Mismatch Decoding of a Compound Timing Channel

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***Abstract*— A compound exponential-server timing channel consists of a number of infinite-buffer single-server queues in tandem. The message is coded in the sequence of arrival times to the first queue. We consider the performance of a structurally simple mismatch decoder that is the maximum-likelihood decoder for a single queue.**

I. Introduction

A timing channel is a channel in which a message is coded in a sequence of arrival times to a given system. The decoder estimates the input message based on a sequence of departure times. When considering the exponential-server timing channel [1] arrivals occur to an infinite-buffer single-server queue with independent and exponentially distributed service times. The queue is work-conserving, and, without loss of generality, one can assume that the service discipline is first-come first-served (FCFS); this follows from the memoryless property of the exponential distribution. In [1] it was shown that the capacity of such a channel is $\mu/e \approx 0.368\mu$ nats per second, where μ is the service rate.

We consider a tandem queueing network consisting of K single-server FCFS queues. Each queue is work conserving, and the waiting room is of infinite size. Service times at server i are exponential with mean $1/\mu_i$ and form an i.i.d. sequence. Sequences of service times at different servers are independent. We examine the performance of a decoder that is optimal for a single exponential-server timing channel. Such a decoder is attractive due to its structural simplicity, and the fact that it does not depend on the number of queues the channel consists of. This property is desirable in the cases when the number of queues in the channel is unknown and/or might change over time.

Timing channels have been studied in the context of covert communications [2], computation [3], and neuron models [4]. Our study is motivated by a possible communication over a multi-hop network with links (nodes) modeled as timing channels. Potential advantages of timing (pulse) communication include energy efficiency, low peak transmission power, and simplicity of network elements. We note that even very simple (timing) network devices allow for asynchronous network coding [5]. A network code for a node consists of a rule when to forward and when to drop arriving packets (pulses). For example, a node that receives two pulse sequences from two different sources can combine those into a single pulse sequence that contains partial information about the both sequences.

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An (n, M, T, ε)-code is defined by M codewords and a decoder. Each codeword is a vector of n nonnegative interarrival times $(x_1, \ldots, x_n)$. Under the equiprobable codewords assumption, the decoder selects the correct codeword with probability at least $(1 - \varepsilon)$ after observing n departures. Under the same assumption, the expected nth departure occurs no later than T. Then the code rate is $\log M/T$. A rate r is achievable if, for every $\delta > 0$, there exists a sequence of $(n, M_n, T_n, \varepsilon_n)$-codes such that $\log M_n/T_n > r - \delta$ for all sufficiently large n, and $\varepsilon_n \to 0$ as $n \to \infty$.

It is assumed that at the beginning of transmission the system of queues is in statistical equilibrium induced by a Poisson arrival process to the first queue. The transmission starts with encoder adding a packet to the first queue. This special packet can be thought of an a synchronization packet [1]. Thus, it takes $(n+1)$ packets (or, equivalently, n interarrival times) to transmit each codeword in an (n, M, T, ε)-code. This assumption will facilitate the analysis. We note that the assumption is not very limiting, especially in the context of random Poisson codes. Given that the channel is in equilibrium at the beginning of transmission of the first message, the channel will remain in equilibrium at the end of the transmission if the sequence of arrivals is Poisson. Hence, the transmission of the next message can start after an exponential amount of time.

The capacity of related discrete-time queues was studied in [6], [7]. Studies of the error exponent of the exponential timing channel can be found in [8], [9]. Robust and sequential decoding of timing channels were discussed in [10], [11]. In [12] the authors present a point-process channel framework for studying timing capacities of queues; they provide bounds and estimates for capacities of multi-server queues, queues with spurious departures, and a two exponential-server queues in tandem. The exponential-server timing channel is closely related to the Poisson [13], [14], [15], [16] and telephone-signaling channels [1].

II. Random Walk

This section contains combinatorial results on a symmetric random walk that will be used in the following section. Let $\{Z_k\}_{k=0}^{\infty}$ be the symmetric discrete-time random walk on nonnegative integers. By $c_{n,k}$ we denote the (n, k)-element of the Catalan's triangle [17]:

$$c_{n,k} := (n + 1 - k)\frac{(n + k)!}{(n + 1)!k!},$$

$n \geq k \geq 0$. The definition of $c_{n,k}$ is extended by setting $c_{n,k} \equiv 0$ for all other values of (n, k). Throughout the paper, we use $1_{\{\cdot\}}$ to denote the indicator function.

Lemma 1: For $n \geq j \geq 0$ and $0 \leq i \leq n-j$ let

$$\alpha_{n,j}(i) := \mathbb{P}\left[\sum_{k=0}^{2n-j-1} 1_{\{Z_k=0\}} = i, \min_{k=0,\dots,2n-j} Z_k \geq 0 \,\middle|\, \mathcal{E}_{n,j}\right],$$

where

$$\mathcal{E}_{n,j} = \{Z_0 = j,\, Z_{2n-j} = 0\}.$$

Then

$$\alpha_{n,j}(i) = \frac{c_{n-1,n-i-j}}{\binom{2n-j}{n}}.$$

Proof: The increments of the process $\{Z_k\}_k$ are independent and identically distributed implying that all realizations of $\{Z_k\}_{k=0}^{2n-j}$ with $\{Z_0 = j\}$ and $\{Z_{2n-j} = 0\}$ have equal probabilities. Therefore, it is sufficient to consider the number of realizations with $\{Z_0 = j\}$ and $\{Z_{2n-j} = 0\}$, or equivalently, the number of paths from $(0, j)$ to $(2n-j, 0)$ on $\mathbb{Z}^2$ with each step being either $(x, y) \to (x+1, y+1)$ or $(x, y) \to (x+1, y-1)$. In particular,

$$\alpha_{n,j}(i) = \eta_{n,j}(i)/\eta_{n,j},$$

where $\eta_{n,j}$ is the number of nonnegative paths from $(0, j)$ to $(2n-j, 0)$, and $\eta_{n,j}(i)$ is the number of nonnegative paths from $(0, j)$ to $(2n-j, 0)$ that visit the interval $[0, 2n-j-1]$ on the abscissa exactly i times. Given that $\eta_{n,j} = \binom{2n-j}{n}$, it remains to prove

$$\eta_{n,j}(i) = c_{n-1,n-i-j}. \tag{1}$$

The proof is by double induction. Since there exits only one path from $(0, n)$ to $(n, 0)$, it is straightforward to conclude that $\eta_{n,n}(i) = 1_{\{i=0\}}$ for $n \geq 1$, i.e., (1) holds for $j = n$. Assume that (1) holds for some $j = j^* < n$ and all n, i such that $0 \leq i \leq n-j$. Then we argue that the inductive assumption (1) holds for $j = j^* - 1$ and all n, i such that $0 \leq i \leq n - j^* + 1$. To this end, we invoke the second induction. First, since all the paths are nonnegative, we have

$$\eta_{j^*-1,0}(i) = \eta_{j^*-1,1}(i-1) = c_{j^*-2,j^*-i-1},$$

where the second equality is due to the inductive assumption, i.e., (1) is satisfied for $n = j^*$, $j = 0$ and all i. Now, for the second assumption, assume that (1) holds for $n = j^*-1+k$, $j = k$, all i and some $k > 1$. Then, we have that (1) holds when k is incremented by one:

$$\begin{aligned}
\eta_{j^*+k,k+1}(i) &= \eta_{j^*+k,k+2}(i) + \eta_{j^*+k-1,k}(i) \\
&= c_{j^*+k-1,j^*-i-2} + c_{j^*+k-2,j^*-i-1} \\
&= c_{j^*+k-1,j^*-i-1},
\end{aligned}$$

where the first equality is obtained by conditioning on the next step of the random walk, the second equality follows from the inductive assumption, and the last equality is due to the definition of elements of the Catalan's triangle as partial sums.

This completes the second induction and, thus, the first one. Hence, (1) holds and with it the statement of the lemma. ∎

The following lemma provides an upper bound on a sum of conditional probabilities $\alpha_{n,j}(i)$ defined in Lemma 1.

Lemma 2: Probabilities $\alpha_{n,j}(i)$ obey the following inequality for $0 \leq i \leq n$

$$\sum_{j=0}^{n-i} \alpha_{n,j}(i) \leq 2n^3 \frac{n!(2n-i)!}{(n-i)!(2n)!}.$$

Proof: Lemma 1 and the definition of $c_{n,k}$ render

$$\begin{aligned}
\sum_{j=0}^{n-i} \alpha_{n,j}(i) &= \sum_{j=0}^{n-i} (i+j) \frac{(n-j)!}{(2n-j)!} \frac{(2n-i-j-1)!}{(n-i-j)!} \\
&\leq 2n^2 \sum_{j=0}^{n-i} \frac{(n-j)!}{(2n-j)!} \frac{(2n-i-j)!}{(n-i-j)!},
\end{aligned} \tag{2}$$

where the inequality is due to $(i+j) \leq n$. Next, we argue that the first element ($j = 0$) in the preceding sum is the largest, i.e.,

$$\frac{n!}{(2n)!} \frac{(2n-i)!}{(n-i)!} \geq \frac{(n-j)!}{(2n-j)!} \frac{(2n-i-j)!}{(n-i-j)!}. \tag{3}$$

Indeed, (3) can be rewritten as

$$\frac{(2n-i)!(2n-j)!}{(2n)!(2n-i-j)!} \geq \frac{(n-i)!(n-j)!}{n!(n-i-j)!},$$

so that Lemma 3 of the appendix is applicable. Combining (2) and (3) with the fact that the sum of interest contains at most n elements results in the statement of the lemma. ∎

III. Mismatch Decoding

An input vector of nonnegative interarrival times $\boldsymbol{x} = (x_1, \dots, x_n)$ generates a random vector of interdeparture times $(Y_1, \dots, Y_n)$. In order to define the relationship between the input and output sequences, let Y_i^k be the time between departures of the ith and $(i-1)$th customers from the kth queue with understanding that $Y_i^0 = x_i$ and $Y_i = Y_i^K$. Without loss of generality, we assume that the zeroth (synchronization) packet enters the system at time $t = 0$. At that time the system is in statistical equilibrium induced by a Poisson process of rate $\lambda < \min_i \mu_i$. Given that S_i^k is the ith customer's service time in the kth queue, the output sequence is defined by the following recursive relationship:

$$\begin{cases} Y_i^k &= W_i^k + S_i^k, \\ W_i^k &= \left(\sum_{j=0}^{i} Y_j^{k-1} - \sum_{j=0}^{i-1} Y_j^k\right)^+, \end{cases}$$

$k = 1, \dots, n$, where Y_0^k is the zeroth packet's departure time from the kth queue (by definition $Y_0^0 \equiv 0$), and $(\cdot)^+ = \max\{0, \cdot\}$. The quantity W_i^k is the time elapsed between the $(i-1)$th departure and ith arrival in the kth queue [1]. We note that Y_0^1 is exponentially distributed with mean $1/(\mu_1 - \lambda)$ due to the fact that at $t = 0$ the first queue is in stationarity, i.e., the number of customers in the fist queue is geometrically distributed with parameter λ/μ_1.

Next, for a channel input sequence $\boldsymbol{x}$ and an output sequence $\boldsymbol{y} := (y_0, y_1, \dots, y_n)$ we introduce *signed channel-idling times* as

$$\xi_i = \sum_{j=1}^{i} x_j - \sum_{j=0}^{i-1} y_j,$$

$i=1,\dots,n$. We say that the channel is idle if *all* K queues are empty. Given the input and output sequences, $\boldsymbol{x}$ and $\boldsymbol{y}$, it is straightforward to express the amount of time $T_{\downarrow}(\boldsymbol{x},\boldsymbol{y})$ the channel is idle during the time interval $[y_0, \sum_{i=0}^{n} y_i]$ under these sequences

$$T_{\downarrow}(\boldsymbol{x},\boldsymbol{y}) = \left(\prod_{i=1}^{n} 1_{\{\xi_i<y_i\}}\right)\sum_{i=1}^{n}\xi_i^{+};$$

if the output sequence $\boldsymbol{y}$ is not feasible under the input sequence $\boldsymbol{x}$, i.e., $\xi_i > y_i$ for at least one i, we set $T_{\downarrow}(\boldsymbol{x},\boldsymbol{y}) = 0$. The amount of time the channel is not idle (busy) during the time interval $[y_0, \sum_{i=0}^{n} y_i]$ is defined as

$$T_{\uparrow}(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{n} y_i - T_{\downarrow}(\boldsymbol{x},\boldsymbol{y}).$$

Given the codebook $\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_M\}$, where $\boldsymbol{x}_i \in \mathbb{R}_+^n$, $i = 1,\dots,M$, the mismatch decoder $\phi : \mathbb{R}_+^{n+1} \to \{0,1,\dots,M\}$ operates as follows:

$$\phi(\boldsymbol{y}) = \begin{cases} i, & \text{if } \{T_{\downarrow}(\boldsymbol{x}_i,\boldsymbol{y}) > \max_{j\neq i} T_{\downarrow}(\boldsymbol{x}_j,\boldsymbol{y})\} \\ & \qquad \cap \{T_{\uparrow}(\boldsymbol{x}_i,\boldsymbol{y}) < \min_{j\neq i} T_{\uparrow}(\boldsymbol{x}_j,\boldsymbol{y})\} \\ 0, & \text{otherwise.} \end{cases}$$

The output 0 of the decoder is interpreted as an error. We note that for a single exponential timing channel ($K = 1$) the decoder ϕ is optimal [1]. Furthermore, in [10] it was shown that ϕ performs well even when service times in the single-server timing channel are not exponential. Throughout the paper we interpret $\log$ as the natural logarithm. The following proposition establishes an achievable rate with the decoder ϕ.

Proposition 1: Consider a K-compound exponential timing channel with service rates $\{\mu_i\}_{i=1}^{K}$. Rate

$$r = \sup_{0<\lambda<\min_i \mu_i} \lambda \log\left(1 - \prod_{i=1}^{K}\left(1-\frac{\lambda}{\mu_i}\right)\right)^{-1}$$

nats per second is achievable using the decoder ϕ.

Remark 1: For $K = 1$ the expression for r reduces to the capacity of a single exponential timing channel [1]:

$$r = \sup_{0<\lambda<\mu} \lambda \log\frac{\mu}{\lambda} = \frac{\mu}{e}.$$

Remark 2: When $K = 2$ and $\mu_1 = \mu_2 = \mu$, the proposition indicates that the rate $\approx 0.207\mu$ nats per second is achievable with the decoder ϕ.

Proof: The proof is based on the standard random coding argument. We consider a random codebook that consists of M i.i.d. codewords. Each codeword $\boldsymbol{X}_i = (X_1(i), X_2(i),\dots,X_n(i))$ is a vector of n i.i.d. exponential random variables with mean $1/\lambda$. The intensity of the Poisson process λ is chosen in such a way that the queueing system remains stable, i.e., $\lambda < \min_i \mu_i$. Since by construction the system is in stationarity at time $t = 0$ (the time the zeroth packet enters the channel), according to Burke's theorem [18, p. 218], the interdeparture times $\boldsymbol{Y} = (Y_1,\dots,Y_n)$ are independent and exponentially distributed with mean $1/\lambda$.

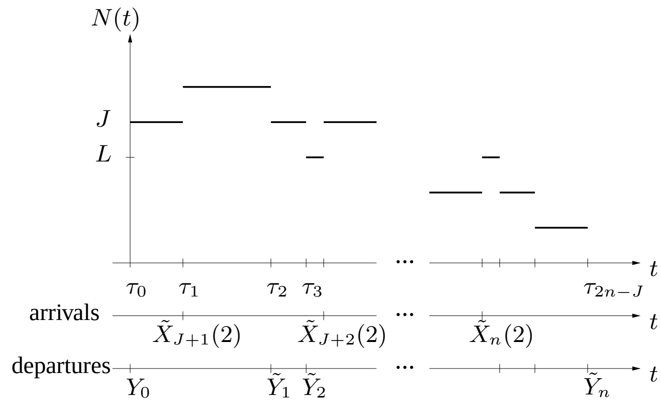

Fig. 1. Number of customers $N(t)$ in the channel. The number of customers in the channel at times $t = Y_0$ and $t = \tilde{X}_n(2)$ is equal to J and L, respectively.

Due to the choice of the arrival rate λ, the quantity

$$\theta_K \equiv \theta_K(\lambda) := \prod_{i=1}^{K}\left(1-\frac{\lambda}{\mu_i}\right) \in [0,1) \tag{4}$$

is the probability that a departing customer leaves an idle channel. This is a simple consequence of the considered queueing system being a Jackson network [18, Sec. 3.8] and the arrivals-see-time-averages (ASTA) property [19, p. 128]. In order to apply the ASTA property it is sufficient to add an imaginary $(K+1)$ queue to the system, so that departures from the Kth queue are arrivals to the $(K+1)$st queue.

For the purpose of estimating the average probability of error p_λ, we assume, without loss of generality, that $m = 1$ message is sent across the channel. Defining events

$$\begin{aligned} E_i := \{&\lambda T_{\downarrow}(\boldsymbol{X}_i,\boldsymbol{Y}) > (\theta_K - \varepsilon)n, \\ &\lambda T_{\uparrow}(\boldsymbol{X}_i,\boldsymbol{Y}) < (1-\theta_K+\varepsilon)n\}, \end{aligned}$$

for $\varepsilon > 0$, $i = 1,\dots,M$, applying the union bound, and exploiting the symmetry of the random code results in

$$p_\lambda \le \mathbb{P}[\overline{E}_1] + M\mathbb{P}[E_2], \tag{5}$$

where $\overline{E}_1$ is the complement of event E_1. The fact that the channel is ergodic and (4) yield

$$\mathbb{P}[\overline{E}_1] \to 0, \tag{6}$$

as $n \to \infty$, since

$$\frac{\lambda}{n} T_{\downarrow}(\boldsymbol{X}_1,\boldsymbol{Y}) \to \theta_K \quad \text{and} \quad \frac{\lambda}{n} T_{\uparrow}(\boldsymbol{X}_1,\boldsymbol{Y}) \to 1-\theta_K$$

in probability as $n \to \infty$, e.g., see [20, p. 111].

For notational simplicity set $\tilde{X}_i(2) := \sum_{j=1}^{i} X_j(2)$ and $\tilde{Y}_i := \sum_{j=0}^{i} Y_j$, i.e., $\tilde{X}_i(2)$ is the ith packet's arrival time to the first queue in the second codeword. In order to proceed, we need to introduce two more quantities. Let J be the number of customers in the channel just after time $t = Y_0$, i.e., $J := \max\{i :\ \tilde{X}_i(2) < Y_0\}$, and let L be the number of customers in the system just after the arrival of the nth customer, i.e., $L := \max\{i :\ \tilde{Y}_{n+1-i} > \tilde{X}_n(2)\}$. Note that

$L \geq 1$ since at least the last customer will be present at the system, i.e., $\tilde{Y}_n > \tilde{X}_n(2)$.

Consider the right-continuous process $N(t)$ that represents the number of customers in the channel for $t \in [Y_0, \tilde{Y}_n]$, and the sequence of arrival and departure times $\{\tau_i\}_{i=0}^{2n-J}$ in $[Y_0, \tilde{Y}_n]$ such that $\tau_0 = Y_0$, $\tau_1 = \min\{\tilde{X}_{J+1}(2), \tilde{Y}_1\}$, ... (see Figure 1). The embedded process $\{\hat{N}_i\}_{i=0}^{2n-J}$ is defined by

$$\hat{N}_i = N(\tau_i).$$

On the event $\{N(t) \geq 0, t \in [Y_0, \tilde{Y}_n]\}$ the quantities $T_\downarrow(\boldsymbol{X}_2, \boldsymbol{Y})$ and $T_\uparrow(\boldsymbol{X}_2, \boldsymbol{Y})$ used in the definition of event E_2 can be expressed as

$$T_\downarrow(\boldsymbol{X}_2, \boldsymbol{Y}) = \sum_{i=0}^{2n-J-L-1} (\tau_{i+1} - \tau_i) 1_{\{\hat{N}_i = 0\}},$$

$$T_\uparrow(\boldsymbol{X}_2, \boldsymbol{Y}) = \sum_{i=0}^{2n-J-1} (\tau_{i+1} - \tau_i) 1_{\{\hat{N}_i > 0\}}.$$

Define two sets of points on the real line: $\mathcal{X}_2 := \{\tilde{X}_i(2),\ J < i \leq n\}$ and $\mathcal{Y} := \{\tilde{Y}_i,\ 0 < i \leq n - L\}$. Based on the superposition/decomposition property of the Poisson process and Burke's theorem, we have that $\mathcal{X}_2 \cup \mathcal{Y}$ is a set of the first $(2n - J - L)$ points of a Poisson process with intensity 2λ on $[Y_0, \infty)$ and $\mathbb{P}[x \in \mathcal{X}_2 \,|\, x \in \mathcal{X}_2 \cup \mathcal{Y}] = 1/2$. Given this two conclusions follow:

- $\{\tau_{i+1} - \tau_i\}_{i=0}^{2n-J-1}$ is a sequence of independent and exponentially distributed random variables with $\mathbb{E}[\tau_{i+1} - \tau_i] = 1/(2\lambda)$ for $0 \leq i < 2n - J - L$ and $\mathbb{E}[\tau_{i+1} - \tau_i] = 1/\lambda$ for $2n - J - L \leq i < 2n - J$,
- process $\hat{N}_i$ is a symmetric random walk subject to the constraints $\hat{N}_0 = J$ and $\hat{N}_{2n-J} = 0$.

Conditioning first on the value of J, and then on the number of times the random walk $\hat{N}_i$ visits the abscissa, we obtain

$$\mathbb{P}[E_2] \leq \sum_{j=0}^{n} \mathbb{P}[J = j] \sum_{i=0}^{n-j} \Bigg[\alpha_{n,j}(i)\ [1 - \pi_i((\theta_K - \varepsilon)n/\lambda)] \times \pi_{2n-i}((1 - \theta_K + \varepsilon)n/\lambda)\Bigg], \quad (7)$$

where $\alpha_{n,j}(i)$ is the conditional probability defined in the statement of Lemma 1, and $\pi_i(\cdot)$ is the Erlang-i distribution, i.e., for $x \geq 0$

$$\pi_i(x) := \int_0^x \frac{\lambda(\lambda u)^{i-1}}{(i-1)!} e^{-\lambda u}\, du. \quad (8)$$

In the preceding bound we also used the fact that the amount of time the channel is busy after the last (nth) arrival is equal in distribution to a sum of L independent exponential random variables with intensity λ. Therefore, this amount of time stochastically dominates the sum of L independent random variables with the higher intensity 2λ.

Next, the change of the order of summation in (7), Lemma 2, and (8) render

$$\begin{aligned}\mathbb{P}[E_2] &\leq \sum_{i=0}^{n} [1 - \pi_i((\theta_K - \varepsilon)n/\lambda)] \times \\ &\qquad \pi_{2n-i}((1 - \theta_K + \varepsilon)n/\lambda) \sum_{j=0}^{n-i} \alpha_{n,j}(i) \\ &\leq \frac{16n^5}{(2n)!} \int_{(\theta_K - \varepsilon)n/\lambda}^{\infty} \int_0^{(1-\theta_K+\varepsilon)n/\lambda} \lambda^2 e^{-2\lambda(u+v)} \times \\ &\qquad (2\lambda v)^{2n} \sum_{i=0}^{n} \binom{n}{i} \left(\frac{u}{v}\right)^{i-1} dv\, du,\end{aligned}$$

and, therefore,

$$\mathbb{P}[E_2] \leq \frac{16n^7(2n)^{2n}}{(2n)!} \frac{1 - \theta_K + \varepsilon}{\theta_K - \varepsilon} \times \int_{\theta_K - \varepsilon}^{\infty} \int_0^{1-\theta_K+\varepsilon} \left(e^{-2(u+v)} v(u+v)\right)^n dv\, du, \quad (9)$$

where the inequality follows from a change of variables. An asymptotic logarithmic estimate of $\mathbb{P}[E_2]$ follows from (9), Stirling's approximation, and Lemma 4 from the appendix:

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}[E_2] \leq \log(1 - \theta_K + \varepsilon).$$

Finally, (5), (6), and the preceding inequality yield that $p_\lambda \to 0$ as $n \to \infty$ for all $M < e^{-n\log(1-\theta_K+2\varepsilon)}$. The statement of the proposition follows. ■

IV. Concluding Remarks

Pulse communication across network is of interest due to energy efficiency and simplicity of network elements. We considered mismatch decoding of a compound exponential-server timing channel and obtained a lower bound on the achievable rate.

Appendix

Lemma 3: Let

$$\beta_n(i,j) := \frac{(n-i)!(n-j)!}{n!(n-i-j)!}, \quad i,j \geq 0,\ n \geq i + j.$$

For fixed i an j the sequence $\{\beta_n(i,j)\}_{n \geq i+j}$ is non-decreasing in n.

Proof: It is sufficient to consider the following difference for $n \geq i + j$:

$$\begin{aligned}\beta_{n+1}(i,j) &- \beta_n(i,j) \\ &\geq \beta_n(i,j)\left(\frac{(n+1-i)(n+1-j)}{(n+1)(n+1-i-j)} - 1\right) \\ &= \beta_n(i,j)\frac{ij}{(n+1)(n+1-i-j)} \geq 0.\end{aligned}$$

■

Lemma 4: For $\theta \in (0,1)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log \int_\theta^\infty \int_0^{1-\theta} \left(e^{-2(u+v)} v(u+v)\right)^n dv\, du = -2 + \log(1-\theta).$$

Proof: The limit is a straightforward consequence of

$$\sup_{u\geq\theta,\,0\leq v\leq 1-\theta}\left\{e^{-2(u+v)}v(u+v)\right\} = e^{-2}(1-\theta).$$

■

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