
Inverse reliability measures and reliability-based design optimisation

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Abstract: Several inverse reliability measures (e.g. Probabilistic Performance Measure (PPM) and Probabilistic Sufficiency Factor (PSF)) that are essentially equivalent have been introduced in recent years as measures of safety. The different names for essentially the same measure reflect the fact that different researchers focused on different advantages of inverse measures. These advantages include improved computational efficiency of Reliability-Based Design Optimisation (RBDO), accuracy in Response Surface Approximations (RSAs) and easy estimates of resources needed for achieving target safety levels. This paper surveys these inverse measures and describes their advantages compared with the direct measures of safety such as probability of

failure and reliability index. Methods to compute the inverse measures are also described. RBDO with inverse measure is demonstrated with a beam design example.

Keywords: inverse reliability measures; Reliability-Based Design Optimisation (RBDO); Monte Carlo Simulation (MCS); First-Order Reliability Method (FORM); Probabilistic Sufficiency Factor (PSF); Probabilistic Performance Measure (PPM).

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1 Introduction

Inverse reliability measures are becoming popular in reliability community. Various inverse reliability measures have been developed. This paper focuses on discussing the interconnection of various inverse reliability measures, examining methods to compute inverse reliability measures and the advantages of using them over direct reliability measures.

Traditionally, structural safety was defined in terms of safety factors, which were used to compensate for uncertainties in loading and material properties and for inaccuracies in geometry and theory. Safety factors permit design optimisation using computationally inexpensive deterministic methods. In addition, it is relatively easy to estimate the change in structural weight of over or under designed structures needed to satisfy a target safety factor requirement (Qu and Haftka, 2003, 2004).

Probabilistic approaches in design optimisation allow incorporation of available uncertainty data and thus provide more accurate measures of safety. Structural safety is measured in terms of probability of failure to satisfy some performance criterion. The probability of failure is often expressed in terms of a reliability index. This reliability index is the ratio of the mean to the standard deviation of the safety margin distribution, which is the difference between the capacity and the resistance of the system.

Optimisation using probabilistic approaches called Reliability-Based Design Optimisation (RBDO) enables to gauge the structural safety better but is computationally significantly more expensive compared to deterministic approaches. In addition, the difference between the computed probability of failure or reliability index and their target values does not provide the designer with easy estimates of the change in the design cost needed to achieve these target values.

Safety factors are defined as the ratio of the capacity to resistance of a system. In deterministic approaches, safety factors are determinate, typically calculated for mean values whereas in probabilistic approaches it is a random number. In probabilistic approaches, one safety measure that combines the advantages of safety factors and probability of failure was proposed by Birger (1970, as reported by Elishakoff, 2001 and 2004). Birger relates the safety factor and the fractile of the safety factor distribution corresponding to a target probability of failure level. It belongs to a class of inverse reliability measures, which carry that name because they require use of the inverse of the Cumulative Distribution Function (CDF). Several researchers developed equivalent inverse reliability methods (Du et al., 2004; Kiureghian et al., 1994; Lee and Kwak, 1987; Lee et al., 2002; Li and Foschi, 1998; Tu et al., 1999; Qu and Haftka, 2003) that are closely related to the Birger measure. These measures quantify the level of safety in terms of the change in structural response needed to meet the target probability of failure.

Lee and Kwak (1987) used the inverse formulation in RBDO and showed that it is preferable for design when the probability of failure is very low in some region of the

design space so that the safety index approaches infinity. Kiureghian et al. (1994) addressed the inverse reliability problem of seeking to determine one unknown parameter in the limit state function such that a prescribed first-order reliability index is attained. To solve the inverse reliability problem, they proposed an iterative algorithm based on the Hasofer-Lind-Rackwitz-Fiessler algorithm. Li and Foschi (1998) employed the inverse reliability strategy in earthquake and offshore applications to solve for multiple design parameters. They show that it is an efficient method to estimate design parameters corresponding to target reliabilities. Kirjner-Neto et al. (1998) reformulated the standard RBDO formulation similar to Lee and Kwak (1987) except that they use an inequality constraint. They developed a semi-infinite optimisation algorithm to solve the reformulated problem. This formulation does not require second-order derivatives of the limit state functions and obviates the need for repeated reliability indices computation. However, they found that the approach can result in conservative design. Royset et al. (2001) extended the reformulation technique discussed in Kirjner-Neto et al. (1998) for reliability-based design of series structural systems. The required reliability and optimisation calculations are completely decoupled in this approach. Tu et al. (1999) dubbed the inverse measure approach the Performance-Measure Approach (PMA) and called the inverse measure Probability Performance Measure (PPM). Lee et al. (2002) adopted the same procedure as Tu et al. (1999) and named it target performance-based approach, calling the inverse measure the target performance. They compared the Reliability Index Approach (RIA) and inverse measure-based approach and found that the latter was superior in both computational efficiency and numerical stability. Youn et al. (2003) showed that PMA allows faster and more stable RBDO compared to the traditional RIA.

Qu and Haftka (2003, 2004) called the inverse measure Probability Sufficiency Factor (PSF) and explored its use for RBDO with multiple failure modes through Monte Carlo Simulation (MCS) and Response Surface Approximation (RSA). They showed that PSF leads to more accurate RSA compared to RSA fitted to failure probability, provides more effective RBDO and that it permits estimation of the necessary change in the design cost needed to meet the target reliability. Moreover, PSF enables performing RBDO in variable-fidelity fashion and sequential deterministic optimisation fashion to reduce the computational cost (Qu and Haftka, 2004 and Qu et al., 2004). An initial study of using PSF to convert RBDO to sequential deterministic optimisation was performed to solve problems with reliability constraints on individual failure modes (Qu and Haftka, 2003). An improved version for system reliability problems with multiple failure modes was developed for reliability-based global optimisation of stiffened panels (Qu et al., 2004).

Du et al. (2004, 2005) employed PMA to formulate RBDO, but used percentile levels of reliability (1 minus failure probability) in the probabilistic constraint and called the inverse measure as percentile performance. Traditionally, design for robustness involves minimising the mean and standard deviation of the performance. Here, Du et al. (2004) proposed to replace the standard deviation by percentile performance difference, which is the difference between the percentile performance corresponding to the left tail of a CDF and the right tail of that CDF. They demonstrated increased computational efficiency and more accurate evaluation of the variation of the objective performance. In an effort to address reliability-based designs when both random and interval variables are present, Du et al. (2005) proposed the use of percentile performance with worst-case combination of the interval variables for efficient RBDO solutions.

Du and Chen (2004) developed the Sequential Optimisation and Reliability Assessment (SORA) to improve the efficiency of the probabilistic optimisation. The method is a serial single loop strategy, which employs percentile performance and the key is to establish equivalent deterministic constraints from probabilistic constraints. This method is based on evaluating the constraint at the Most Probable Point (MPP) of the inverse measure (see Section 4) based on the reliability information from the previous cycle. This is referred to as ‘design shift’ (Chiralaksanakul and Mahadevan, in press; Youn et al., 2005). They show that the design quickly improves in each cycle and is computationally efficient. The SORA, however, is not guaranteed to lead to an optimal design. Single-level (or unilevel) techniques that are equivalent to the standard RBDO formulation are based on replacing the RIA or PMA inner loop by the corresponding Karush-Kuhn-Tucker conditions. Here again, Agarwal et al. (2004) showed that the PMA approach is more efficient than the unilevel RIA approach due to Kuschel and Rackwitz (2000).

The several inverse measures discussed above are all based on the common idea of using the inverse of the CDF. The numerous names for the inverse measures contemplate that they were developed by different researchers for different applications. As these inverse measures come under various names, it is easy to fail to notice the commonality among them. The main purpose of this paper is to highlight the relationship between the inverse measures. The objectives of this work include:

- 1 to discuss the relationship (or minor differences) between the various inverse measures
- 2 to discuss methods available for calculating these inverse measures and
- 3 to explore the advantages of using inverse measures instead of direct measures.

Section 2 describes inverse reliability measures. Calculation of inverse measures by MCS is discussed in Section 3. Section 4 describes calculation of inverse measures using moment-based techniques, followed by discussion of using inverse measures in RBDO in Section 5. Section 6 demonstrates the concepts with the help of a beam design example and Section 7 provides concluding remarks.

2 Inverse reliability measures

2.1 Birger safety factor

The safety factor, S is defined as the ratio of the capacity of the system G_c (e.g. allowable strength) to the response G_r with a safe design satisfying $G_r \leq G_c$. To account for uncertainties, the design safety factor is greater than one. For example, a load safety factor of 1.5 is mandated by FAA in aircraft applications. To address the probabilistic interpretation of the safety factor, Birger (1970) proposed to consider its CDF F_s :

$$F_s(s) = \text{Prob}\left(\frac{G_c}{G_r} \leq s\right) \quad (1)$$

Note that unlike the deterministic safety factor, which is normally calculated for the mean value of the random variables, G_c/G_r in Equation (1) is a random function. Given a target probability, P_{target} , Birger suggested a safety factor s^* (which we call here the Birger safety factor) defined in the following equation

$$F_s(s^*) = \text{Prob}\left(\frac{G_c}{G_r} \leq s^*\right) = \text{Prob}(S \leq s^*) = P_{\text{ftarget}} \quad (2)$$

That is, the Birger safety factor is found by setting the CDF of the safety factor equal to the target probability. That is, we seek to find the value of the safety factor that makes the CDF of the safety factor equal to the target failure probability. This requires the inverse of the CDF, hence the terminology of inverse measure.

2.2 Probabilistic Sufficiency Factor

Qu and Haftka (2003, 2004) developed a similar measure to the Birger safety factor, calling it first the probabilistic safety factor and then the Probabilistic Sufficiency Factor (PSF). They obtained the PSF by MCS and found that the response surface for PSF was more accurate than the response surface fitted to failure probability. Later, they found the reference to Birger's work in Elishakoff's review (2001) of safety factors and their relations to probabilities. It is desirable to avoid the term safety factor for this entity because the common use of the term is mostly deterministic and independent of the target safety level. Therefore, while noting the identity of the Birger safety factor and the PSF, we will use the latter term in the following.

Failure happens when the actual safety factor S is less than one. The basic design condition that the probability of failure should be smaller than the target probability for a safe design may then be written as

$$P_f = \text{Prob}(S \leq 1) = F_s(1) \leq P_{\text{ftarget}} \quad (3)$$

Using inverse transformation, Equation (3) can be expressed as

$$1 \leq F_s^{-1}(P_{\text{ftarget}}) = s^* \quad (4)$$

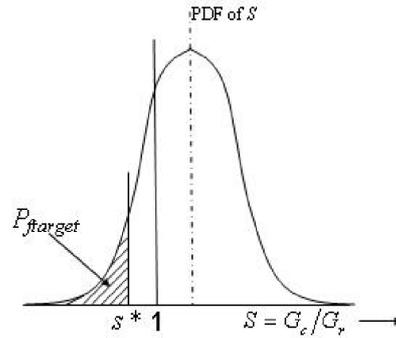
The PSF concept is illustrated in Figure 1. The design requirement P_{ftarget} is known and the corresponding area under the probability density function of the safety factor is the shaded region in Figure 1. The upper bound of the abscissa s^* is the value of the PSF. The region to the left of the vertical line $S = 1$ represents failure. To satisfy the basic design condition s^* should be larger than or equal to one. To achieve this, it is possible to either increase G_c or decrease G_r . The PSF s^* , represents the factor that has to multiply the response G_r or divide the capacity G_c , so that the safety factor be raised to 1.

For example, a PSF of 0.8 means that G_r has to be multiplied by 0.8 or G_c be divided by 0.8 so that the safety factor ratio increases to one. In other words, this means that G_r has to be decreased by 20% ($1-0.8$) or G_c has to be increased by 25% ($(1/0.8)-1$) to achieve the target failure probability. It can be observed that PSF is a safety factor with respect to the target failure probability and is automatically normalised in the course of its formulation.

PSF is useful in estimating the resources needed to achieve the required target probability of failure. For example, in a stress-dominated linear problem, if the target probability of failure is 10^{-5} and a current design yields a probability of failure of 10^{-3} , one cannot easily estimate the change in the weight required to achieve the target failure probability. Instead, if the failure probability corresponds to a PSF of 0.8,

this indicates that maximum stresses must be lowered by 20% to meet the target. This permits the designers to readily estimate the weight required to reduce stresses to a given level.

Figure 1 Schematic probability density of the safety factor S . The PSF is the value of the safety factor corresponding to the target probability of failure



2.3 Probabilistic Performance Measure

In probabilistic approaches, instead of the safety factor it is customary to use a performance function or a limit state function to define the failure (or success) of a system. For example, the limit state function and the safety criterion can be expressed as:

$$G(\mathbf{X}) = G_c(\mathbf{X}) - G_r(\mathbf{X}) \geq 0 \quad (5a)$$

where \mathbf{X} is a vector of random variables. In terms of safety factor S , another form of the limit state function is:

$$\tilde{G}(\mathbf{X}) = S - 1 \geq 0 \quad (5b)$$

Here, $G(\mathbf{X})$ and $\tilde{G}(\mathbf{X})$ are the ordinary and normalised limit state functions, respectively. Failure happens when $G(\mathbf{X})$ or $\tilde{G}(\mathbf{X})$ is less than zero, so the probability of failure P_f is

$$P_f = \text{Prob}(G(\mathbf{X}) \leq 0) \quad (6a)$$

$$P_f = \text{Prob}(\tilde{G}(\mathbf{X}) \leq 0) \quad (6b)$$

Using Equation (6), Equation (3) can be rewritten as:

$$P_f = \text{Prob}(G(\mathbf{X}) \leq 0) = F_G(0) \leq P_{\text{ftarget}} \quad (7a)$$

$$P_f = \text{Prob}(\tilde{G}(\mathbf{X}) \leq 0) = F_{\tilde{G}}(0) \leq P_{\text{ftarget}} \quad (7b)$$

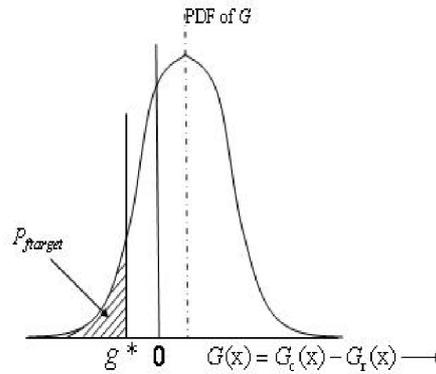
where F_G and $F_{\tilde{G}}$ are the CDF of $G(\mathbf{X})$ and $\tilde{G}(\mathbf{X})$, respectively. Inverse transformation allows us to write Equations (7a) and (7b) as:

$$0 \leq F_G^{-1}(P_{\text{ftarget}}) = g^* \quad (8a)$$

$$0 \leq F_{\tilde{G}}^{-1}(P_{\text{ftarget}}) = \tilde{g}^* \quad (8b)$$

Here, g^* and \tilde{g}^* are the ordinary and normalised Probabilistic Performance Measure (PPM) (Tu et al., 1999), respectively. PPM can be defined as the solution to Equation (7). That is, the value in (\bullet) (instead of zero) which makes the inequality an equality. Hence PPM is the value of the limit state function that makes its CDF equals the target failure probability. Figure 2 illustrates the concept of PPM. The shaded area corresponds to target failure probability. The area to the left of the line $G = 0$ indicates failure. g^* is the factor that has to be subtracted from Equation (5a) to make the vertical line g^* move further to the right of the $G = 0$ line and hence facilitate a safe design.

Figure 2 Schematic probability density of the limit state function. The PPM is the value of the limit state function corresponding to the target probability of failure



For example, a PPM of -0.8 means that the design is not safe enough and -0.8 has to be subtracted from $G(\mathbf{X})$ to achieve the target probability of failure.

A PPM value of 0.3 means that we have a safety margin of 0.3 to reduce while improving the cost function to meet the target failure probability.

Considering \tilde{g}^* as the solution for Equation (7b), it can be rewritten in terms of the safety factor as:

$$\text{Prob}(\tilde{G}(\mathbf{X}) = S - 1 \leq \tilde{g}^*) = P_{\text{target}} \tag{9}$$

Comparing Equations (4), (8b) and (9), we can observe a relationship between s^* and \tilde{g}^* . PSF (s^*) is related to the normalised PPM (\tilde{g}^*) as:

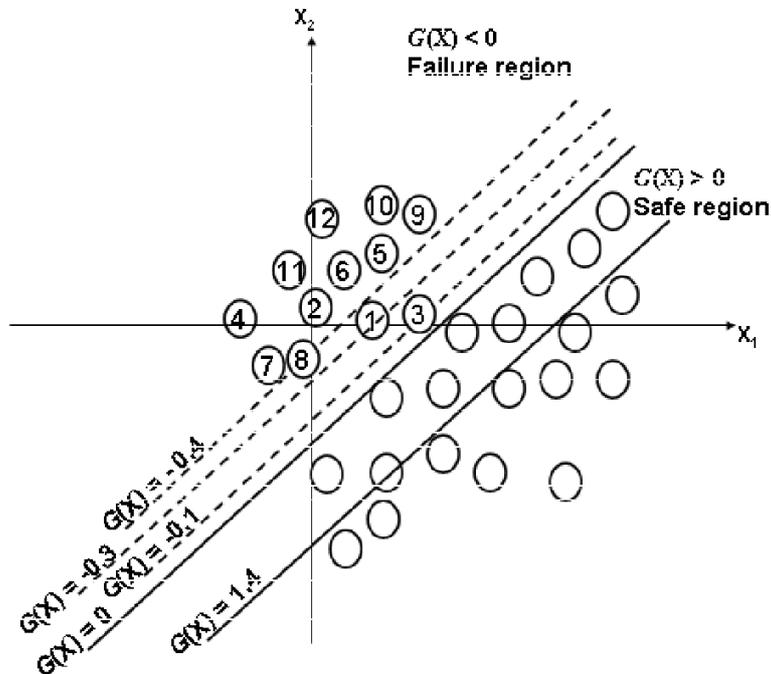
$$s^* = \tilde{g}^* + 1 \tag{10}$$

This simple relationship between PPM and PSF shows that they are closely related and the difference is only in the way the limit state function is written. If the limit state function is expressed as the difference between capacity and response as in Equation (5a), failure probability formulation allows us to define PPM. Alternatively, if the limit state function is expressed in terms of the safety factor as in Equation (5b), the corresponding failure probability formulation allows us to define PSF. PSF can be viewed as PPM derived from the normalised form of Equation (5a). The PMA notation may appeal because of its generality, while the PSF notation has the advantage of being automatically scaled and being expressed in terms that are familiar to designers who use safety factors.

3 Inverse measure calculation by MCS

Conceptually, the simplest approach to evaluate PSF or PPM is by MCS, which involves the generation of random sample points according to the statistical distribution of the variables. The sample points that violate the safety criteria in Equation (5a) are considered failed. Figure 3 illustrates the concept of MCS. A two-variable problem with a linear limit state function is considered. The straight lines are the contour lines of the limit state function and sample points generated by MCS are represented by small circles, with the numbered circles representing failed samples. The zero value of the limit state function divides the distribution space into a safe region and a failure region. The dashed lines represent failed conditions and the continuous lines represent safe conditions.

Figure 3 Illustration of the calculation of PPM with MCS for the linear performance function



The failure probability is estimated as the ratio of the number of samples failed to the total number of samples N :

$$P_f \approx \frac{\text{num}(G(\hat{\mathbf{X}}) \leq 0)}{N} \tag{11}$$

where $\hat{\mathbf{X}}$ is the randomly chosen sample point, $\text{num}(G(\hat{\mathbf{X}}) \leq 0)$ denotes the number of samples for which $(G(\hat{\mathbf{X}}) \leq 0)$ and N is the total number of trials. For example, in Figure 3, the number of sample points that lie in the failure region above the $G = 0$ curve is 12. If the total number of samples is 100,000, the failure probability is estimated at 1.2×10^{-4} . For a fixed number of samples, the accuracy of MCS deteriorates with the decrease in failure probability.

For example, with only 12 failure points out of the 100,000 samples, the standard deviation of the probability estimate is 3.5×10^{-5} , more than a quarter of the estimate. When the probability of failure is significantly smaller than one over the number of sample points, its calculated value by MCS is likely to be zero.

PPM is estimated by MCS as the n th smallest limit state function among the N sampled functions, where $n = N \times P_{\text{target}}$. For example, considering the example illustrated in Figure 3, if the target failure probability is 10^{-4} , to satisfy the target probability of failure, no more than 10 samples out of the 100,000 should fail. So, the focus is on the two extra samples that failed. PPM is equal to the value of the highest limit state function among the n (in this case, $n = 10$) lowest limit state functions. The numbered small circles are the sample points that failed. Of these, the three highest limit states are shown by the dashed lines. The tenth smallest limit state corresponds to the sample numbered 8 and has a limit state value of -0.4 , which is the value of PPM. Mathematically this is expressed as:

$$g^* = \text{nth min}_{i=1}^N (G(\mathbf{X}_i)) \quad (12a)$$

where, n th min is the n th smallest limit state function. So, the calculation of PPM in MCS requires only sorting the lowest limit state function in the MCS sample. Similarly, PSF can be computed as the n th smallest factor among the N sampled safety factors and is mathematically expressed as: ¹

$$s^* = \text{nth min}_{i=1}^N (S(\mathbf{X}_i)) \quad (12b)$$

Finally, probabilities calculated through MCS using a small sample size are computed as zero in the region where the probability of failure is lower than one over the number of samples used in MCS. In that region, no useful gradient information is available to the optimisation routine. On the other hand, PSF and PFM values varies in this region and thus provides guidance to optimisation.

MCS generates numerical noise due to limited sample size. Noise in failure probability may cause RBDO to converge to a spurious minimum. To filter out the noise, Response Surface Approximations (RSAs) are fitted to failure probability to create a so-called Design Response Surface (DRS). It is difficult to construct a highly accurate DRS because of the huge variation and uneven accuracy of failure probability. To overcome these difficulties, Qu and Haftka (2003, 2004) discuss the usage of PSF to improve the accuracy of the DRS. They showed that DRS based on PSF is more accurate compared to DRS based on failure probability and this accelerates the convergence of RBDO. For complex problems, RSAs can also be used to approximate the structural response to reduce the computational cost. Qu et al. (2004) employ PSF with MCS based on RSA to design stiffened panels under system reliability constraint.

4 Inverse measure calculation by moment-based methods

Moment-based methods provide for less expensive calculation of the probability of failure compared to MCS, although they are limited to a single failure mode. These methods require a search for the Most Probable Point (MPP) on the failure surface in the standard normal space. The First-Order Reliability Method (FORM) is the most widely used moment-based technique. FORM is based on the idea of the linear approximation of

the limit state function and is accurate as long as the curvature of the limit state function is not too high. When the limit state has a significant curvature, second-order methods can be used. The Second-Order Reliability Method (SORM) approximates the measure of reliability more accurately by considering the effect of the curvature of the limit state function (Melchers, 1999, pp.127–130). All the random variables are to be transformed to the standard normal variables with zero mean and unit variance.

Moment-based methods are employed to calculate the reliability index, which is denoted by β and related to the probability of failure as:

$$P_f = \Phi(-\beta) \quad (13)$$

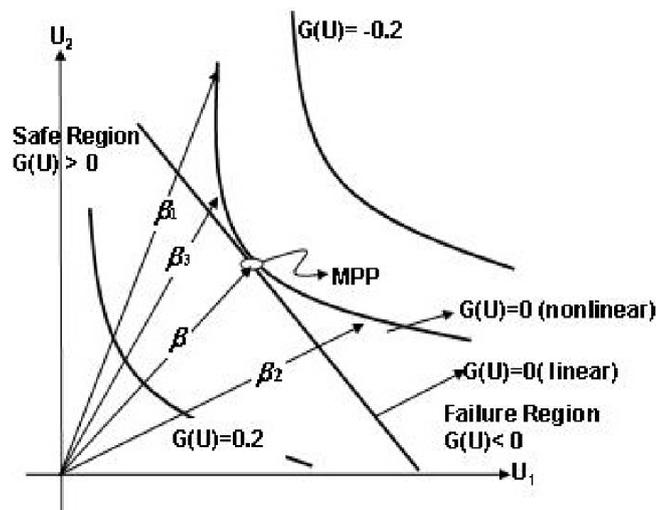
where Φ is the standard normal CDF. Respective target values of β and failure probabilities are also related in the same manner. In FORM, β can be calculated as $\beta = \|\mathbf{U}\|$, where \mathbf{U} is the vector of standard normal variates (variables with normal distribution of zero mean and unit variance). The standard normal variates are obtained from a transformation of the basic random variables \mathbf{X} , which could be non-normal and dependent. In the standard normal space the point on the first-order limit state function at which the distance from the origin is minimum is the Most Probable Point (MPP).

Figure 4 illustrates the concept of the reliability index and MPP search for a two-variable case in the standard normal space. In reliability analysis, concerns are first focused on the $G(\mathbf{U}) = 0$ curve. Next, among the various β values possible (denoted by $\beta_1, \beta_2, \beta_3$), the minimum β is sought. The corresponding point is the MPP. This process can be mathematically expressed as:

$$\begin{aligned} &\text{To find } \beta = \|\mathbf{u}^*\| \\ &\text{Minimise } \sqrt{\mathbf{U}^T \mathbf{U}} \\ &\text{s.t.: } G(\mathbf{U}) = 0 \end{aligned} \quad (14)$$

where \mathbf{u}^* is the MPP. The calculation of the failure probability is based on linearisation of the limit function at the MPP.

Figure 4 Reliability analysis and MPP

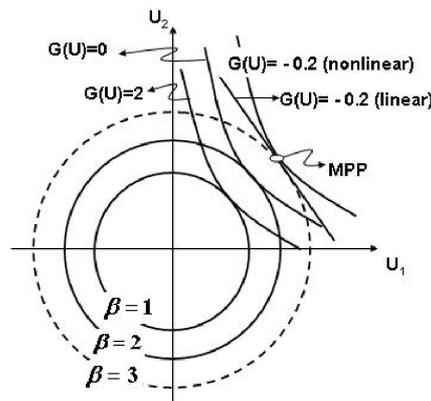


Inverse reliability measures can also be computed through moment-based methods. Figure 5 illustrates the concept of inverse reliability analysis and MPP search. The circles represent the β curves with the target β curve represented by a dashed circle. Here, among the different values of limit state functions that pass through the β_{target} curve, the one with minimum value is sought. The value of this minimal limit state function is the PPM as shown by Tu et al. (1999). The point on the target circle with the minimal limit state function is sought. This point is also an MPP and to avoid confusion between the usual MPP and MPP in inverse reliability analysis, Du et al. (2003) coined the term Most Probable Point of Inverse Reliability (MPPIR) and Lee et al. (2002) called it the Minimum Performance Target Point (MPTP). Du et al. (2003) developed the sequential optimisation and reliability analysis method in which they show that evaluating the probabilistic constraint at the design point is equivalent to evaluating the deterministic constraint at the MPP of the inverse reliability. This facilitates in converting the probabilistic constraint to an equivalent deterministic constraint. That is, the deterministic optimisation is performed using a constraint limit, which is determined based on the inverse MPP obtained in the previous iteration. Kiureghian et al. (1994) proposed an extension of the Hasofer-Lind-Rackwitz-Fiessler algorithm that uses a merit function and search direction to find the MPTP. In Figure 5, the value of the minimal limit state function or the PPM is -0.2 . This process can be expressed as:

$$\begin{aligned} &\text{Minimise: } G(\mathbf{U}) \\ &\text{s.t.: } \|\mathbf{U}\| = \sqrt{\mathbf{U}^T \mathbf{U}} = \beta_{\text{target}} \end{aligned} \tag{15}$$

In reliability analysis the MPP is on the $G(\mathbf{U}) = 0$ failure surface. In inverse reliability analysis, the MPP search is on the β_{target} curve.

Figure 5 Inverse reliability analysis and MPP for target probability of failure of 0.00135 ($\beta = 3$)



5 RBDO with inverse measures

Generally, RBDO problems are formulated as:

$$\begin{aligned} &\text{Minimise: cost function (design variables)} \\ &\text{s.t.: probabilistic constraint} \end{aligned} \tag{16}$$

The probabilistic constraint can be prescribed by several methods like the RIA, the PMA, the PSF approach, (see Table1).

Table 1 Different approaches to prescribe the probabilistic constraint

Method	RIA	PMA	PSF
Probabilistic constraint	$\beta \geq \beta_{\text{target}}$	$g^* \geq 0$	$s^* \geq 1$
Quantity to be computed	Reliability index (β)	PPM (g^*)	PSF (s^*)

In RIA, β can be computed as the product of reliability analysis as discussed in the Section 4. The PPM or PSF can be computed through inverse reliability analysis or as a byproduct of reliability analysis using MCS.

To date, most researchers have used RIA to prescribe the probabilistic constraint. However, the advantages of the inverse measures, illustrated in Section 6, have led to its growing popularity.

6 Beam design example

The cantilever beam shown in Figure 6, taken from Wu et al. (2001), is a commonly used demonstration example for RBDO methods. The length L of the beam is 100". The width and thickness is represented by w and t . It is subjected to end transverse loads \mathbf{X} and \mathbf{Y} in orthogonal directions as shown in the Figure 6.² The objective of the design is to minimise the weight or equivalently the cross sectional area: $A = wt$ subject to two reliability constraints, which require the safety indices for strength and deflection constraints to be larger than three. The first two failure modes are expressed as two limit state functions:

Stress limit:

$$G_s = \mathbf{R} - \sigma = \mathbf{R} - \left(\frac{600}{wt^2} \mathbf{Y} + \frac{600}{w^2 t} \mathbf{X} \right) \quad (17)$$

Tip displacement limit:

$$G_d = D_0 - D = D_0 - \frac{4L^3}{\mathbf{E}wt} \sqrt{\left(\left(\frac{\mathbf{Y}}{t^2} \right)^2 + \left(\frac{\mathbf{X}}{w^2} \right)^2 \right)} \quad (18)$$

where \mathbf{R} is the yield strength, \mathbf{E} is the elastic modulus, D_0 the displacement limit and w and t are the design parameters. \mathbf{R} , \mathbf{X} , \mathbf{Y} and \mathbf{E} are uncorrelated random variables and their means and standard deviations are defined in Table 2.

Figure 6 Cantilever beam subjected to horizontal and vertical random loads

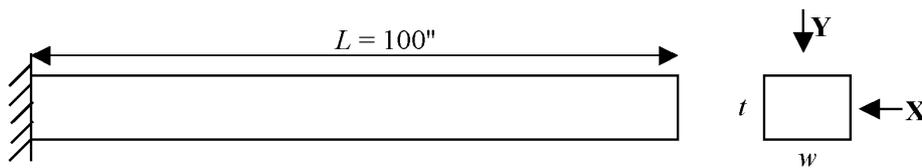


Table 2 Random variables for beam problem

Random variables	X (lb)	Y (lb)	R (psi)	E (psi)
Distribution (μ, σ)	Normal (500, 100)	Normal (1000, 100)	Normal (40,000, 2000)	Normal ($29 \times 10^6, 1.45 \times 10^6$)

6.1 Design for stress constraint

The design with strength reliability constraint is solved first, followed by the design with a system reliability constraint. The results for the strength constraint are presented in Table 3. The yield strength case has a linear limit state function and FORM gives reasonably accurate results for this case. The MCS is performed with 100,000 samples. The standard deviation in the failure probability calculated by MCS is:

$$\sigma_p = \sqrt{\frac{P_f(1-P_f)}{N}} \quad (19)$$

In this case, the failure probability of 0.0013 calculated from 100,000 samples has a standard deviation of 1.14×10^{-4} . It is seen from Table 3 that the designs obtained from RIA, PMA and PSF match well. Since the stress equation (17) is a linear function of random variables, the RIA and PMA are exact. The more conservative design from PSF is due to limited sampling of MCS.

Table 3 Comparison of optimum designs for the stress constraint. Minimise objective function $A = wt$ such that $\beta \geq 3$

Method	Reliability analysis FORM (RIA)	Inverse reliability analysis		Exact optimum (Wu et al., 2001)
		FORM (PMA)	MCS (PSF) (Qu and Haftka, 2003)	
Optima				
w	2.4460	2.4460	2.4526	2.4484
t	3.8922	3.8920	3.8884	3.8884
Objective function	9.5202	9.5202	9.5367	9.5204
Reliability index	3.00	3.00	3.0162	3.00
Failure probability	0.00135	0.00135	0.00128	0.00135

6.2 Comparison of inverse measures

The relation between PSF and PPM in Equation (10) is only approximate when PPM is calculated by FORM and PSF by MCS. PSF suffers from sampling error and PPM from error due to linearisation. For the linear stress constraint and with a large Monte Carlo sample, the difference is small, as seen in Table 4. It may be expected that the Minimum Performance Target Point (MPTP) should also be close to the point used to calculate PSF. This result may be useful because when a response surface is used for an approximation to the response, it is useful to centre it near the MPTP. To check the

accuracy of the MPTP estimation, the MPP of PPM and the point corresponding to the PSF are compared and the results are tabulated in Table 5. The coordinates corresponding to the PSF, computed by a million-sample MCS, deviate considerably from the MPTP. Since the accuracy of the points computed by MCS depends on the number of samples used, an increased number of samples lead to more accurate results, albeit at increased computational cost. Alternative approaches to obtain the points with better accuracy without increasing the number of samples are to average the coordinates computed by repeated MCS simulations with fewer numbers of samples. Alternatively, we can average a number of points that are nearly as critical as the PSF point. That is, instead of using only the \mathbf{x}_i corresponding to the S_n in Equation (12), we use also the points corresponding to $S_{n-p}, S_{n-p+1}, \dots, S_{n+p}$ in computing the PSF, where $2p$ is the total number of points that is averaged around the PSF. It can be observed from Table 5 that averaging 11 points around the PSF matches well with the MPTP, reducing a Euclidean distance of about 0.831 for the raw PSF to 0.277 with 11-point average. The values of \mathbf{X} , \mathbf{Y} and \mathbf{R} presented in Table 5 are in the standard normal space.

Table 4 Comparison of inverse measures for $w = 2.4526$ and $t = 3.8884$ (stress constraint)

Method	FORM	MCS (1×10^7 samples)
P_f	0.001238	0.001241
Inverse measure	PPM: 0.00258	PSF: 1.002619

Table 5 Comparison of MPP obtained from calculation of PSF for $w = 2.4526$ and $t = 3.8884$ (stress constraint)

Coordinates	\mathbf{X}	\mathbf{Y}	\mathbf{R}
MPTP	2.1147	1.3370	-1.6480
PSF			
1) 10^6 samples	2.6641	1.2146	-1.0355
2) Average: 10 runs of each 10^5 samples	2.1888	1.8867	-1.1097
3) Average: 11 points around PSF of 10^6 Samples	2.0666	1.5734	-1.5128

6.3 Use of PSF in estimating the required change in weight to achieve a safe design

The relation between the stresses, displacement and weight for this problem is presented to demonstrate the utility of PSF in estimating the required resources to achieve a safe design. Consider a design with the dimensions of $A_0 = w_0 t_0$ with a PSF of s_0^* less than one. The structure can be made safer by scaling both w and t by the same factor c . This will change the stress and displacement expressed in Equations (17) and (18) by a factor of c^3 and c^4 , respectively and the area by a factor of c^2 . If stress is most critical, it will scale as c^3 and the PSF will vary with the area, A as:

$$s^* = s_0^* \left(\frac{A}{A_0} \right)^{1.5} \quad (20)$$

Equation (20) indicates that a 1% increase in area will increase the PSF by 1.5%. As non-uniform scaling of width and thickness may be more efficient than uniform scaling, this is a conservative estimate. Thus, for example, considering a design with a PSF of 0.97, the safety factor deficiency is 3% and the structure can be made safer with a weight increase of less than 2%, as shown by Qu and Haftka (2003). For a critical displacement state, s^* will be proportional to A^2 and a 3% deficit in PSF can be corrected in under 1.5% weight increase. While for more complex structures we do not have analytical expressions for the dependence of the displacements or the stresses on the design variables, designers can usually estimate the weight needed to reduce stresses or displacements by a given amount.

6.4 Design for system reliability by MCS and PSF

MCS is a good method to use for system reliability analysis with multiple failure modes. The allowable deflection was chosen to be 2.25" to have competing constraints (Wu et al., 2001). The results are presented in Table 6. It can be observed that the contribution of the stress mode to failure probability dominates the contribution due to displacement and the interaction between the modes. The details of the design process are provided in Qu and Haftka (2004). They demonstrated the advantages of using PSF as an inverse safety measure over probability of failure or the safety index as a normal safety measure. They showed that the DRS of the PSF was much more accurate than the DRS for the probability of failure. For a set of test points the error in the probability of failure was 39.11% from the DRS to the PSF, 96.49% for the DRS to the safety index and 334.78% for the DRS to the probability of failure.

Table 6 Design for system reliability by PSF, Qu and Haftka (2004)^a

<i>Optima</i>	<i>Objective function</i>	P_f	<i>Safety index</i>	P_{f1}	P_{f2}	$P_{f1} \cap P_{f2}$	PSF_{s^*}
$w = 2.604$ 1	9.5691	0.001289	3.01379	0.001133	0.000208	0.000052	1.0006
$t = 3.6746$							

^a100,000 samples, P_{f1} , P_{f2} , $P_{f1} \cap P_{f2}$ – failure probabilities due to stress constraint, displacement constraint and intersection between the modes, respectively.

6.4 Design for system reliability by moment-based method and PPM

Moment-based methods are restricted to address single mode failures. A common way of addressing multimode failures using moment-based methods is to prescribe reliability indices for each failure modes. Tu et al. (1999) adopted this procedure to solve system reliability problems using PMA. They showed that the PPM helps accelerate the convergence. The system reliability for the beam design example with separate reliability indices for the stress mode and displacement mode are performed by RIA and PMA and the results are presented in Table 7 for RIA and Table 8 for PMA. The convergence of the objective function is shown in Figure 7. Comparing Tables 7 and 8, it is clear that the optimal values converge more quickly in PMA compared to RIA. Figure 7 illustrates that there is a smoother convergence in PMA compared to RIA.

Table 7 Design for system reliability using RIA. Minimise objective function $A = wt$ such that $\beta_1 \geq 3$ and $\beta_2 \geq 3$

Iteration	Objective function	w	t	G_s	G_d
0	25.00000	5.00000	5.00000	17.0000	14.4737
1	10.37935	3.22170	3.22170	1.05001	1.80263
2	9.388570	3.01922	3.10961	-0.51283	-0.22049
3	9.837700	3.05509	3.22010	0.25643	0.93133
4	9.566030	2.93753	3.25648	-0.12697	0.55007
5	9.319250	2.33623	3.98900	-0.32482	-0.70387
6	9.343100	3.33128	2.80466	-0.92653	-1.79320
7	10.35260	3.26945	3.16647	0.96194	1.52101
8	9.484850	3.16068	3.00089	-0.47394	-0.54209
9	9.881950	3.16180	3.12541	0.24075	0.63704
10	9.634850	3.07532	3.13296	-0.11994	0.27740
11	9.261750	2.44827	3.78298	-0.41620	-0.36921
12	9.654420	2.52917	3.81722	0.20960	0.45939
13	9.456370	2.48175	3.81037	-0.10400	0.01784
14	9.556000	2.49435	3.83106	0.05513	0.20102
15	9.503700	2.47832	3.83474	-0.02757	0.07240
16	9.531700	2.48254	3.83950	0.01704	0.12597
17	9.515450	2.47788	3.84016	-0.00870	0.08711
18	9.525750	2.47901	3.84257	0.00775	0.10534
19	9.518280	2.47638	3.84363	-0.00405	0.08576
20	9.524250	2.47683	3.84534	0.00548	0.09559
21	9.518870	2.47318	3.84884	-0.00294	0.07522
Optimum	9.518870	2.47318	3.84884	-0.00294	0.07522

Figure 7 Convergence of objective function in (a) RIA and (b) PMA

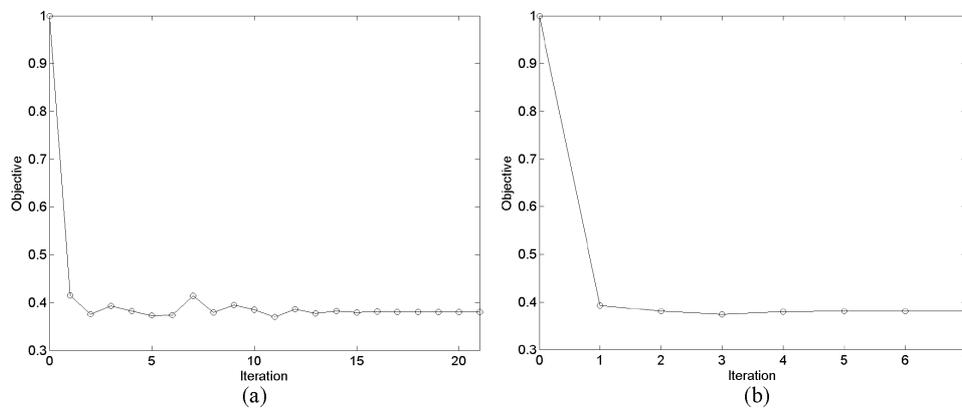


Table 8 Design for system reliability using PMA. Minimise objective function
 $A = wt$ such that $g_1^* \geq 0$ and $g_2^* \geq 0$

Iteration	Objective function	w	t	G_s	G_d
0	25.0000	5.00000	5.00000	0.75742	0.85363
1	9.81975	3.13366	3.13366	0.01462	0.05128
2	9.51375	2.84095	3.34881	-0.01442	0.05584
3	9.34600	2.52977	3.69435	-0.02924	-0.00178
4	9.50050	2.42565	3.91663	-0.00317	-0.01491
5	9.52350	2.44235	3.89926	0.00049	-0.00374
6	9.51975	2.45319	3.88055	0.00009	0.00044
7	9.51769	2.45354	3.87917	0.00041	0.00028
Optimum	9.51769	2.45354	3.87917	0.00041	0.00028

7 Concluding remarks

This paper described various inverse measures and their usage for RBDO. In particular, the relationship between two inverse safety measures, PPM and PSF was established. The computation of inverse measure by MCS and moment-based techniques was discussed. Several advantages of inverse measures were illustrated. They can accelerate convergence in RBDO, increase the accuracy of DRSs and maintain accuracy with MCS even when the failure probability is very low. Moreover, inverse measures can be employed to estimate the additional cost required to achieve the target reliability. These features of inverse measure make it a valuable resource in RBDO. A simple beam example was used to demonstrate some of the benefits.

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Note

¹A more accurate estimate of PPM or PSF will be obtained from the average of the n th and $(n+1)$ th smallest values. So in the case of Fig. 3, PPM is more accurately estimated as 0.35.

²For this example, the random variables are shown in bold face.