ON APPROXIMATING THE DENSITY FUNCTION OF RELIABILITIES OF THE MAX-LOG-MAP DECODER

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***Abstract*— Reliabilities at the output of soft-decision decoders are random variables and hence are characterized by their density function. In this paper, we present a closed-form approximation for the density function of reliabilities associated with the Max-Log-MAP decoding of convolutional codes. Density functions of reliabilities have been computed based on probabilities involving the projection of the noise in directions corresponding to different error events. Each such projection results in a random variable, and two approaches have been taken to compute probabilities involving these random variables. In the first approach, the random variables are treated as if they are independent; in the second approach, correlations between the random variables are taken into account. The former approach results in conservative estimates and a relatively simpler expression for the PDF, whereas the latter approach produces good estimates but results in a complicated expression. The mathematical expressions found using either approach are generally too complicated for further use in analytical work. In this paper, we propose a simple approach to account for the correlation between the random variables resulting from the projection of noise onto directions specified by different error events. Under this approach, we reduce the number of random variables that are considered in the computation of the PDF by eliminating those that are highly correlated. Working with this condensed set of random variables produces results that are close to the true values even if the independence assumption is used. A mathematically tractable estimate of the PDF is also presented, and the validity of this estimate is demonstrated by comparing the estimate of the PDF with simulation results. This PDF estimate can be used to analyze several communication schemes that utilize reliabilities as a design tool.**

I. INTRODUCTION

Decoders that operate on floating-point inputs and produce floating-point outputs instead of hard-decisions are called soft-input soft-output (SISO) decoders. The sign of the soft-output gives the hard-decision value, while the magnitude of the soft-output is called the reliability of the bit decision [1]. Until recently, the soft-outputs were typically used only to produce hard-decisions. The utilization of soft-information as a useful tool was largely ignored until the emergence of turbo codes. The turbo decoder makes explicit use of the soft-outputs to compute extrinsic information, and this extrinsic information is exchanged between the constituent decoders.

In recent years, a number of algorithms have been proposed that make explicit use of soft-information or reliability. For example, reliability-based hybrid ARQ techniques have been proposed [2], [3], [4], wherein the reliability or mean reliability is used to decide the retransmission size or the set of bits to be retransmitted. Cooperative diversity is another area that is receiving a lot of attention from researchers. Techniques in which cooperating nodes exchange reliability information [5], [6], [7] or information based on reliability [8] have been proposed.

The main drawback in the analysis of these techniques is that sufficient tools to model the reliability are not available. For example, in order to analyze techniques that rely on the mean of soft information, it is usual practice to take a semi-analytic approach. The mean of soft information is obtained through simulation and used in the analysis [9]. In [10], the authors address this issue by characterizing the soft information by its probability density function (PDF). The authors of [10] examine the projection of noise in the direction corresponding to an error event and interpret this random variable as a distance in Euclidean space to derive the PDF. Here an *error event* denotes a sequence that translates one codeword into another, where the path through the code trellis that is induced by the error sequence is only in the same state as the original codeword at the endpoints of the sequence. For the rest of the paper, the random variable resulting from the projection of noise onto a direction specified by an error event will be referred to as the *projection random variable* (PRV). In [10], the authors present two approaches to obtain the PDF. In the first approach, the PDF is derived based on the assumption that different PRVs (projection of noise onto directions specified by different error events) are independent. The PDF obtained using the independence assumption results in conservative reliability estimates that are lower than the actual values. The authors suggest incorporating the correlation between the PRVs into the PDF to avoid conservative estimates. In the second approach, the authors obtain a covariance matrix involving the correlation between different PRVs and use it in a joint multivariate distribution to obtain the PDF. Though the PDF obtained using the second approach produces good reliability estimates, the expression for the density function is very complicated. In this paper, we present a simple technique to account for the correlation between the PRVs. Using this technique, we can avoid the use of complicated joint multivariate distributions.

Even with the independence assumption, the PDF obtained using this technique cannot be expressed in closed-form and involves products and summations that depend on the enumeration of all possible error events. Thus, the PDF given in [10] is not attractive for use in the analysis of reliability-based techniques. In this paper, we present an ad hoc estimate of the PDF that is mathematically tractable. This closed-form estimate of the PDF is parameterized by a single quantity that can be numerically evaluated. We show that our technique produces an accurate approximation of the true PDF.

II. The Decoder

Among all trellis-based decoding algorithms for linear codes, the BCJR maximum *a posteriori* (MAP) decoder [11] achieves the optimum bit error probability. The inputs to a BCJR MAP decoder are *a priori* probabilities and likelihoods for the received symbols, and the output consists of *a posteriori* probabilities (APPs). The decoder is usually implemented in the log-domain for fast operation. For each information bit u_i, the Log-MAP decoder computes the log-likelihood ratio (LLR) of the APP as follows,

$$L(u_i|\mathbf{r}) = \ln \frac{\mathrm{P}(u_i=0|\mathbf{r})}{\mathrm{P}(u_i=1|\mathbf{r})}, \quad (1)$$

$$= \ln \frac{\sum_{\mathbf{c}\in C_+} \mathrm{P}(\mathbf{c}|\mathbf{r})}{\sum_{\mathbf{c}\in C_-} \mathrm{P}(\mathbf{c}|\mathbf{r})}, \quad (2)$$

where $\mathbf{r}$ is the received codeword, C_+ is the set of all codewords with $u_i = 0$ and C_- is the set of all codewords with $u_i = 1$. Note that $\mathbf{c}_k \in \{+1,-1\}$. The output LLR is also referred to as the soft-information or soft-output. Assuming that all the codewords are equally likely and using Baye's rule, the soft information for codewords transmitted on a additive white Gaussian channel (AWGN) with noise variance $\sigma^2 = N_0/2$ can be written as

$$\begin{aligned} L(u_i|\mathbf{r}) &= \ln \sum_{\mathbf{c}\in C_+} \mathrm{P}(\mathbf{r}|\mathbf{c}) - \ln \sum_{\mathbf{c}\in C_-} \mathrm{P}(\mathbf{r}|\mathbf{c}), \\ &= \ln \Bigg[\sum_{\mathbf{c}\in C_+} \exp\Big(-\frac{\|\mathbf{r}-\mathbf{c}\|^2}{2\sigma^2}\Big)\Bigg] \\ &\quad -\ln \Bigg[\sum_{\mathbf{c}\in C_-} \exp\Big(-\frac{\|\mathbf{r}-\mathbf{c}\|^2}{2\sigma^2}\Big)\Bigg], \end{aligned} \quad (3)$$

A suboptimal implementation of the Log-MAP decoder called the Max-Log-MAP decoder is obtained by using the approximation $\ln(\sum x_i) = \max(\ln(x_i))$ to evaluate the log-APP in (3). Thus, for a Max-Log-MAP decoder the soft-output is given by,

$$L(u_i|\mathbf{r}) = \min_{\mathbf{c}\in C_+}\Big(\frac{\|\mathbf{r}-\mathbf{c}\|^2}{2\sigma^2}\Big) - \min_{\mathbf{c}\in C_-}\Big(\frac{\|\mathbf{r}-\mathbf{c}\|^2}{2\sigma^2}\Big), \quad (4)$$

Since the union of C_+ and C_- spans the space of all valid codewords, one of the terms in (4) corresponds to the Euclidean distance between $\mathbf{r}$ and the maximum-likelihood (ML) decoding solution.

The magnitude of the soft information is called the *reliability* and is a measure of the correctness of the bit decision. The reliability for bit i (Λ_i) can be expressed as

$$\Lambda_i \triangleq |L(u_i|\mathbf{r})| = \frac{1}{2\sigma^2}\min_j \Big\{ \|\mathbf{r}-\mathbf{c}_i^{(j)}\|^2 - \|\mathbf{r}-\mathbf{c}_{ML}\|^2 \Big\}, \quad (5)$$

where $\mathbf{c}_i^{(j)}$ is a codeword corresponding to an input sequence that differs from the ML input sequence in the ith bit. Since the distance between $\mathbf{r}$ and the ML codeword is smaller than the distance between $\mathbf{r}$ and any other codeword, the difference in (5) is always positive. Thus, the Max-Log-MAP decoder associates with the ith bit, the minimum difference between the metric associated with the ML path and the best path that differs from the ML path in the input of the ith trellis section [12]. A high value of reliability implies that the ML path and the next best path are far apart, hence there is a lower probability of choosing the other path and making a bit error. Thus, reliability is a measure of the correctness of the bit decision, as has also been shown via simulation results in [2], [3]. A bit with high reliability is more likely to have decoded correctly than a bit with low reliability.

Note that the scaling of the reliability by the noise variance in (5) does not affect the performance of the Max-Log-MAP decoder and is just an implementation consideration. If channel estimates are available to the decoder, the scaling can be performed.

III. The Density Function of Reliability

In [10], the authors derive the density function of soft-information without the scaling of the reliability by the noise variance (see (5)). The density function for any noise variance can then be obtained by a simple transformation. The authors of [10] work in Euclidean space and use a geometrical interpretation of the reliability is used to obtain the PDF. In this section, we provide a more streamlined derivation by working in the more conventional Hamming space and use a high signal-to-noise ratio (SNR) approximation to obtain the PDF.

Since $\mathbf{c}_i^{(j)}$ is a codeword corresponding to an input sequence that differs from the ML input sequence in the ith bit, $\mathbf{c}_i^{(j)}$ can be expressed as,

$$\mathbf{c}_i^{(j)} = \mathbf{c}_{ML} + \mathbf{e}_i^{(j)}, \tag{6}$$

where $\mathbf{e}_i^{(j)}$ is an error event generated by an input sequence with bit i equal to 1. Since the symbols of $\mathbf{c}_i^{(j)}$ and $\mathbf{c}_{ML}$ take on values in $\{+1,-1\}$ and the error event transforms one codeword into another, the components of $\mathbf{e}_i^{(j)}$ take on values in $\{+2,0,-2\}$. Using (6) in (5), we get

$$\Lambda_i = \min_j \big\{\|\mathbf{e}_i^{(j)}\|^2 - 2(\mathbf{r}-\mathbf{c}_{ML})^{\mathrm{T}} \cdot \mathbf{e}_i^{(j)}\big\}. \tag{7}$$

At high SNRs, the ML decoder will find the correct codeword (input sequence). Thus for high SNRs we can express the received sequence as

$$\mathbf{r} = \mathbf{c}_{ML} + \mathbf{e}, \tag{8}$$

where $\mathbf{e} \sim \mathcal{N}\big(0, \frac{N_0}{2}\mathbf{I}\big)$. This assumption is similar to the approach in [10], in which the authors obtain the conditional density function given correct decoding of a bit. Using (8) in (7) we get

$$\Lambda_i = \min_j \big\{\|\mathbf{e}_i^{(j)}\|^2 + 2\mathbf{e}^{\mathrm{T}} \cdot \mathbf{e}_i^{(j)}\big\}. \tag{9}$$

Note that according to our terminology, $\mathbf{e}_i^{(j)}$ is an error event, whereas $\mathbf{e}^{\mathrm{T}} \cdot \mathbf{e}_i^{(j)}$ is the projection of the noise in the direction of the error event $\mathbf{e}_i^{(j)}$. Let d_j be the Hamming weight (number of non-zero elements) of $\mathbf{e}_i^{(j)}$. Also, define

$$Z_j \triangleq \|\mathbf{e}_i^{(j)}\|^2 + 2\mathbf{e}^{\mathrm{T}} \cdot \mathbf{e}_i^{(j)}. \tag{10}$$

Since Z_j is just a linear combination of Gaussian noise samples, Z_j is also a Gaussian random variable. It is easy to see that

$$Z_j \sim \mathcal{N}\bigg(4d_j, 16d_j^2\frac{No}{2}\bigg). \tag{11}$$

Thus, the reliability can be expressed as the minimum over a sequence of Gaussian random variables with distributions given by (11). Assuming that all the Z_js are independently distributed, the cumulative density function (CDF) of Λ can be written as

$$\begin{aligned} F_\Lambda(\lambda) &= 1 - \prod_j \mathrm{Prob}(Z_j > \lambda) \\ &= 1 - \prod_{d=d_{min}}^{d_{max}} \bigg\{Q\bigg(\frac{\lambda - 4d}{\sqrt{16d\sigma^2}}\bigg)\bigg\}^{a(d)}, \end{aligned} \tag{12}$$

where $a(d)$ is the multiplicity of error events of weight d and $Q(x)$ represents the Gaussian complementary distribution function. The PDF can be obtained by differentiating the CDF. Using the product rule of differentiation, the PDF is obtained as

$$\begin{aligned} f_\Lambda(\lambda) = \sum_{d_j=d_{min}}^{d_{max}} \bigg\{&\frac{a(d_j)}{4\sqrt{(2\pi d_j\sigma^2)}} \exp\bigg(-\frac{(\lambda-4d_j)^2}{32d_j\sigma^2}\bigg) \\ &\times Q\bigg(\frac{\lambda-4d_j}{\sqrt{16d_j\sigma^2}}\bigg)^{a(d_j)-1} \prod_{\substack{d_i=d_{min}\\ d_i\neq d_j}}^{d_{max}} Q\bigg(\frac{\lambda-4d_i}{\sqrt{16d_i\sigma^2}}\bigg)^{a(d_i)}\bigg\}. \end{aligned} \tag{13}$$

Thus, even under the simplifying assumption of independent output error events, the density function obtained from first principles is very complicated and not suited for use in the analysis of techniques involving reliabilities. For the Max-Log-MAP decoder with the noise scaling implemented (as in (5)), the CDF and PDF of the reliability can be obtained by a simple transformation as

$$F_{\Lambda,\sigma}(\lambda) = F_\Lambda\big(2\sigma^2\lambda\big), \qquad f_{\Lambda,\sigma}(\lambda) = 2\sigma^2 f_\Lambda\big(2\sigma^2\lambda\big) \tag{14}$$

The subscript σ is used in the above expressions to indicate that the soft-information is scaled by the noise variance in the Max-Log-MAP decoder. Since Λ is non-negative and continuous, the mean of the reliability can then be evaluated numerically as

$$\begin{aligned} E[\Lambda] &= \int_\lambda \big[1 - F_{\Lambda,\sigma}(\lambda)\big] d\lambda \\ &= \int_0^\infty \prod_{d=d_{min}}^{d_{max}} \bigg\{Q\bigg(\frac{2\sigma^2\lambda - 4d}{\sqrt{16d\sigma^2}}\bigg)\bigg\}^{a(d)} d\lambda. \end{aligned} \tag{15}$$

IV. On the Correlation Between Output Error Events

In Section III, we model the reliability as the minimum of a number of Gaussian random variables that are assumed to be independent. This assumption is valid only if all the PRVs are independent. However, this is not a valid assumption because different output error events depend on common noise samples and are thus correlated. Because of this correlation, the expressions for the PDF and mean of the reliabilities given by (14) and (15) can significantly differ from the simulation results, as will be shown in Section VI. Thus, the correlations among the output error events should be considered in order for the analytical expressions to agree with the simulation results. In [10], the authors account for this correlation by obtaining the joint multivariate distribution of Z_j and using this distribution to compute the density function. However, this approach would involve computing a covariance matrix involving different pairs of error events and using this covariance matrix in the density function. This approach results in a very complicated expression. Even with the independence assumption, the density function in (13) is complicated. Further, the approach using the multivariate distribution offers no further insight into the behavior of the reliabilities.

Note that the correlation between different PRVs arise because they share common noise samples, which is a consequence of the associated error events differing from the correct codeword in a common set of symbols. We introduce a simple approach to account for the correlation between PRVs by computing the correlation among different error events. We first define the correlation between two error events $\mathbf{e}_1$ and $\mathbf{e}_2$ of lengths l_1 and l_2 respectively as,

$$C_{\mathbf{e}_1,\mathbf{e}_2} = \frac{\sum_{i=1}^{\min(l_1,l_2)} \mathbf{e}_{1,i} \odot \mathbf{e}_{2,i}}{\max(l_1,l_2)}, \tag{16}$$

where $\mathbf{e}_{j,i}$ refers to the ith bit of error event $\mathbf{e}_j$ and the '$\odot$' operator denotes the XOR operation. For example, '11 10 10 11' and '11 10 10 00 01 11' are two error events of length 8 and 12 respectively, and the correlation between the two events can be computed using (16) to be 0.5. We account for the correlation between output error events by eliminating some of the error events that are highly correlated and using the reduced set of error events to compute the PDF/CDF of reliabilities. We define a correlation threshold T_{Corr}, and whenever two error events have a correlation value greater than T_{Corr}, the longer of the two events is eliminated from the event set. The longer of the two events is removed from the event set because performance is usually dominated by the low weight error events. This process is continued until all remaining pairs of error events have correlation less than T_{Corr}. We normalize the correlation by the longer of the two error event lengths to ensure that events with very dissimilar lengths have a low value of the correlation. This eliminates the possibility of discarding a long event which may share a common initial path through the trellis with a small error event. Thus, a condensed event set is obtained within which the events have low correlation value. We expect the small correlation between the events in the condensed set to have a negligible effect on the independence assumption used in deriving the PDF. It will be shown in Section VI that if the summation in (13) is performed over the condensed event set, the resulting values are strikingly close to the simulation results for properly chosen values of T_{Corr}. Thus, the need for joint multivariate distributions involving the covariance matrix of output error events is avoided using this technique.

V. A Mathematically Tractable Density Function

The expressions for the density function of the reliability given by (13), (14) are complicated and not convenient for use in mathematical analysis of reliability-based techniques. We address this issue with an ad hoc estimate of the PDF based on the following observations:

- The mean of the reliabilities obtained from (15) is very close to the simulation results. (This fact will be substantiated in Section VI).
- Given the correct decoder output, the conditional distribution of the soft output for a bit is approximately Gaussian with variance approximately equal to twice the mean (cf. [9]).

Thus, we suggest modeling the reliability as the absolute value of a Gaussian random variable that satisfies the symmetry condition, i.e.,

$$\Lambda = |X|\ , X \sim \mathcal{N}(\mu, 2\mu), \tag{17}$$

where μ is the mean obtained by numerically evaluating (15). The cumulative distribution function (CDF) can easily found to be

$$F_\Lambda(\lambda) = \begin{cases} Q\left(\frac{\mu-\lambda}{\sqrt{2\mu}}\right) - Q\left(\frac{\mu+\lambda}{\sqrt{2\mu}}\right), & \lambda > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Differentiating with respect to λ, the PDF of the reliability is,

$$f_\Lambda(\lambda) = \begin{cases} \frac{\exp\left(-\frac{(\mu-\lambda)^2}{4\mu}\right) + \exp\left(-\frac{(\mu+\lambda)^2}{4\mu}\right)}{2\sqrt{\pi\mu}}, & \lambda > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Unlike the expression in (14), the density function in (19) does not involve summations and products and is expressed in closed-form. Indeed, we have to resort to numerical computation to obtain the mean, μ, but for problems involving explicit probabilities of reliabilities, (19) is mathematically more tractable than (14). In Section VI we provide results that show this Gaussian approximation is extremely accurate.

VI. Numerical Results

In this section, the analytical expressions derived in the previous sections are compared with simulation results. For all results, non-recursive, non-systematic convolutional codes (CC) with a block size of 1000 bits are used. The Max-Log-MAP implementation of the BCJR algorithm is used for decoding. In our implementation the Max-Log-MAP metric is scaled by the noise variance as in (5). In Figure 1, the means of the reliabilities obtained using (15) are compared with the actual means obtained from simulation. The comparison is shown for a rate $1/2$, constraint length 3 convolutional code and a rate $1/3$, constraint length 7 convolutional code. The constraint length 3 CC has generator polynomials $1 + D^2$ and $1 + D + D^2$ or $(5, 7)_8$ in octal notation. The constraint

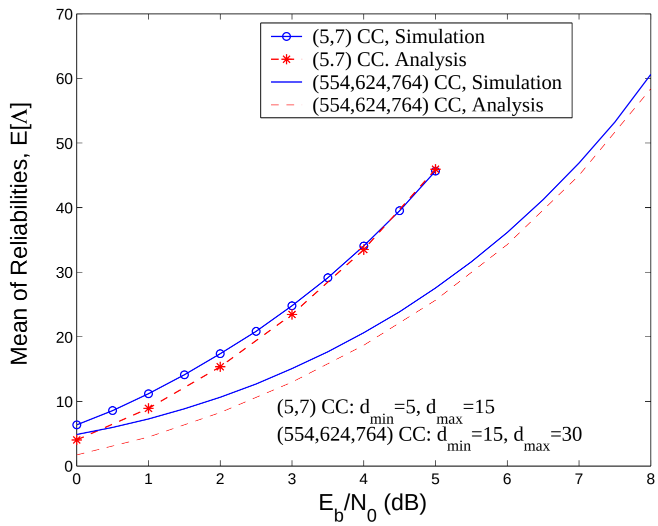

Fig. 1. The mean of reliabilities as a function of the signal-to-noise ratio when the correlation between the output error events are ignored.

length 7 CC has generator polynomials $(554, 624, 764)_8$. It is observed that the analytical expression produces the estimates that are smaller than the actual values. As explained in Section IV and in [10], the assumption that all the output error events are independent leads to over-counting which causes the analytical results to produce conservative results. At low SNRs, there is larger gap between analytically obtained values and the simulation results when compared to high SNRs. This is because the performance at low SNRs is dominated by blocks that decode incorrectly and hence the assumption (8) is violated.

To tighten the gap between the analytical and simulation results, it is required to consider the correlation between error events. The number of error events with weight d (event multiplicity) is shown in Table I for the $(5, 7)_8$ CC It is seen that eliminating events that have a correlation value higher than the correlation threshold ($T_{Corr} = 0.7$ in this case) results in a condensed set of error events. We expect that using this condensed set of events with low correlation will reduce the over-counting problem caused by the independence assumption.

The mean of the reliabilities after accounting for the correlation between output error events (as explained in Section IV) is shown in Figure 2. If the summation in (13) is performed over a condensed set of error events (as shown in Table I), and the mean then computed using (15), it can be seen from Figure 2 that the analytical results are very close to the true values even at low SNRs.

The PDF of reliabilities (eqn. (14)) for the $(5, 7)_8$ code is compared with the true PDF in Figure 3. The true PDF was obtained experimentally by simulating the decoding of a number of blocks that were transmitted over an AWGN channel. The reliability of each bit was recorded and the true PDF was estimated from the histogram of the recorded reliabilities. It can be seen from Figure 3 that results close to

d	a(d) All Events	a(d) T_{Corr}=0.7
d_{min}=5	1	1
6	2	2
7	4	4
8	8	8
9	16	9
10	32	5
11	64	11
12	128	5
13	256	14
14	512	13
d_{max}=15	1024	20

TABLE I

ERROR EVENT MULTIPLICITY OF THE $(5, 7)$ CONVOLUTIONAL CODE

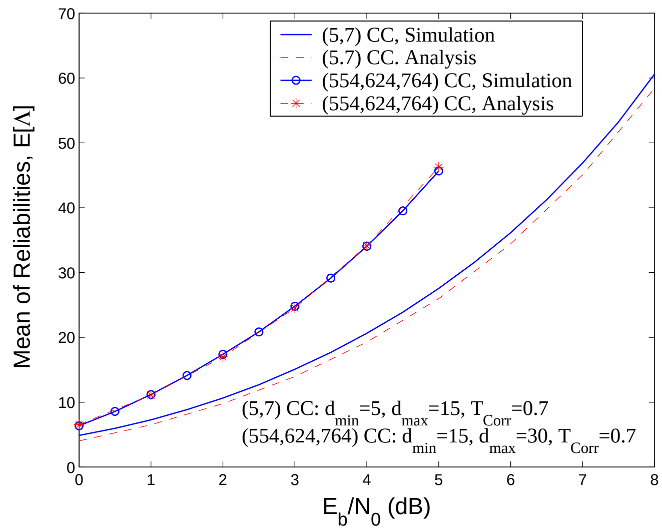

Fig. 2. The mean of reliabilities as a function of signal-to-noise ratio after taking into account the correlation between output error events

the true PDF can be obtained when the correlation between output error events is considered in the computation of the density function. The correlation is considered by evaluating the PDF in (13) over the condensed set of error events shown in Table I. Note that the analytical PDF is much closer to the true PDF at higher SNRs.

The PDF obtained using the simple, ad hoc estimate in (19) is shown in Figure 4. The mean, μ, that specifies the PDF is obtained numerically from (15). It is observed that this ad hoc expression produces results that are closer to the true PDF when compared to the expression in (13). Unlike the PDF given in (13), the ad hoc estimate produces results that are very close to the true PDF even at low SNRs. The correlation between error events can be accounted for in the ad hoc PDF estimate by evaluating μ over a condensed set of low-correlation error-events (as shown in Table I). As before, accounting for the correlation produces better results when compared to treating the PRVs as independent random variables.

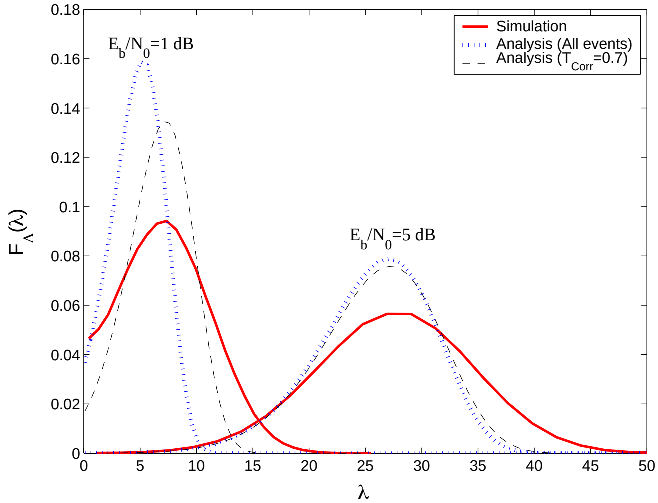

Fig. 3. The PDF of reliabilities of the $(5,7)_8$ CC for two different signal-to-noise ratios

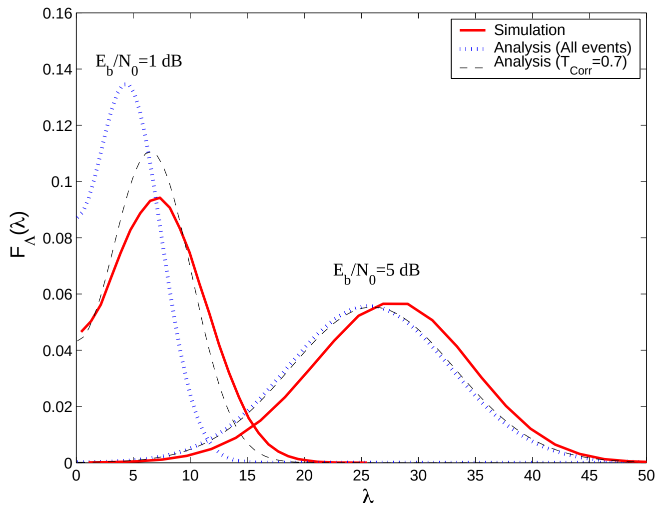

Fig. 4. The PDF of reliabilities of the $(5,7)_8$ CC obtained using the simpler, mathematically tractable expression given in (19)

VII. CONCLUSIONS

In this paper we develop simple yet accurate techniques to approximate the density function for the bit reliabilities (the magnitudes of the soft information). A streamlined derivation of the PDF that is based on the interpretation of reliability in Hamming space is presented. Correlation between output error events tends to make the analytical reliability estimates conservative. It has been demonstrated that an effective method to reduce the effect of correlation is to disregard error events with high correlation in the computation of the density function. It is shown that eliminating events with correlation values larger than 0.7 produces good results. By using this technique, we can avoid resorting to the use of joint-multivariate distributions that tend to produce complicated expressions.

A mathematically tractable expression for the PDF has also been proposed. Unlike existing density functions, this PDF estimate can be expressed in closed-form. The PDF estimate is parameterized by a single quantity that can be numerically evaluated. This alternative form could prove useful in analysis that require the computation of explicit probabilities of reliabilities. Analysis of reliability-based hybrid ARQ or reliability-based cooperative diversity are typical examples where the ad hoc PDF estimate could prove useful.

REFERENCES

[1] J. Hagenauer, E. Offer, and L. Papke, "Iterative decoding of binary block and convolutional codes," *IEEE Trans. Inform. Theory*, vol. 42, pp. 429–445, Mar. 1996.

[2] J. M. Shea, "Reliability-based hybrid ARQ," *IEE Electronics Letters*, vol. 38, pp. 644–645, June 2002.

[3] A.Roongta and J. M. Shea, "Reliability based hybrid ARQ using convolutional codes," in *Proc. Int. Conf. Commun. (ICC)*, (Anchorage, AK), pp. 2889 –2893, May 2003.

[4] V. Tripathi, E. Visotsky, R. Peterson, and M. Honig, "Reliability-based type II hybrid ARQ schemes," in *Proc. Int. Conf. Commun. (ICC)*, (Anchorage, AK), pp. 2899 –2903, May 2003.

[5] A. Avudainayagam, J. M. Shea, T. F. Wong, and X. Li, "Reliability exchange schemes for iterative packet combining in distributed arrays," in *Proc. 2003 Wireless Communications and Networking Conference*, vol. 1, (New Orleans, LA), pp. 832–837, Mar. 2003.

[6] T. F. Wong, X. Li, and J. M. Shea, "Iterative decoding in a two-node distributed array," in *Proc. 2002 IEEE Military Communications Conference (MILCOM)*, vol. 2, (Anaheim, CA), pp. 704.2.1–5, Oct. 2002.

[7] T. F. Wong, X. Li, and J. M. Shea, "Distributed decoding of rectangular parity-check code," *IEE Electronics Letters*, vol. 38, pp. 1364–1365, Oct 2002.

[8] A. Avudainayagam, J. M. Shea, T. F. Wong, and X. Li, "Collaborative decoding on block fading channels," *IEEE Trans. Commun.*, Submitted to.

[9] D. Divsalar, S. Dolinar, and F. Pollara, "Iterative turbo decoder analysis based on density evolution," *IEEE J. Select. Areas Commun.*, vol. 19, pp. 891–907, May 2001.

[10] L. Reggiani and G. Tartara, "Probability density functions of soft information," *IEEE Commun. Letters*, vol. 6, pp. 52–54, Feb. 2002.

[11] L. R. Bahl, J. Cocke, F. Jelinek, and J. Raviv, "Optimal decoding of linear codes for minimizing symbol error rates," *IEEE Trans. Inform. Theory*, vol. IT-20, pp. 284–287, Mar. 1974.

[12] M. P. C. Fossorier, F. Burkert, S. Lin, and J. Hagenauer, "On the equivalence between SOVA and max-log-MAP decodings," *IEEE Commun. Letters*, vol. 2, pp. 137–139, May 1998.