode. Comparison of 1E7 sample based PSF and PSF estimated by the quadratic fit for 1 simulation. Reliability index is presented in parentheses in the abscissa.
(a) Interpolation region. (b) Extrapolated region

index of 3, with a sample size of 10^5 in MCS is 0.004 which is comparable to the extrapolation error. The results for the stress failure mode are presented in Figure 7. The first plot shows the curves for the quadratic fit and the 1E7 samples along with the data used for the fit in the range of reliability indices from 2 to 3. The second plot presents the same plot for a reliability index range of 3 to 4.2. Similarly, the extrapolation technique is applied for the displacement failure mode, and results are presented in Table 7. It is observed that the computed high sample PSF and the extrapolated values match very well. The comparative results are illustrated in Figure 8 for displacement constraint.

Table 7: Cantilever beam deflection constraint. Comparison of PSF estimated based on 1E7 samples and PSF estimated based on quadratic fit
w=2.4526, t=3.8884 (Optimum design obtained by 100,000 sample MCS)

		PSF	
β_{target}	$P_{ftarget}$	1E7 samples	Quadratic Fit* (1E5 samples)
2	2.27E-02	1.1276	1.1270
2.2	1.39E-02	1.1007	1.1000
2.4	1.89E-03	1.0747	1.0741
2.6	4.66E-03	1.0494	1.0498
2.8	2.55E-03	1.0251	1.0257
3	1.35E-03	1.0011	1.0027
			Extrapolated
3.2	6.87E-04	0.9780	0.9804
3.4	3.37E-04	0.9561	0.9587
3.6	1.59E-04	0.9347	0.9385
3.8	7.23E-05	0.9143	0.9191
4	3.17E-05	0.8930	0.9002
4.2	1.33E-05	0.8746	0.8817

* Median of 10 simulations



(a) (b)

Figure 8: Cantilever beam deflection failure mode. Comparison of 1E7 sample based PSF and PSF estimated by the quadratic fit for 1 simulation. Reliability index is presented in parentheses in the abscissa. (a) Interpolation region. (b) Extrapolated region

Circular limit state example:

This example is used to demonstrate the proposed method on a circular limit state function, which would not be amenable to FORM. The limit state function is function of two standard normal variables *x* and *y* given as

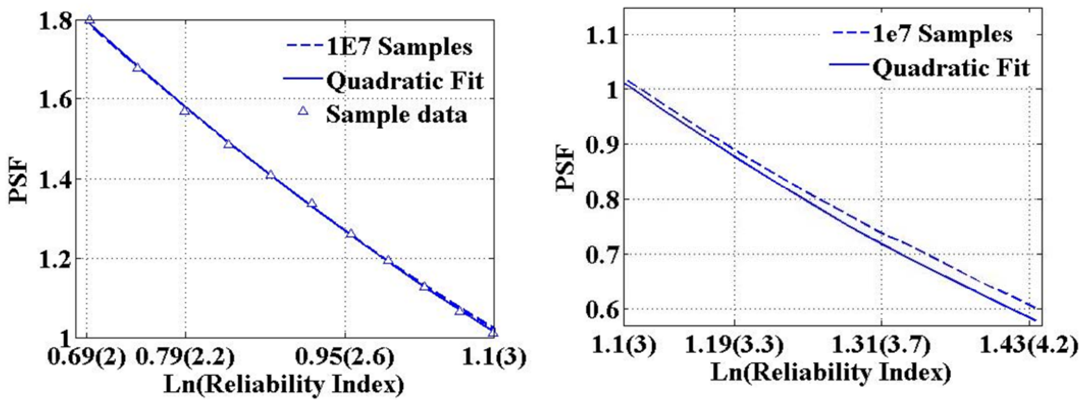
$$G = \frac{12}{x^2 + y^2} - 1 \tag{14}$$

FORM or SORM cannot be used for this example, as the failure region is not well approximated by a half plane. Similar to the earlier example, the PSF corresponding to reliability indices in the range of 2 to 3 are estimated using MCS with 100,000 samples. A quadratic polynomial is fit to these PSF data in terms of the natural logarithm of the reliability index. The fitted polynomial is further used to estimate PSF at higher reliability indices. The results are presented in Table 8. The comparison of the high sample based PSF and extrapolated PSF is presented in Figure 9. It is evident that the proposed technique can be used to extrapolate accurately the PSF corresponding to target failure probability 2 orders of magnitude lower than a particular target failure probability for which the PSF is known even for circular limit state functions that cannot be solved by analytical methods like FORM or SORM.

Table 8: Circular limit state function. Comparison of PSF estimated based on 1E7 samples and PSF estimated based on quadratic fit

		PSF	
β_{target}	$P_{ftarget}$	1E7 samples	Quadratic Fit* (1E5 samples)
2	2.27E-02	1.7856	1.7839
2.2	1.39E-02	1.5795	1.5813
2.4	1.89E-03	1.4066	1.4072
2.6	4.66E-03	1.2589	1.2584
2.8	2.55E-03	1.1313	1.1286
3	1.35E-03	1.0229	1.0174
			Extrapolated
3.2	6.87E-04	0.9282	0.9180
3.4	3.37E-04	0.8457	0.8320
3.6	1.59E-04	0.7740	0.7601
3.8	7.23E-05	0.7124	0.6949
4	3.17E-05	0.6532	0.6376
4.2	1.33E-05	0.6010	0.5873

*Median of 10 simulations



(a) (b)

Figure 9: Circular limit state function. Comparison of 1E7 sample based PSF and PSF estimated by the quadratic fit for 1 simulation. Reliability index is presented in parentheses in the abscissa.
(a) Interpolation region. (b) Extrapolated region

Multiple-mode beam example:

This case investigates the system reliability for the cantilever beam design example discussed in Example 1. Here, both the stress and deflection failure modes are considered simultaneously. The allowable deflection for the displacement mode was changed to 2.145. This allows equal contribution from both the modes to the total failure probability. As explained in the flow chart in Figure 2, the system PSF was estimated for reliability indices 2 to 3. A quadratic polynomial was fit to PSF in terms of natural logarithm of reliability index. The polynomial was further used to predict PSF at higher reliability indices. The results are presented in Table 9 and the comparison plot is presented in Figure 10. As observed in earlier examples, the extrapolated value matches well with the PSF obtained with higher samples.

Table 9: Cantilever beam system reliability. Comparison of PSF estimated based on 1E7 samples and PSF estimated based on quadratic fit. w = 2.6041 t = 3.6746

$D_0 = 2.145,\ P_{f1} = 0.00099,\ P_{f2} = 0.00117,\ P_{f1} \cap P_{f2} = 0.00016,\ P_{fsys} = 0.002$

β_{target}	$P_{f\,target}$	PSF	
		1E7 samples	Quadratic Fit* (1E5 samples)
2	2.27E-02	1.092	1.0918
2.2	1.39E-02	1.0702	1.0697
2.4	1.89E-03	1.0488	1.0481
2.6	4.66E-03	1.0281	1.0273
2.8	2.55E-03	1.0079	1.0077
3	1.35E-03	0.9881	0.9888
			Extrapolated
3.2	6.87E-04	0.9688	0.9706
3.4	3.37E-04	0.9500	0.9530
3.6	1.59E-04	0.9328	0.9359
3.8	7.23E-05	0.9147	0.9193
4	3.17E-05	0.8976	0.9028
4.2	1.33E-05	0.8784	0.8867

* Median of 10 simulations

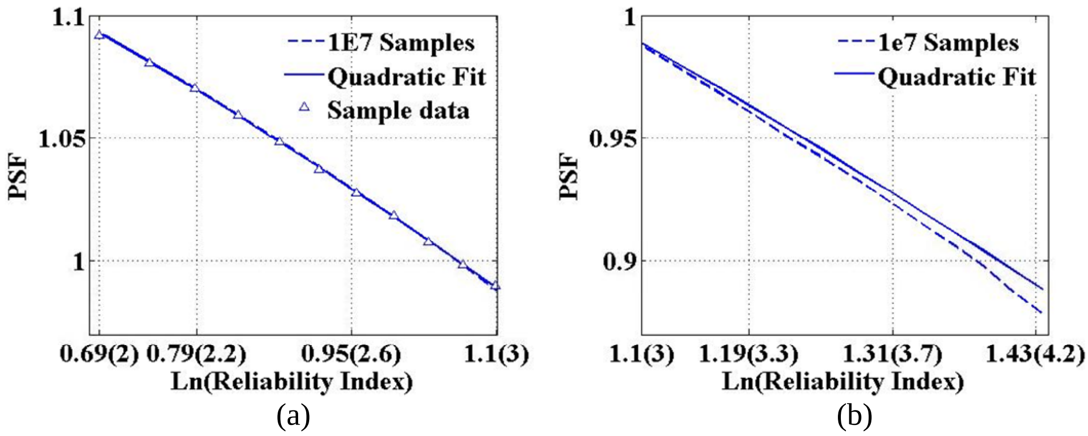


Figure 10: Cantilever beam system failure mode. Comparison of 1E7 sample based PSF and PSF estimated by the quadratic fit for 1 simulation. Reliability index is presented in parentheses in the abscissa.
(a) Interpolation region. (b) Extrapolated region

5. Usage of PSF to estimate change in weight required to achieve a safe design

The extrapolated PSF allows performing reliability analysis in a RBDO process characterized by a high reliability index with fewer MCS samples. Alternately, the predicted PSF can be used to estimate change in weight required to achieve a safe design. This notion of using PSF to estimate the required resources is similar to using safety factor in deterministic approaches. This section presents a discussion on how to estimate the required resources using PSF to achieve a safe design. For the cantilever beam example, if there is a design with the dimensions of $A_0= w_0\ t_0$ with a PSF of s_0^* less than one. A PSF less than 1 indicates that the structure is not safe. The structure can be made safer by scaling both *w* and *t* by the same factor *c*. This will change the stress and displacement expressed in Eqs. (12) and (13) by a factor of c^3 and c^4, respectively, and the area by a factor of c^2. The PSF is inversely proportional to the most critical stress or displacement. So, it is easy to obtain the following relationships from Eqs. (12) and (13):

Stress:
$$s^* = s_0^* \left(\frac{A}{A_0}\right)^{1.5} \tag{15}$$

Displacement:
$$s^* = s_0^* \left(\frac{A}{A_0}\right)^{2} \tag{16}$$

It is evident from the above equations that an increase of 1 percent in area will lead to a 1.5 percent increase in PSF for the stress constraint and a 2 percent increase in PSF for the displacement constraint. Hence, the stress constraint is more critical here. Uniform scaling in width and thickness are considered here, but non uniform scaling may be more efficient. For example, consider a design in the system reliability case at a target reliability index of 4.2 with an extrapolated PSF of 0.8867 (see Table 9). There is a deficiency of 11.33% in the PSF and the structure can be made safer with a weight increase of less than 7.55% (corresponding to 3.7% increase in *w* and *t*). The width and thickness are scaled to 2.7004 (2.6041×1.037) and 3.8105 (3.6746×1.037) respectively. Thus the objective function of the scaled design is A=10.29. The corresponding failure probability is 1.9E-05 (reliability index of 4.12) and the PSF is 1.0002 evaluated using 1E7 samples. Thus, it is possible to estimate the required resources to achieve a safe design with the value of a PSF corresponding to unsafe design. It is to be noted that such an estimation of required resources are not readily available with failure probability of reliability index are used as reliability estimators. For more complex structures where the limit late functions are not known explicitly, designers can estimate the weight required to reduce the stress and displacement by a certain amount.

6. Conclusions

An extrapolation approach of the probabilistic sufficiency factor (PSF) to address reliability estimation of high target reliability index is proposed. This extrapolation technique uses the robustness of MCS in estimating the PSF inverse measure at several higher target probability of failure to compute the PSF corresponding to a low target probability of failure. The proposed approach was demonstrated on a various numerical examples. In all cases, it was shown that the PSF estimated by the proposed approach matched closely the PSF obtained from high sample. Further, this extrapolated PSF can be used to estimate change in weight required to achieve a safe design. Future avenues include incorporating this reliability estimation method with fewer samples to estimate PSF corresponding to a higher reliability index in a RBDO loop.

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