Micromechanical Analysis of Composite Truss-core Sandwich Panels for Integral Thermal Protection Systems

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A composite truss-core sandwich panel is investigated as a potential candidate for an Integral Thermal Protection System (ITPS). This multi-functional ITPS concept will protect the space vehicle from extreme reentry temperatures and will possess load carrying capabilities. The truss core is composed of two thin flat sheets that are separated by two inclined plates. Advantages of this new ITPS concept are discussed. The sandwich structure is idealized as an equivalent orthotropic thick plate continuum. The extensional stiffness matrix [A], coupling stiffness matrix [B], bending stiffness [D] and the transverse shear stiffness terms $A_{44}$ and $A_{55}$ are calculated through a strain energy and axes transformation approach. Using the Shear Deformable Plate Theory (SDPT) a closed form solution of the plate response was derived. The behavior of the stiffness and maximum plate deflection due to changing the web angle inclination is discussed. The calculated results, which require significantly less computational effort and time, agree well with the 3D finite-element analysis. The study indicates that panels with rectangular webs resulted in a weak extensional, bending, and $A_{55}$ stiffness and that maximum plate deflection was greatest for $48^\circ$ web angle configuration. The micromechanical analysis procedures developed in this study is to determine the unit cell stresses for each component (isotropic or composite) of the truss (face or web) that is caused by a uniform pressure load.

Nomenclature

2p = unit cell length

$\bar{d}$ = height of the sandwich panel (centerline to centerline)

t_{TF} = top face sheet thickness

t_{TB} = bottom face sheet thickness

t_w = web thickness

$\theta$ = angle of web inclination

e = component index of the truss

$\{D\}^{(e)}$ = deformation vector of the $i^{\text{th}}$ component (micro deformation)

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\[ \{D\}^M = \text{deformation vector of the unit cell (macro deformation)} \]
\[ [T_p]^y = \text{Deformation transformation matrix of the } i^{th} \text{ component of the truss} \]
\[ P_z = \text{Pressure load acting on the 2D orthotropic panel} \]
\[ l = \text{length of the cantilever beam} \]
\[ F_i^{(m)} = \text{nodal force on the FEM model} \]
\[ \psi_y(x, y) = \text{panel assumed rotation in the } y^-\text{direction} \]
\[ \psi_x(x, y) = \text{panel assumed rotation in the } x^-\text{direction} \]
\[ w(x, y) = \text{panel assumed deflection} \]
\[ \tau_{yy} = \text{local shear stress on the webs} \]
\[ \varepsilon_o = \text{midplane strain} \]
\[ \kappa = \text{curvature} \]
\[ \bar{y} = \text{local axis of the web} \]
\[ s = \text{web length} \]
\[ U = \text{unit cell strain energy} \]
\[ Q_{o}, Q_{y} = \text{shear force on the unit cell} \]
\[ Q_{ij} = \text{transformed lamina stiffness matrix} \]

**I. INTRODUCTION**

Reducing the cost of launching a spacecraft is one of the critical needs in the space industry. Government and private corporations use space for various objectives such as reconnaissance, communications, weather monitoring, military, and experimental purposes. With every launch the government or corporation spends a significant amount of money to launch their payload into space. One of NASA’s goals is to reduce the cost of delivering a pound of payload into a low earth orbit by an order of magnitude [1]. One of the most expensive systems of a space vehicle is the thermal protection system (TPS), which protects the vehicle from the high thermal loads during re-entry, therefore deserves some special attention [2].

There is a considerable interest to develop a launch vehicle by the end of the decade. The goal of the vehicle is to lower the cost of access to space so as to promote the creation and delivery of new space services and other activities that will improve economic competitiveness [3]. The future space vehicle requires a more advanced TPS than the one currently used. The TPS has to be lightweight and multifunctional. Not only does the TPS have to offer protection from the extreme temperatures during reentry but it also must offer structural integrity and load bearing capabilities. Other desirable characteristics of a TPS are that it must be low maintenance and have a low life-cycle cost. Currently the TPS on the Space Shuttle requires 40,000 man-hours of maintenance between typical flights [1]. This new TPS concept can be accomplished by using recently developed metallic foams and also innovative core materials, for example, truss core in sandwich construction. The Integral TPS/structure (ITPS) design can significantly reduce the overall weight of the vehicle as the TPS/structure performs the load-bearing function also.

A structural sandwich panel is a three-layer plate, consisting of two face sheets and a core. Two thin, stiff and strong faces are separated by a thick, light and weaker core [4]. Such construction provides high strength-to-weight ratio and it promises high stiffness. Traditionally, sandwich structures are made up of two face sheets and a core that is made from expanded materials such as, metallic foil, plastic foams and composite in the shape of hexagons with vertical walls (honeycomb). Several types of conventional sandwich structures have been proposed and investigated by various researchers (Libove, Fung, and Hubka [5, 6]). More recently developed truss-core sandwich panels have been investigated by Lok et al [7] and Valdevit et al [8]. Lok et al [7], Valdevit et al [8] investigated and analyzed metallic truss-core sandwich panels. Lok [9, 7] developed and investigated the behavior of equivalent stiffness parameters for a truss core sandwich panel, as

![Figure 1. Truss-core Sandwich Panel](image)
well as verified and compared his results with FEM and conventional sandwich forms that were investigated by Libove, Fung and Hubka [5,10]. Evans, Hutchinson and Wicks analyzed and optimized metallic sandwich panels with prismatic cores [8] and trusses [11, 12] subjected to shear and moment forces.

An understanding as to the composite TPS’s performance and behavior needs to be investigated. Composite truss core sandwich structures will be investigated in this paper for use in multifunctional structures for future space vehicles (Fig. 1). This type of ITPS would insulate the vehicle from aerodynamic heating as well as carry primary vehicle loads. The advantages of using such a structure are that it is lightweight, multifunctional, for example offer insulation as well as load bearing capabilities, low maintenance, and low life-cycle cost. The truss-core sandwich panel is comprised of several unit cells. The unit cell consists of two thin face sheets and an inclined web made up of composite laminates. Commonly used materials for facings and core are ceramic matrix composite laminates and metals [13]. The composite truss core will be filled with Saffil®, which is a non-load-bearing insulation made of aluminum fibers. The finite element method (FEM) is frequently used to analyze sandwich structures. Shell elements are often preferred for the truss core to construct a 3D FEM model. However, the number of elements and nodes needed to adequately represent the sandwich panel can be excessive; as a result a 3-D FEM model is uneconomic to conduct. Such panels may also be represented as a thick plate that is continuous, orthotropic, and homogenous for which analytical and 2D FEM solutions [14] are available.

The objective of the current research work is to establish an analysis procedure that can be used on the design of the ITPS. The analytical models will be compared with expensive and detailed finite element analysis. The analytical models were refined so that the errors in prediction of the critical metrics are within 5%. A detailed formulation and description of the extensional, coupling, bending, and shearing stiffness of the ITPS panel are presented for a unit cell by representing the sandwich panel as an equivalent thick plate, which is homogeneous, continuous, and orthotropic with respect to the \(x\)- and \(y\)-directions. A strain energy approach is used in deriving the analytical equations of the extensional, bending, coupling and shearing stiffness. Fung et al [15, 16] and Libove et al [6] used the equivalent homogenous model approach to derive the elastic constants of Z-core, C-core, and corrugated core sandwich panels. By using bending stiffness in the shear deformable plate theory, the maximum deflection of the truss-core panel is obtained as well as strains and stresses in the unit cell. The stiffness of the unit cell was verified with finite element formulation. By selecting appropriate shape, dimensions and material of the face sheets and core, outstanding stiffness and strength at low weight of the sandwich are achieved [17, 18]. The behavior of stiffness constants and maximum plate deflection to a change in web angle inclination are presented.

II. GEOMETRIC PARAMETERS

Consider the truss-core unit cell in Fig. 2 below. The \(z\)-axis is in the thickness direction of the ITPS panel. The stiffer longitudinal direction is parallel to the \(y\)-axis, and \(x\)-axis is in the transverse direction. The unit cell consists of two inclined webs and two thin face sheets. The unit cell is symmetric with respect to the \(yz\)-plane. The upper face plate thickness, \(t_{\text{TP}}\), can be different from the lower plate thickness, \(t_{\text{BF}}\), as well as the web thickness, \(t_w\). The unit cell can be identified by six geometric parameters \((p, d, t_{\text{TP}}, t_{\text{BF}}, t_w, \theta)\), (Fig. 2). Four other dimensions \((b_c, d_c, s, f)\) are obtained from geometric considerations. The equations for these relationships are given below:

\[
d_c = d - \frac{1}{2} t_{\text{TP}} - \frac{1}{2} t_{\text{BF}}
\]

\[
f = \frac{1}{2} \left( p - \frac{d_c}{\tan \theta} \right)
\]

\[
b_c = p - 2f
\]

\[
s = \sqrt{d_c^2 + b_c^2} = \frac{d_c}{\sin \theta} = \frac{b_c}{\cos \theta}
\]
The ratio $\frac{f}{p} = 0$, corresponds to a triangular truss core, and $\frac{f}{p} = 0.5$, corresponds to a rectangular truss core.

![Figure 2. Dimensions of the unit cell.](image)

III. ANALYSIS

Our objective is to determine the equivalent stiffness properties of the ITPS panel by idealizing it as a homogeneous orthotropic plate. The extensional stiffness matrix $[A]$, coupling stiffness matrix $[B]$, bending stiffness $[D]$ and the transverse shear stiffness terms $A_{44}$ and $A_{55}$ are calculated by analyzing the unit cell. Then they can be used in the first order shear deformable plate theory (FSDT) to determine their response to mechanical and thermal loads. Typically, plate analyses yield information on deflections, and force and moment resultants at any point on the plate. We will again use the micromechanical analysis procedures developed in this study to determine the stresses (micro-stresses) in the face sheets and the webs. Then failure theories such as Tsai-Hill criterion can be used to determine if the stresses are acceptable or not.

In the derivation of the stiffness parameters the following assumptions were made:

1. The deformation of the panel is small.
2. The panel dimensions in the $y$-direction are much larger than the width $2p$.
3. The face sheets are thin with respect to the core thickness.
4. The core contributes to bending stiffness in about $x$-axis but not the $y$-axis.
5. During distortion of the plate, straight lines normal to the mid-plane of the plate remain straight, but not necessarily normal to the mid-plane. That is due to the transverse strains, which can be significant for the sandwich panel because of the relatively flexible core [16].
6. The face and web plate laminates are symmetric with respect to their own mid-plane.
7. The top and bottom face sheet thicknesses and laminate stacking sequence are equal.
8. The core is sufficiently stiff so that the elastic modulus in the $z$-direction is assumed to be infinite for the equivalent plate. Local buckling of the facing plates does not occur and the overall thickness of the panel is constant.

Previous researchers adopted these assumptions in the derivation of stiffness parameters of sandwich panels with corrugated core (Libove and Hubka [19], C-core, Fung et al. [15], and Z-core [16]). Assumptions 6, and 7 were made for simplicity purposes but the analysis can be easily extended to unsymmetric constructions. The in-plane and out-of-plane stiffness governing the elastic response of a shear-deformable sandwich panel are defined in the context of laminated plate theory incorporating FSDT described by Vinson [17] and Whitney [20]. The appropriate stiffness of the orthotropic plate may be obtained by comparing the behavior of a unit cell of the truss-core sandwich panel with that of an element of the idealized homogeneous orthotropic plate, (Fig. 3).
Figure 3. Equivalent Orthotropic thick plate for the unit cell truss-core sandwich panel.

The in-plane extensional and shear response and out-of-plane (transverse) shear response of an orthotropic panel are governed by the following constitutive relation:

\[
\begin{bmatrix}
N_x \\
N_y \\
Q_x \\
Q_y \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & 0 & 0 & 0 & 0 & 0 \\
A_{12} & A_{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_{44} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_{55} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & D_{11} & D_{12} & 0 \\
0 & 0 & 0 & 0 & D_{12} & D_{22} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & D_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy} \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]

or \( \{F^e\} = [K]\{D\} \)  

Where \( \varepsilon \) and \( \gamma \) are the normal and shear strains, \( \kappa \) are the bending and twisting curvatures. Advanced knowledge of the bending, twisting, and shearing stiffness of the orthotropic thick plate is essential for the successful implementation of the FSDT. From assumptions 6 and 7 the coupling stiffness matrix \( [B] = 0 \).

A. Extensional and Bending Stiffness

An analytical method was determined to calculate the stiffness matrix of the truss-core sandwich panel. Consider a unit cell made up of four composite laminates (two face sheets and two webs). Each laminate has its respective material properties, and \( ABD \) matrix. The \( ABD \) matrix of each component needs to be combined together in an appropriate manner to create an overall stiffness of the sandwich panel. The determination of the \( ABD \) matrix of a composite laminate is given below [21].

\[
\begin{bmatrix}
A^e \gamma, B^e \gamma, D^e \gamma
\end{bmatrix} = \int_{-\frac{1}{2}}^{\frac{1}{2}} (Q^e \gamma)_k [1, z, z^2] dz = \sum_{k=1}^{N} (Q^e \gamma)_k \left[ \left( z_k - z_{k-1} \right), \left( \frac{z_k^2 - z_{k-1}^2}{2} \right), \left( \frac{z_k^3 - z_{k-1}^3}{3} \right) \right]
\]

\( N \) is the number of lamina in the composite and \( Q^e \gamma \) are the components of the transformed lamina stiffness matrix. Where \( e = 1 \) to 4, (1 = top face sheet, 2 = bottom face sheet, 3 = left web, 4 = right web). The overall stiffness of the unit truss-core was determined by imposing unit mid-plane strains and curvature (macro deformation) to the unit cell and then calculating the corresponding mid-plane strains and curvatures (micro deformations) in each component. The unit truss-core components are the two face sheets and two webs. A transformation matrix relates the macro- and micro-deformations as follows:

\[
\{D\}^{(e)} = [T_D]^{(e)} \{D\}^{M}
\]

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\( T_D^{(e)} \) is the deformation transformation matrix that relates macro deformations to micro deformation, \( \{D\}^{(e)} \) is the micro deformation in each component, and \( \{D\}^M_D \) is the macro deformation in the unit cell.

**B. Formulation of Deformation Transformation Matrix, Face Sheets**

The deformation transformation matrix of the top face sheet was determined by first considering the unit cell under the action of mid-plane macro strains, \( \varepsilon_{xo}, \varepsilon_{yo}, \gamma_{xyo} \) and macro curvature \( \kappa_x, \kappa_y, \kappa_{xy} \). Each strain and curvature was considered by itself and the resulting micro strains and curvatures were derived. Shown below is the transformation matrix of the top and bottom face sheets.

**Top face sheet:**

\[
\{D\}^{(1)} = T_D^{(1)} \{D\}^M_D
\]

\[
\begin{bmatrix}
\varepsilon_{xo} \\
\varepsilon_{yo} \\
\gamma_{xyo} \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}^{(1)} =
\begin{bmatrix}
1 & 0 & 0 & \frac{d}{2} & 0 & 0 \\
0 & 1 & 0 & 0 & \frac{d}{2} & 0 \\
0 & 0 & 1 & 0 & 0 & \frac{d}{2} \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xo} \\
\varepsilon_{yo} \\
\gamma_{xyo} \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}^{(M)}
\]

**Bottom face sheet:**

\[
\{D\}^{(2)} = T_D^{(2)} \{D\}^M_D
\]

\[
\begin{bmatrix}
\varepsilon_{xo} \\
\varepsilon_{yo} \\
\gamma_{xyo} \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}^{(2)} =
\begin{bmatrix}
1 & 0 & 0 & -\frac{d}{2} & 0 & 0 \\
0 & 1 & 0 & 0 & -\frac{d}{2} & 0 \\
0 & 0 & 1 & 0 & 0 & -\frac{d}{2} \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xo} \\
\varepsilon_{yo} \\
\gamma_{xyo} \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}^{(M)}
\]

There was a 1:1 relationship between mid-plane macro and micro strain as well as a 1:1 relationship between macro and micro curvature as indicated by unity along the diagonal of the transformation matrices. Using the assumptions that the in-plane displacements \( u \) and \( v \) are linear functions of the z-coordinate and that the transverse normal strain \( \varepsilon_z \) is negligible [21] the \( d/2 \) factor was used to relate the macro curvatures to the mid-plane micro strains.

**C. Formulation of Deformation Transformation Matrix, Webs**

1. **Right Web**

Formulation of the deformation transformation matrix for the webs is relatively complicated because of the need for coordinate transformation due to the inclination of the webs. Consider a global \( XYZ \) coordinate axis and a local \( XYZ \) coordinate axis (Fig. 4).
The transformation from global to local coordinate axes requires a rotation and translation. The transformation from global to local displacements only requires a rotation.

\[
\begin{bmatrix}
 x \\
 y \\
 z
\end{bmatrix} = \begin{bmatrix}
 1 & 0 & 0 \\
 0 & \cos \theta & \sin \theta \\
 0 & -\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
 \bar{x} \\
 \bar{y} \\
 \bar{z}
\end{bmatrix} + \begin{bmatrix}
 0 \\
 f \\
 \frac{d_f}{\bar{z}}
\end{bmatrix} \tag{9a}
\]

\[
\begin{bmatrix}
 \bar{u} \\
 \bar{v} \\
 \bar{w}
\end{bmatrix} = \begin{bmatrix}
 1 & 0 & 0 \\
 0 & \cos \theta & -\sin \theta \\
 0 & \sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
 u \\
 v \\
 w
\end{bmatrix} \tag{9b}
\]

where \(\theta\) is the angle of web inclination of the right web, the first matrix is the rotation matrix and the second vector is a translation vector.

Consider the unit cell of the panel under the action of mid-plane macro strains, \(\varepsilon_{x0}, \varepsilon_{y0}, \gamma_{xy}\), and macro curvature \(\kappa_x, \kappa_y, \kappa_{xy}\). From assumption 4 above we can be noted that \(\varepsilon^{(3,4)}_{x0} = 0\) and \(\varepsilon^{(3,4)}_{y0} = 1\) when the unit truss core is subjected to \(\varepsilon_{y0}^M = 1\) and \(\varepsilon_{x0}^M = 1\).

The micro strains on the webs due to a macro curvature are more complex to determine; therefore a detailed discussion is appropriate, see Appendix A. The micro strains and curvature in the right web due to a macro unit curvature along the \(y\)-axis (\(\kappa_y\)) will be derived. All other curvatures will be zero. The definition of curvature is shown below.

\[
\kappa_x = -\frac{\partial^2 w}{\partial x^2} = 0 \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2} = 1 = \kappa_o \quad \kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} = 0 \tag{10}
\]

Starting with Eq. (10) and following the detailed derivation in Appendix A leads to the transformation matrix for the left and right web.
From Eq. (12) and Eq. (11) we observed that micro mid-plane strains are a function of $y$. 

D. Stiffness Matrix Determination Through Strain Energy Approach

As the material is deformed by the unit macro strains and curvature, it stores energy internally throughout its volume. The total strain-energy in the unit cell is the sum of all the individual strain energy (i.e. faces and webs),

$$ U^M = \frac{1}{2} (2p)^2 \begin{bmatrix} \{D\}^M \end{bmatrix}^T [K] [D]^M = \sum_{e=1}^{4} U^{(e)} $$

where $U^M$ is the total strain-energy and $U^{(e)}$ is the strain energy of the $e^{th}$ component.

$$ U^{(e)} = \int_{A^{(e)}} \frac{1}{2} \begin{bmatrix} \{D\}^{(e)} \end{bmatrix}^T [K]^{(e)} [D]^{(e)} dA^{(e)} $$

where $A^{(e)}$ denotes the area of the $e^{th}$ component. The determination of the total strain energy in the right web will be derived. Since the deformation of the webs is a function of $\bar{y}$, integration in Eq. (14) is performed with respect to $\bar{y}$,

$$ U^{(4)} = (2p) \int_{0}^{\bar{y}} \frac{1}{2} \begin{bmatrix} \{D\}^{(4)} \end{bmatrix}^T [K]^{(4)} [D]^{(4)} d\bar{y} $$
\( \{D\}^{(4)} \) is the deformation vector of the right web and \([K]^{(4)} \) is the ABD matrix of the right web composite laminate. The integration bounds are from \( \theta \) to \( s \), \( s \) being the length of the webs. Substituting Eq. (12) into Eq. (15) yields

\[
U^{(4)} = \frac{1}{4} (2p) \int_0^s \left[ T_D^{\tau} \{D\}^M \right] \left[ K \right] \left[ T_D^{\tau} \{D\}^M \right] dy
\]

In general we write the strain energy in each laminate in terms of the global deformation \( \{D\}^M \) as

\[
U^{(c)} = \frac{1}{2} (D^M)^T K^{(c)} D^M
\]

Then the stiffness matrix \( K \) of the idealized orthotropic panel can be derived as

\[
K = \sum_{e=1}^4 K^{(e)}
\]

### E. Formulation of Transverse Shear Stiffness \( A_{55} \)

For a corrugated core sandwich structure loaded in shear transverse to the corrugations (by shear stress \( \tau_{xz} \) or shear force \( Q_x \)), it is recognized that the face sheets and core will undergo bending deformation \([6, 22]\). For the determination of \( A_{55} \), the shear stress in the face sheets are neglected because of its small thickness. To determine the shearing stiffness due to \( Q_x \), we must first identify the shear stresses in the webs due to \( Q_x \). Figure 5 shows a free body diagram of the truss-core panel unit of length \( dx \) in the \( x \)-direction where only the stress and body-force components which act in the \( x \)-direction are shown and considered. The stress values shown are average stresses over the faces of an element which is assumed to be very small. A summation of the forces in the \( x \)-direction yields

\[
(F + \frac{\partial F}{\partial x} \Delta x - F) \Delta y \Delta z + 2(\tau_{xz} \Delta x (\frac{L_z}{d})) \Delta y \Delta z = 0
\]

Following the procedure in Appendix B, we determined the shear stresses in the webs \( \tau_{xy} \) due to \( Q_x \).

The shear strain energy density (strain energy per unit area of the sandwich panel) can be calculated either from the micro shear stresses given in Eq. (B6), Appendix B, or from the shear force \( Q_x \) and yet to be determined shear stiffness \( A_{55} \). By equating the two shear strain energy density terms we obtain

\[
U_s = \frac{t}{P} \int_0^s \frac{1}{2} \left( \frac{\tau_{xy}}{G_{xy}} \right)^2 dy = \frac{Q_x^2}{2A_{55}}
\]

Using Eq. (19), the equivalent shear stiffness, \( A_{55} \), of the sandwich panel was solved.
F. Formulation of the Transverse Shear Stiffness, $A_{44}$

Formulation of the transverse shear stiffness $A_{44}$ of the panel is relatively complicated [5] because certain conditions need to be fulfilled. Figure 6(a) shows a sandwich panel of unit length in the $x$-direction subjected to unit transverse shear, $Q_y=1$. The horizontal force $Y=p/d$ provides equilibrium.

Point A in Fig. 6b is assumed to be fixed to eliminate rigid body movements of the unit cell. The relative displacements $\delta_y$ and $\delta_z$ will result from the transverse shearing and horizontal force. Because the force is small the displacements will be proportional to $Q_y$, thus an average shear strain is represented as

$$\gamma_y = \frac{\delta_y}{d} + \frac{\delta_z}{p}$$

Due to antisymmetry only half of the unit cell needs to be considered for analysis, Fig 7a. The unit shear force resultant is divided into a force $P$ acting on the top face sheet, and $R$ acting on the lower face sheet. A shear force $F$ is assumed to act on the top face sheet at point A where there are no horizontal forces due to antisymmetry, and a force $(1-F)$ was determined through a summation of the forces in the $z$-direction. Under the action of all these forces in the half unit cell the displacements are represented in Fig 7b.

From Figs. 7a,b we observed that there are three unknown forces and five displacements that need to be solved. These forces and displacements can be solved through the energy method. The total strain energy in half the unit cell is the sum of the individual strain energies from each member (i.e. AB, BC, DE, BE, etc…). The strain energy due to a bending moment is considered while the strain energy due to shear and normal forces are neglected. The total strain energy in Fig. 7a is shown in Appendix C.

Using Castigliano’s theorem, Eq. (21), which states that displacement is equal to the first partial derivative of the strain energy in the body with respect to the force acting at the point and in the direction of displacement [19], the unknown forces and displacements can be determined.

$$\delta_i = \frac{\partial U_i}{\partial P_i}$$
Because the overall thickness of the sandwich panel remains constant during distortion, the boundary conditions are $\delta_z^C = \delta_z^G$ and $\delta_z^I = 0$. Since half the unit cell is under unit shear then $P + R = 1$. The two boundary conditions along with Castigiliano’s second theorem lead to a system of two linear equations with two unknowns.

$$\frac{\partial U_z}{\partial F} = 0$$  \hspace{1cm} (22)

$$\frac{\partial U_z}{\partial P} = \frac{\partial U_z}{\partial R}$$  \hspace{1cm} (23)

Substituting Eq. (C1) from Appendix C into Eq. (22), Eq. (23) and solving the system of linear equations the unknown forces $P$, $F$, $R$ were determined. The solution to the forces $P$, $F$, and $R$ is quite lengthy so it was omitted in this abstract. Using Eq. (C1) from Appendix C and Eq. (21) along with the values of the unknown forces, the displacements were determined, refer to Appendix C. The displacement of the half the unit cell are $\delta_y = \delta_y^C + \delta_y^G$ and $\delta_z = \delta_z^C = \delta_z^G$ in the $y$ and $z$ directions. Utilizing the force distortion relationships developed by Libove et al [10] to describe the elastic behavior of an orthotropic thick plate; the transverse shear stiffness $A_{44}$ was obtained as follows:

$$A_{44} = \frac{Q_y}{\gamma_y} \frac{1}{d} \left( \frac{\delta_y}{\delta_p} + \frac{\delta_z}{\delta_p} \right) = \frac{1}{d} \left( \frac{\delta_y^C + \delta_y^G}{\delta_p} + \frac{\delta_z^C}{\delta_p} \right)$$  \hspace{1cm} (24)

IV. DEFLECTION OF THE ORTHOTROPIC PANEL

ITPS panels form the outer skin of the vehicle, which covers the crew compartment. The ITPS panel experiences thermal forces and moments due to the extreme reentry temperatures as well as a pressure load which comes from the pressurized crew compartment. All those conditions cause the panel to deflect. Knowing the panel deflection and behavior is important because excessive deflection of the panel can lead to extremely high local aerodynamic heating. A 2D plate analysis is needed to determine the behavior of the ITPS when it is subjected to those various conditions. Consider a simply supported orthotropic sandwich panel of width $b$ ($y$-direction) and length $a$ ($x$-direction) as illustrated in Figure 3, the boundary conditions may be described as [24]:

$$x = [0, a]: w = 0, \frac{\partial w}{\partial y} = 0$$  \hspace{1cm} (25a)

$$y = [0, b]: w = 0, \frac{\partial w}{\partial x} = 0$$  \hspace{1cm} (25b)

The panel is subject to a pressure load

$$P_z = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right)$$  \hspace{1cm} (26)

where $P_{mn} = \frac{16P_0}{\pi^2 mn}$ for uniform loads. The panel is also assumed to have the following deflections:

$$w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right)$$  \hspace{1cm} (27a)
\[ \psi_x(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cos \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \]  
\quad (27b)

\[ \psi_y(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \]  
\quad (27c)

Where \( w(x, y) \) is the out-of-plane displacement, \( \psi_x(x, y), \psi_y(x, y) \) are the rotations along the \( x \) and \( y \) directions. Using FSDT the effect of shear deformation on deflections, and stresses can be investigated. The equations of motion and constitutive relations of FSDT are [20]:

\[
\begin{align*}
\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} - Q_x & = 0 \\
\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y & = 0 \\
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + P_z & = 0
\end{align*}
\]

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
= \begin{bmatrix}
D_{11} & D_{12} & 0 \\
D_{12} & D_{22} & 0 \\
0 & 0 & D_{66}
\end{bmatrix}
\begin{bmatrix}
\psi_{x,x} \\
\psi_{y,y} \\
\psi_{x,y} + \psi_{y,x}
\end{bmatrix}
\]

\[
\begin{bmatrix}
Q_x \\
Q_y
\end{bmatrix}
= k \begin{bmatrix}
A_{44} & 0 \\
0 & A_{55}
\end{bmatrix}
\begin{bmatrix}
\psi_y + w_{y} \\
\psi_x + w_{x}
\end{bmatrix}
\]

(28a, b, c)  
(29a, b)

The unknown constants \( A_{mn}, B_{mn}, C_{mn} \) were obtained by substituting Eq. (27), Eq. (29) into Eq. (28). Doing so will yield a system of three linear equations and three unknowns; refer to Eq. (C5) Appendix C. Solving for the unknown constants one can now determine the deflections at a particular \( x \) and \( y \) coordinate on the two-dimensional orthotropic unit truss-core sandwich panel. The results in the series converged after 23 terms \( (m=n=23) \) [7].

V. RESULTS

G. Extensional, Bending and Coupling Stiffness Verification

For verification of the effectiveness of the analytical models, consider a truss-core sandwich panel unit with the following dimensions: \( p=80\text{mm}, d=80\text{mm}, t_{TF}=1\text{mm}, t_{BF}=1\text{mm}, t_w=1\text{mm}, \theta=75^\circ, a=0.65\text{m}, b=0.65\text{m} \). An AS/3501 graphite/epoxy composite, \( E_1=138\text{GPa}, E_2=9\text{GPa}, v_{12}=0.3, G_{12}=6.9\text{GPa}, \) with four lamina’s in each component and a stacking sequence of \( [(0/90)_2] \) was used to verify the analytical models. A representative volume element, or unit cell (Fig. 8), approach was adopted for obtaining the stiffness properties. A finite element analysis was conducted on the panels using the commercial ABAQUS\textsuperscript{TM} finite element program. Eight node shell elements were used to model the face sheets and webs of the unit truss-core sandwich panel. The shell elements have the capability to include multiple layers of different material properties and thicknesses. Five integration points were used through the thickness of the shell elements. The FEM model consisted of 18,240 nodes and 6,000 elements.

Known strains and curvatures were imposed on the unit cell and force and moment resultants were calculated from the resulting stresses. Strains were imposed by enforcing periodic displacement boundary conditions on the unit cell, Table 1. To prevent rigid body motion and translation the unit-cell (Fig 9) was subjected to minimum support constraints. The top and bottom surfaces were assumed to be free of traction. The faces \( x=0 \) and \( x=a \) have the identical nodes on each side as well as the other faces \( y=0 \) and \( y=b \). The identical nodes on the opposite faces are constrained to enforce the periodic

\begin{figure}
\centering
\includegraphics[width=\textwidth]{image.png}
\caption{(a)Finite element unit-cell mesh. (b) Typical unit-cell}
\end{figure}
boundary conditions. Figure 10 shows the deformations of the unit cell as a result of imposing the periodic boundary conditions.

Table 1. Periodic displacement boundary conditions imposed on the lateral faces of unit cell for in-plane strains and curvatures.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Condition</th>
<th>Condition</th>
<th>Condition</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_{x0} = 1$</td>
<td>$v_a(y) - v(\theta_0,y)$</td>
<td>$w(x,b) - w(x,0)$</td>
<td>$\theta_x(a) - \theta_x(\theta_0)$</td>
<td>$\theta_y(x) - \theta_y(0)$</td>
</tr>
<tr>
<td>$\varepsilon_{y0} = 1$</td>
<td>$a$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\gamma_{xy0} = 1$</td>
<td>$0$</td>
<td>$a/2$</td>
<td>$b/2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\kappa_x = 1$</td>
<td>$az$</td>
<td>$0$</td>
<td>$-a^2/2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\kappa_y = 1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$bz$</td>
</tr>
<tr>
<td>$\kappa_{xy} = 1$</td>
<td>$0$</td>
<td>$a/2$</td>
<td>$-ay/2$</td>
<td>$bz/2$</td>
</tr>
</tbody>
</table>


<table>
<thead>
<tr>
<th>Stiffness</th>
<th>$A_{11}$ [N/m]</th>
<th>$A_{12}$ [N/m]</th>
<th>$A_{22}$ [N/m]</th>
<th>$A_{66}$ [N/m]</th>
<th>$D_{11}$ [N/m]</th>
<th>$D_{12}$ [Nm]</th>
<th>$D_{22}$ [Nm]</th>
<th>$D_{66}$ [Nm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical</td>
<td>2.23E+08</td>
<td>5.43E+06</td>
<td>1.48E+08</td>
<td>1.43E+07</td>
<td>2.76E+05</td>
<td>8790</td>
<td>2.37E+05</td>
<td>22327</td>
</tr>
<tr>
<td>FE</td>
<td>2.20E+08</td>
<td>5.43E+06</td>
<td>1.48E+08</td>
<td>1.41E+07</td>
<td>2.78E+05</td>
<td>8690</td>
<td>2.37E+05</td>
<td>22200</td>
</tr>
<tr>
<td>% diff.</td>
<td>1.35</td>
<td>0</td>
<td>0</td>
<td>0.89</td>
<td>0.63</td>
<td>1.09</td>
<td>0.13</td>
<td>0.6</td>
</tr>
</tbody>
</table>

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The finite element result indicates that using Eq. (17) to determine the extensional, coupling, and bending stiffness is adequate. The finite element results are excellent in agreement with the formulation of the derived stiffness parameters of the truss-core sandwich panel.

H. Transverse Shearing Stiffness verification, $A_{44}$

The finite element verification of the $A_{44}$ stiffness term consisted of a two part finite element procedure. First we assumed that the truss behaves like a cantilever beam. Assuming that the truss behaves like a cantilever beam one can determine the equivalent cross sectional properties of the beam from finite element results. The three beam properties are: axial rigidity $EA$, flexural rigidity $EI$, and shear rigidity $GA$. The truss beam consisted of ten unit cells and was clamped on the left end, Fig. 11. Eight node solid elements were used to model the beam. First an end couple was applied and the corresponding tip deflections were determined from the finite element output after analyses. Using Eq. (31) we determined the flexural rigidity of the beam. The couple was then removed and a transverse force was applied at the tip. The tip deflections were obtained from the finite element output. Using finite element tip deflection in Eq. (32) along with the flexural rigidity result from Eq. (31) we determined the shear rigidity. This finite element verification procedure was done for various web angles. The finite element result along with the analytical result from Eq. (24) is shown in Fig. 12. The finite element results are in good agreement with the analytical formulation of $A_{44}$. The percentage difference between the finite element results and the analytical result does not exceed 7%.

I. Two-dimensional Orthotropic Plate Results and Stiffness Behavior

To determine the optimal web angle inclination for greatest stiffness and minimum panel deflection we investigated. The behavior of the stiffness' and panel deflection to a change of web angle of inclination. Changes in $A_{44}$ are important because certain applications depend on the behavior in this plane. Maximum panel deflection is important because excessive deflection of the ITPS panel can lead to high local aerodynamic heating. Consider a truss-core sandwich panel with the following dimensions: $p=80mm, d=80mm, t_{PF}=1mm, t_{BF}=1mm, t_w=1mm,$
The truss-core sandwich panel is made out of graphite/epoxy T300/934: $E_1=131\text{Gpa}$, $E_2=10.3\text{Gpa}$, $G_{12}=6.9\text{Gpa}$, $\nu_{12}=0.22$, with four lamina’s in each component and a stacking sequence of $[90/0]_s$. By prescribing an internal web angle the thickness of the faces and webs is defined such that the truss-core panel cross-sectional area (thus the weight) remains the same to the 90° web angle configuration. Doing so will allow us to only get the behavior of stiffness to a change in angle rather than a change in angle and area. The results are shown in the figures below.

![D-matrix behavior](image1)
![Shearing stiffness behavior](image2)
![Maximum deflection behavior](image3)
![Bending stiffness behavior](image4)

Figure 13. Behavior analysis of (a) D-matrix, (b) Shear stiffness, (c) Maximum deflection, (d) Bending stiffness, as a function of web inclination angle.

From the figure above the following conclusions can be made:

- The highest bending, extensional, and $A_{55}$ shear stiffness are provided by the truss-core panel with vertical webs.
- While the bending, extensional, and $A_{55}$ shear stiffness decreases with decreasing web angle inclination, the transverse shear stiffness $A_{44}$ increases.
- Maximum deflection occurred at the 48° web angle inclination, and minimum deflection occurred at the triangular web configuration.
- $A_{44}$ is the most dominant stiffness when the truss-core sandwich panel has triangular webs.
- For panels with rectangular web configurations, its behavior is dominated by shear deformation in the y-direction.

In design a triangular truss core may be preferred because the influence on shear can be neglected due to the high stiffness, however that type of configuration posses problems such as buckling because of the long unsupported lengths of the webs.

VI. CONCLUSIONS AND FUTURE WORK

The truss-core sandwich panel is similar to other types of conventional sandwich systems but with absent discrete spot-welds or rivets connecting the facing plates to the core [16]. Thus the advantage of low maintenance, load bearing capabilities, and low life-cycle cost make it an excellent candidate for use as a thermal protection system in launch vehicles.

Transforming the sandwich panel into an equivalent orthotropic thick plate continuum has been shown. Detailed formulation of the bending, extensional, coupling, and shear stiffness for the unit truss-core sandwich panel was presented and verified. Panels with rectangular webs resulted in a weak extensional, bending, and $A_{55}$ stiffness. The analytical method proved to be robust and efficient because it can be used to determine stiffness for isotropic and orthotropic materials. The stiffness results between the analytical model and the finite element analysis were
within 2% thus validating the method. The computational time and effort in determining stiffness and plate behavior of the ITPS is significantly reduced in comparison with FEM.

The equivalent stiffness parameters were used in the closed-form solution to evaluate the maximum deflection of orthotropic thick plate. Maximum deflection was greatest for 48° configurations. Maximum deflection was fairly constant for the web angle range of 80°-90°. Panels with triangular web configuration have negligible shear deformation effect because of the high shearing stiffness in both directions, but this leads to other problems such as local buckling. Global buckling of the panel is not expected because the ITPS panel is expected to be thick. However local buckling is a factor because the face sheets and the webs are made of thin plates. A triangular web configuration will results in the web length to be long as well as the length between the adjoining unit cell. The increased length will lead to a lower critical buckling value.

Future work will include critical thermal and mechanical buckling analysis of the unit cell of the panel. Local and global buckling will also be investigated. A 45° shearing terms will be verified with FEM results. First-ply failure using Tsai-Hill failure criterion due to mechanical and thermal load will be investigated. An optimization procedure will be performed to minimize mass, which is a function of the geometric parameters, subjected to maximum plate deflection, local buckling, and Tsai-Hill first ply failure constraints.

**APPENDIX**

A. Right Web Transformation Matrix Determination

A detailed procedure of analysis is presented in this section for the derivation of the deformation transformation matrices of the left and right web. Integrating Eq. (10) twice with respect to \( y \) results in the out of place displacement

\[
w(y) = \frac{1}{2} \kappa_y y^2
\]

(A1)

From Classical Lamination Theory the \( u \) and \( v \) displacements of the sandwich panel in the global coordinates are determined as shown below:

\[
\begin{align*}
  u(x, y, z) &= u_o(x, y) - z \frac{\partial w}{\partial x} \\
  u &= 0 \\
  v(x, y, z) &= v_o(x, y) - z \frac{\partial w}{\partial y} \\
  v(y, z) &= \kappa_y y z
\end{align*}
\]

(A2a, b)

To determine the strains in the web the displacements from Eq. (A2) must be in the local coordinate system. Substitution of Eq. (9a) into Eq. (A2), and then Eq. (A2) into Eq. (9b) resulted in

\[
\begin{align*}
  \varepsilon_x &= \frac{\partial u}{\partial x} = 0 \\
  \varepsilon_y &= \frac{\partial v}{\partial y} = \kappa_y \cos^2 \theta ( -\bar{y} \sin \theta + z \cos \theta + \frac{d_s}{2} ) \\
  \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0
\end{align*}
\]

(A3a)

\[
\begin{align*}
  \kappa_x &= -\frac{\partial^2 w}{\partial x^2} = 0 \\
  \kappa_y &= -\frac{\partial^2 w}{\partial y^2} = 2\kappa_y \cos \theta \sin^2 \theta + \kappa_s \cos^3 \theta \\
  \kappa_{xy} &= -2 \frac{\partial^2 w}{\partial x \partial y} = 0
\end{align*}
\]

(A3b)

Equations Eq. (A3a) and Eq. (A3b) describe the micro mid-plane strains and curvatures in the right web. From Eq. (A3a) we observed that the mid-plane strain in the \( \bar{Y} \)-direction is
\[ \epsilon_{\tau_0} = -\kappa_\theta \cos^2 \theta \left( \frac{d_c}{2} - \bar{y} \sin \theta \right) \]  
\hspace{1cm} (A4)

The same procedure applies to unit curvature along the \( x \)-direction, unit twist \( \kappa_y \) and unit shear strain in the \( xy \) plane. Shown below are the deformation transformation matrices for the left and right webs.

Left web:

\[ \{D\}_l^{(c)} = T_D \{D\}_l^{(M)} \]

\[
\begin{bmatrix}
\epsilon_{\tau_0} \\
\epsilon_{\tau_0} \\
\gamma_{\tau_0} \\
\kappa_x \\
\kappa_y \\
\kappa_{\tau_0}
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & (\frac{d_c}{2} - \bar{y} \sin \theta) & 0 & 0 \\
0 & 0 & 0 & \cos^2 \theta (\frac{d_c}{2} - \bar{y} \sin \theta) & 0 & 0 \\
0 & -\cos \theta & 0 & 0 & \cos \theta (\frac{d_c}{2} - \bar{y} \sin \theta) & 0 \\
0 & 0 & 0 & -\cos \theta & 0 & 0 \\
0 & 0 & 0 & \cos^3 \theta - 2 \cos \theta \sin^2 \theta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\epsilon_{x_0} \\
\epsilon_{y_0} \\
\gamma_{x_0} \\
\kappa_x \\
\kappa_y \\
\kappa_{x_0}
\end{bmatrix}
\hspace{1cm} (A5)

Right web:

\[ \{D\}_r^{(c)} = T_D \{D\}_r^{(M)} \]

\[
\begin{bmatrix}
\epsilon_{\tau_0} \\
\epsilon_{\tau_0} \\
\gamma_{\tau_0} \\
\kappa_x \\
\kappa_y \\
\kappa_{\tau_0}
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & (\frac{d_c}{2} - \bar{y} \sin \theta) & 0 & 0 \\
0 & 0 & 0 & \cos^2 \theta (\frac{d_c}{2} - \bar{y} \sin \theta) & 0 & 0 \\
0 & \cos \theta & 0 & 0 & \cos \theta (\frac{d_c}{2} - \bar{y} \sin \theta) & 0 \\
0 & 0 & 0 & \cos \theta & 0 & 0 \\
0 & 0 & 0 & \cos^3 \theta + 2 \cos \theta \sin^2 \theta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\epsilon_{x_0} \\
\epsilon_{y_0} \\
\gamma_{x_0} \\
\kappa_x \\
\kappa_y \\
\kappa_{x_0}
\end{bmatrix}
\hspace{1cm} (A6)

B. Detailed Derivation of \( A_{ss} \)

Starting with Eq. (19) and dividing through by \( Ax, Ay, Az \) gives

\[ 2(\tau_{xz}(\frac{d_c}{d})) = -\frac{\partial F}{\partial x} \]

(B1)

Recognizing that the forces in the \( x \)-direction is a summation of the webs and face sheet forces and that \( \tau_{xz} = \tau_{\tau_0} \sin \theta \) results in

\[ 2(\tau_{xz}(\frac{d_c}{d})) = -\frac{\partial}{\partial x} \left[ N_x^{(1)}(2p) + 2 \int_0^\tau \bar{N}_x^{(4)} d\bar{y} \right] \]

(B2)

Since the force resultant in the webs is a function of \( \bar{y} \) from Eq. (11) and Eq. (12), integration must be done from zero to \( \bar{y} \), where \( \bar{y} \) is an arbitrary length on the web. Substituting Eq. (2) into Eq. (B2) for \( N_x \) we obtain

\[ 2(\tau_{xz}(\frac{d_c}{d})) = -\frac{\partial}{\partial x} \left[ (2p) A_{xz}^{(1)} \epsilon_{x_0}^{(1)} + 2 \int_0^\tau A_{xz}^{(3)} \epsilon_{x_0}^{(3)} d\bar{y} \right] \]

(B3)
Substituting Eq. (7) and Eq. (12) for \( \varepsilon_{\text{xy}}^{(\theta)} \) into Eq. (B3) and noting that when the unit cell is subjected to pure bending moment per unit length \( M_y \), with \( M_y = M_{xy} = 0 \), the resulting curvature is \( \kappa_x = D_{11} M_x \),

\[
2(\tau_{\text{yy}} \sin \theta(K_{\text{xx}})) = \frac{\partial}{\partial x} \left[ (2p)A_{11}^{(3)} d \right] D_{11} M_x + 2D_{11} M_x \int A_{11}^{(3)} \left( \frac{d}{2} - \bar{y} \sin \theta \right) d\bar{y} \]  

(B4)

where \( D_{11} \) comes from the inverse relation

\[
\begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix} =
\begin{bmatrix}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} 
\]  

(B5)

Distributing out \( M_y \) and recognizing \( \frac{\partial M_y}{\partial x} = Q_x \), Equation Eq. (B4) was solved for \( \tau_{\text{yy}} \), which is the average shear stress in the webs due to the transverse force resultant \( Q_y \).

\[
\tau_{\text{yy}}(\bar{y}) = -Q_x \left( \frac{d}{2} \sin \theta \right) \left[ p d A_{11}^{(1)} + 2A_{11}^{(3)} D_{11} \left( \frac{d}{2} - \frac{\bar{y}}{2} \sin \theta \right) \right] 
\]  

(B6)

**C. Transverse shear modulus, \( A_{14} \)**

Total strain energy in half the unit cell, Fig 7a, due to a bending moment.

\[
U_z = \frac{1}{2} D_{11}^{(1)} \int_0^f \int_0^{p-f} M_{AB}^2(y) dy + \frac{1}{2} D_{11}^{(2)} \int_0^f \int_0^{p-f} M_{CB}^2(y) dy + \frac{1}{2} D_{11}^{(3)} \int_0^f \int_0^{p-f} M_{DE}^2(y) dy 
\]  

(C1)

Displacement equations of half the unit cell due to a unit \( Q_y \)

\[
\delta_y^C = -\frac{1}{3} D_{11}^{(3)} (F \cos \theta - (1 - R) \cos \theta - \frac{p \sin \theta}{d}) s^3 \sin \theta - \frac{1}{2} D_{11}^{(3)} (F f + (1 - R)(p - f) s^3 \sin \theta \]  

(C2)

\[
\delta_y^G = \frac{1}{2} D_{11}^{(2)} d (1 - F)(p - f)^2 
\]  

(C3)

\[
\delta_z^G = D_{11}^{(2)} \left[ (1 - F)(p - f) \left[ \frac{1}{3} (p - f) - \frac{1}{2} p \right] + \frac{1}{3} R f^3 \right] 
\]  

(C4)

Substituting the assumed deflections and constitutive relations from Eq. (27) into the equilibrium equations for FSDT yields a system of linear equations. The unknown constants can be determined by inverting the 3x3 matrix.
\[
\begin{bmatrix}
D_{11}\left(\frac{m\pi}{a}\right)^2 + D_{66}\left(\frac{n\pi}{b}\right)^2 + kA_{55} \\
(D_{12} + D_{66})\left(\frac{mn\pi^2}{ab}\right) \\
kA_{55}\left(\frac{m\pi}{a}\right)
\end{bmatrix}
\begin{bmatrix}
D_{22}\left(\frac{n\pi}{b}\right)^2 + D_{66}\left(\frac{m\pi}{a}\right)^2 + kA_{44} \\
kA_{44}\left(\frac{n\pi}{b}\right) \\
kA_{55}\left(\frac{m\pi}{a}\right)^2 + kA_{44}\left(\frac{n\pi}{b}\right)^2
\end{bmatrix}
\begin{bmatrix}
B_{mn} \\
P_{mn}
\end{bmatrix}
\] (C5)

Acknowledgment

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References