A Robust Semi-analytical Method for Calculating the Response Sensitivity of a Time Delay System

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Abstract

It is often necessary to establish the sensitivity of an engineering system's response to variations in the process/control parameters. Applications of the calculated sensitivity include gradient-based optimization and uncertainty quantification, which generally require an efficient and robust sensitivity calculation method. In this paper, the sensitivity of the milling process, which can be modeled by a set of time delay differential equations, to variations in the input parameters is calculated. The semi-analytical derivative of the maximum eigenvalue provides the necessary information for determining the sensitivity of the process stability to input variables. Comparison with the central finite difference derivative of the stability boundary shows that the semi-analytical approach is more efficient and robust with respect to step size and numerical accuracy of the response. An investigation of the source of inaccuracy of the finite difference approximation found that it is caused by discontinuities associated with the iterative process of root finding using the bisection method.

Nomenclature

ε	error tolerance
ε_a	approximate relative error
λ_k	k^{th} eigenvalue of the dynamic map
$\lambda_{\max}$	complex maximum eigenvalue
Ω	spindle speed, rev/min
τ	tooth passing period, seconds
$\mathbf{A}$	state transition matrix
a	radial depth of cut
b	axial depth of cut
b_i	axial depth at iteration i
b_l	initial lower limit of axial depth inteval
b_u	initial upper limit of axial depth inteval
$b_{\lim}$	axial depth limit
$\mathbf{C}$	2x2 modal damping matrix
D	vector which depends on the process parameters
E	number of finite elements
E_a	absolute error
e_C	condition error
e_T	truncation error
$E_{a,d}$	desired absolute error
$\vec{f}_0\left(t\right)$	2x1 vector that represents the component of the cutting forces that are independent of the position vector
h	finite step size in variable of interest
$\mathbf{K_c}$	2x2 matrix representing the component of cutting forces which depend on the position vector

$\mathbf{K}$	2x2 modal stiffness matrix
K_n	normal cutting force coefficient
K_t	tangential cutting force coefficient
$\mathbf{M}$	2x2 modal mass matrix
m	number of tooth passage
N	number of teeth on the cutting tool
n	number of iterations
$\dot{q}$	collocation of x and y velocities for all nodal times in one tooth passage
q	collocation of x and y positions for all nodal times in one tooth passage
s_b	bound on the third derivative in the interval $\Omega \pm h$
u	right eigenvector
v	left eigenvector
$\vec{X}(t)$	two-element position vector for x and y-directions
z	generic variable

1 Introduction

Sensitivity plays an important role in the design of systems. For example, market economy demands an optimal design, which both favors the customer preferences and remains robust to variations in the system or its inputs. The process of searching for an optimal design or quantifying the variations in the system response cannot be effectively carried out without computing the sensitivity. Providing accurate and efficient tools for calculating the sensitivity is therefore an important activity. In this brief, we develop a semi-analytical method for calculating the sensitivity of the response to variations in the input parameters, when the system response is determined using an eigenvalue analysis.

Our application example is milling, which can be modeled as a system of delay differential equations and can exhibit unstable behavior (known as chatter). The region separating the unstable cutting domain from the stable one is characterized by the stability boundary, which can be computed using time domain [11] or frequency domain [1] analyses, or an eigenvalue analysis using the time finite element analysis (TFEA) [2, 9, 10]. The latter method transforms the system of delay differential equations into finite form. A dynamic map is generated using a nonsymmetrical state transition matrix, which relates the vibration of the tool tooth while in the cut, to free vibration while the tooth is out of the cut. Stability of the process is identified from the maximum eigenvalue of the state transition matrix.

In this brief, the semi-analytical sensitivity of the stability boundary is established by computing the sensitivity of the maximum eigenvalue, using the adjoint method [8] in combination with difference methods. Although we demonstrate the method using time finite element analysis, it is applicable to any technique that solves delay differential equations using an eigenvalue analysis of a dynamic map. The efficiency and robustness of the method is compared to a central finite difference derivative of the stability boundary. The brief begins with a description of the milling model in section 2. The analysis method is outlined in section 3. Section 4 gives the stability boundary description. Section 5 details the semi-analytical derivative used in calculating the stability sensitivity to spindle speed and section 6 summarizes the paper conclusions.

2 Milling model

A schematic of a two degree-of-freedom milling tool is shown in Fig. 1a. Tool dynamics and cutting forces are used to formulate the governing delay differential equations for the system (the workpiece is assumed rigid, although this is not a strict requirement). A compact form of the equations is:

$$\mathbf{M}\ddot{\vec{X}}(t) + \mathbf{C}\dot{\vec{X}}(t) + \mathbf{K}\vec{X}(t) = \mathbf{K_c}(t)b\left(\vec{X}(t) - \vec{X}(t-\tau)\right) + \vec{f_0}\,(t)\,b, \quad (1)$$

where $\vec{X}\,(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}^T$ is the two-element position vector; $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the 2x2 modal mass, damping, and stiffness matrices, respectively; $\mathbf{K_c}$ is a 2x2 matrix representing the component of cutting forces which depend on the position vector and $\vec{f_0}\,(t)$ is the 2x1 vector that represents the components of the cutting force that are independent of the position vector; b is the axial depth of cut (Fig. 1b); Ω is the spindle speed in rev/min (rpm); N is the number of teeth on the cutting tool; and $\tau = 60/(N\pi)$ is the tooth passing period in seconds. The difficulty in the solution of Eq. (1) arises from the fact that delay differential equations comprise an infinite dimensional monodromy operator [5]. This operator is approximated by a finite dimensional operator using TFEA [9]. The accuracy of this approximation is improved by increasing the number of elements in time. This transforms the original time periodic delay differential equations into a discrete form and provides a means for predicting the milling process stability (i.e., the absence of the self-excited vibrations).

3 Analysis method

The initial work of applying the time finite element approach to delay equations can be found in reference [2]. While this work considered the equations for turning, the methodology was extended to milling in references [9, 10]. Full details for the milling model are provided in these references. Here, we focus only on the eigenvalues of the solution method for brevity. The dynamic behavior of the milling process, Eq. (1), is described by TFEA as a discrete linear map that relates the vibration while the tool tooth is engaged in the cut, which depends on previous tooth passages and therefore includes the time delay τ, to free vibration while the tooth is not engaged in the cut. The dynamic map is expressed as:

$$\begin{Bmatrix} q \\ \\ \dot{q} \end{Bmatrix}^{m} = \mathbf{A} \begin{Bmatrix} q \\ \\ \dot{q} \end{Bmatrix}^{m-1} + \{D\}, \tag{2}$$

where q and $\dot{q}$ are collocation of x and y positions and velocities for all nodal times in one tooth passage m, respectively; $\mathbf{A}$ is the state transition matrix and D is a vector which depends on the process parameters. Note that the size of $\mathbf{A}$ depends on the number of time finite elements, E, and polynomial order representing one time period. Stability of the milling process is determined from the eigenvalues of $\mathbf{A}$, while the eigenvectors determine the modes of instability. The maximum magnitude of the map eigenvalues is described by:

$$\lambda_{\max}[\mathbf{A}] = \max_{k} |\lambda_k|, \tag{3}$$

where λ_k denotes the k^{th} eigenvalue of the dynamic map and the state transition matrix $\mathbf{A}$ is a function of the cutting conditions including Ω, N, radial depth of cut, a, tangential and normal direction cutting force coefficients, K_t and K_n, which relate the corresponding cutting force components to the uncut chip area, and tool modal parameters contained in the mass, stiffness, and damping matrices. Unstable conditions exist if $\lambda_{\max} > 1$.

4 Stability boundary

Stability of the milling process is affected by the cutting conditions, workpiece material and tool modal parameters. For a specific workpiece/tool combination, the primary cutting conditions that affect the process stability are Ω, b and a (see Fig. 1b). Usually, a is assigned by the selected tool path. Therefore, the stability boundary is defined by the stable space of axial depth and spindle speed. A combination of b and Ω values below the stability boundary, $b_{\lim}$, gives stable cutting conditions, whereas a combination above the stability boundary leads to an unstable cut. The stability boundary corresponds to the cutting conditions at which:

$$\lambda_{\max}[\mathbf{A}(b_{\lim}, \Omega)] = 1. \tag{4}$$

We use the bisection method to find the limiting stable axial depth, $b_{\lim}$ corresponding to Eq. (4). The method does not need an initial guess and is guaranteed to converge provided the root is within the selected interval. In Section 5.2, the method allows us to illustrate difficulties associated with finite difference derivatives when the function (e.g., $b_{\lim}$) is calculated by iterative methods. To terminate the bisection method iterations, an absolute value of relative error is used:

$$\varepsilon_a = \left| \frac{b_i - b_{i-1}}{b_i} \right| \leq \varepsilon, \tag{5}$$

where ε_a is the approximate relative error, ε is the error tolerance and b_i is the axial depth at iteration i. The number of iterations, n, needed to find $b_{\lim}$ can also be used as a stopping criterion after setting a desired absolute error in $b_{\lim}$, $E_{a,d}$ [3]:

$$n = log_2 \left(\frac{b_u - b_l}{E_{a,d}} \right), \tag{6}$$

where b_u and b_l are the initial upper and lower limits of the axial depth interval, respectively. The absolute error E_a can be calculated as:

$$E_a = \frac{b_i - b_{i-1}}{2}. \tag{7}$$

The values of ε or n are set based on the numerical accuracy required in the calculation of $b_{\lim}$. For example the number of iterations required for a tool with a flute length (maximum b value) equal to 100 mm and $E_{a,d} = 0.1$ mm is approximately 10. Additionally, a value of $\varepsilon = 0.001$ is typically adequate for $b_{\lim}$ calculation. However, to obtain an accurate derivative of $b_{\lim}$ by finite differences it may be necessary to use a smaller value of ε or $E_{a,d}$ (see section 5.1).

Using the method, the stability boundary is computed in Fig. 2 for a down milling process (see Fig. 1c) using a 25.4 mm diameter endmill with a 12° helix angle and 114 mm overhang length (from the holder face). Table 1 lists the tool mean modal values, representative cutting force coefficients for 6061-T6 aluminum and machining parameters. The stability boundary (Fig. 2), is seen to exhibit slope discontinuity at the lobe peaks. Since the stability boundary is determined using $\lambda_{\max}$, the discontinuity occurs when two eigenvalues change places in terms of having the largest magnitude. In section 5.3, the sensitivity of this boundary to spindle speed is computed.

5 Sensitivity of stability boundary

The sensitivity of b_{lim} to input parameters is cumbersome to compute analytically using the TFEA method; therefore, a numerical derivative is required. One option, which is available for calculating any derivative, is finite differences. Here, the central difference method is used to calculate the b_{lim} sensitivity to a parameter of interest. Since finite difference calculation is often a source of numerical inaccuracies, we propose an alternative semi-analytical method that calculates the stability boundary sensitivity using the derivative of the system maximum eigenvalue.

5.1 Semi-analytical derivative of the stability boundary

The sensitivity of the stability boundary is calculated using the system maximum eigenvalue. Equation (4) provides an implicit relationship between the limiting axial depth of cut and spindle speed. The process of calculating $\partial b_{\text{lim}}/\partial \Omega$ is the same as calculating derivatives with respect to any other parameter, so the discussion below is limited to this derivative.

The differential form of Eq. (4) is:

$$\frac{\partial \lambda_{\max}}{\partial b}\,\mathrm{d}b_{\mathrm{lim}} + \frac{\partial \lambda_{\max}}{\partial \Omega}\,\mathrm{d}\Omega = 0, \tag{8}$$

which can be viewed as the differential equation of the stability boundary. From Eq. (8), we get

$$\frac{\mathrm{d}b_{\mathrm{lim}}}{\mathrm{d}\Omega} = -\frac{\partial \lambda_{\max}}{\partial \Omega} \Big/ \frac{\partial \lambda_{\max}}{\partial b}. \tag{9}$$

In the milling analysis, the eigenvalue may be complex:

$$|\lambda_{\max}|^2 = \lambda_{\max}\lambda'_{\max}, \tag{10}$$

where $\lambda_{\max}$ denotes the complex maximum eigenvalue and the prime (superscript) denotes its complex conjugate. Differentiating the maximum eigenvalue with respect to a generic variable z we get:

$$\frac{\partial |\lambda_{\max}|}{\partial z} = \frac{\lambda_{\max}\frac{\partial \lambda'_{\max}}{\partial z} + \lambda'_{\max}\frac{\partial \lambda_{\max}}{\partial z}}{2|\lambda_{\max}|}. \tag{11}$$

In order to determine the derivatives in the numerator of the right hand side of Eq. (11), we recognize that the derivative of an eigenvalue λ of a general non-symmetric matrix $\mathbf{A}$ with respect to a parameter is (e.g., see ref. [8] and [4]):

$$\frac{\partial \lambda}{\partial z} = \frac{v^T \frac{\partial \mathbf{A}}{\partial z} u}{v^T u}, \tag{12}$$

where u and v are the right and left eigenvectors associated with the eigenvalue λ, respectively. The derivative of the matrix $\mathbf{A}$ in Eq. (12) is obtained by the central difference method (making our approach semi-analytical). It should be noted that Eq. (12) is valid only if the eigenvalue is not repeated [6]. However, for a repeated eigenvalue, the derivative of $\lambda_{\max}$ is not defined anyway, since there is more than one eigenvalue with the same maximum magnitude. Once the solution to Eq. (12) is obtained, it is substituted into Eq. (11). This result is then inserted in Eq. (9) to determine the desired sensitivity.

A direct calculation of b_{lim} sensitivity with respect to spindle speed can also be performed using central finite difference:

$$\frac{\mathrm{d}b_{\mathrm{lim}}}{\mathrm{d}\Omega} = \frac{b_{\mathrm{lim}}(\Omega+h) - b_{\mathrm{lim}}(\Omega-h)}{2h} + \frac{h^2}{6}\frac{\mathrm{d}^3 b_{\mathrm{lim}}}{\mathrm{d}\Omega^3}(\Omega+\zeta h), \qquad -1 \leq \zeta \leq 1. \tag{13}$$

where h denotes the step size in Ω and last term is the 2^{nd} order truncation error, e_T.

5.2 Error analysis

Factors which affect accurate calculation of sensitivity include: 1) condition error, e_C; and 2) truncation error [4]. Condition error usually occurs in an ill-conditioned numerical computation where the round-off contribution is significant or when the function (e.g., b_{lim}) is calculated using an iterative process such as the bisection method and is terminated early, see Eq. (5). Assuming an error tolerance ε in the calculation of b_{lim}, the condition error for central finite difference can be approximated from Eq. (13) as [4, p. 256]:

$$e_C = \frac{\varepsilon}{h}. \tag{14}$$

The total error becomes,

$$e = e_C + e_T = \frac{h^2}{6}|s_b| + \frac{\varepsilon}{h}, \tag{15}$$

where s_b is the bound on the third derivative in the interval $\Omega \in [\Omega \pm h]$. If s_b is available, we can find an h value which gives a tradeoff of truncation and condition errors. In the following we compare the numerical accuracy of the semi-analytical method with respect to overall finite difference approach and show that the former is robust to an approximate calculation of the response.

5.3 Results

In order to demonstrate the robustness of the semi-analytical method, the b_{lim} sensitivity is computed using different step sizes at some nominal spindle speed. All the computations are made using a converged solution with 10 elements in time. The logarithmic derivative of b_{lim} is calculated to indicate the sensitivity [4] because it gives the percent change in b_{lim} due to a percent change in Ω:

$$\frac{\mathrm{d}ln(b_{\mathrm{lim}})}{\mathrm{d}ln(\Omega)} = \frac{\Omega\,\mathrm{d}b_{\mathrm{lim}}}{b_{\mathrm{lim}}\,\mathrm{d}\Omega}. \tag{16}$$

The central finite difference approach at a nominal $\Omega = 10,000$ rpm yields only a small step size range where accurate calculation of b_{lim} sensitivity is possible [7] (see Fig. 3). For example, with an error tolerance $\varepsilon = 1 \times 10^{-4}$, the spindle speed step size, h, must be in the range of 1×10^{-4} to 1×10^{-2}. This range can be extended by using a smaller ε value. However, to apply a smaller ε value, a larger number of iterations, n, is required; see Eq. (6). For example, the number of iterations for $\varepsilon = 1 \times 10^{-4}$ and $\varepsilon = 1 \times 10^{-7}$ are in the range of 20-22 and 30-32, respectively. The semi-analytical method gives a wider range where the sensitivity is accurate, even for larger ε values. As seen in Fig. 3, the semi-analytical method is accurate in the range of $h = 1 \times 10^{-6}$ to 1×10^{-1} with $\varepsilon = 1 \times 10^{-2}$. Further, because of this larger tolerance, only 15 iterations were required. It should be noted here that the central finite difference needs two function evaluations of b_{lim} ($2n$ iterations) whereas the semi-analytical method requires only one function evaluation (n iterations). As seen in the figure, only for $\varepsilon = 1 \times 10^{-7}$ do the central difference results reach the same level of step size stability as the semi-analytical method. In this case, the central finite difference needed four time more iterations than the semi-analytical method (2×30 versus 15 for the semi-analytical) to reach the same overall step size stability.

Additionally, near C^1 (slope) discontinuities, the accuracy and stable range of the finite difference method is largely affected. Figure 4 compares the two methods near a C^1 discontinuity at 11800 rpm. As can be seen, the stable step size range for the semi-analytical method was minimally affected by being in the vicinity of the discontinuity, although a smaller value of ε was needed for a more accurate evaluation of sensitivity. However, for the central finite difference calculations, the tolerance error of $\varepsilon = 1 \times 10^{-4}$ no longer provided stable calculation of the derivative compared to the case at 10000 rpm and an $\varepsilon = 1 \times 10^{-7}$ value was needed to provide stable calculation of the derivative in even a smaller step size range of 1×10^{-6} to 2×10^{-5}. The minimal effect of the discontinuity on the semi-analytical method makes it more robust than central finite difference.

It is instructive to investigate the source of numerical error encountered in the central finite difference computation. To facilitate this, we report on the computed b_{lim} and $\lambda_{\max}$ as a function of Ω near $\Omega = 11800$, where the response is calculated by incrementing Ω in small steps equal to $h = 1 \times 10^{-4}$ (or 1.18 rpm). In Fig. 6a we note that $\lambda_{\max}$ varies smoothly as Ω is changed, this allows the semi-analytical method to be more accurate. However, in Fig. 6b, constant then staircase variation of b_{lim} is observed. We note here that b_{lim} was computed using an $\varepsilon = 1 \times 10^{-2}$ and fixed number of iterations $n = 15$. The staircase variation in b_{lim} is due to the binary nature of the bisection algorithm. At each iteration the algorithm makes a choice between a left and a right interval. As the frequency changes, the choice in the last iteration can switch from choosing one to the other causing a discontinuity in the calculated b_{lim}. It should be noted that similar difficulties may be expected with finite difference derivatives of quantities calculated by other iterative processes which are terminated early to reduce computational cost. Consequently, for the semi-analytical method, in contrast to central finite difference, see Eq. (15), high accuracy in the stability boundary is not a strict requirement. This is illustrated in Fig. 5, where the sensitivity is compared for $h = 1 \times 10^{-3}$ and different values of tolerance error ε. As can be seen, the sensitivity calculated using the semi-analytical method is accurate even for a coarse calculation of the stability boundary. This illustrates the efficiency and robustness of the semi-analytical method. An example of the sensitivity calculation $\partial b_{\text{lim}}/\partial \Omega$ using both methods is shown in Fig. 7. It is required that the step size be carefully selected (see Table 2) if agreement between the central finite difference and semi-analytical methods is to be achieved. Note that a three order of magnitude smaller tolerance error and one order of magnitude smaller step size was required for the central finite difference calculations to provide stable derivative calculation in the vicinity of the C^1 discontinuity. Considering the tolerance error used in both methods, the number of iterations for central finite difference and semi-analytical methods was about (2×32) and 22, respectively. The semi-analytical solution provided an efficiency improvement of 290% and increased flexibility in the step size and tolerance error selection.

6 Conclusions

In this paper, a new semi-analytical method for calculating the sensitivity of the stability boundary for a system of delay differential equations was presented. The approach was demonstrated for a milling application where the stability limit is defined by the maximum eigenvalue and is expressed as a curve in a plane defined by axial depth and spindle speed. The method was compared to the central finite difference approach and was shown to be more efficient and robust in calculating the stability sensitivity with minimal dependence on the stability accuracy. The loss of accuracy of the finite difference method was traced to be the result of the iterative bisection solution for the stability boundary. It should be noted that similar difficulties may be expected with finite difference derivatives of quantities calculated by other iterative processes which are terminated early to reduce computational cost.

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Table 1

Cutting force coefficients, modal parameters and cutting conditions.

M (kg)		**C** (kg/s)		**K** ($\times 10^6$ N/m)	
0.44	0	83	0	4.45	0
0	0.44	0	91	0	3.55
d (mm)		c (mm)		a (mm)	N
25.4		0.1		0.5	1
K_t(N/m^2)			K_n (N/m^2)		
600$\times 10^6$			180$\times 10^6$		

Table 2

Parameters used in sensitivity calculation in Fig. 7.

Method	$h\%$	E	ε
Central finite difference	0.001	10	1×10^{-7}
Semi-analytical	0.01	10	1×10^{-4}

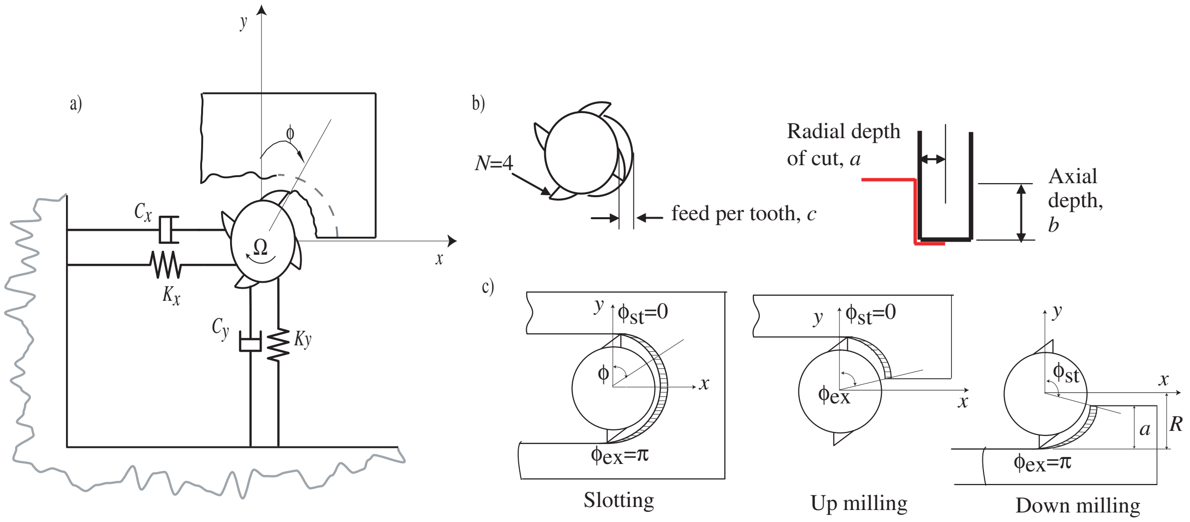

Fig. 1. a) Schematic of 2-DOF milling tool. b) Identification of key variables. c) Various types of milling operations.

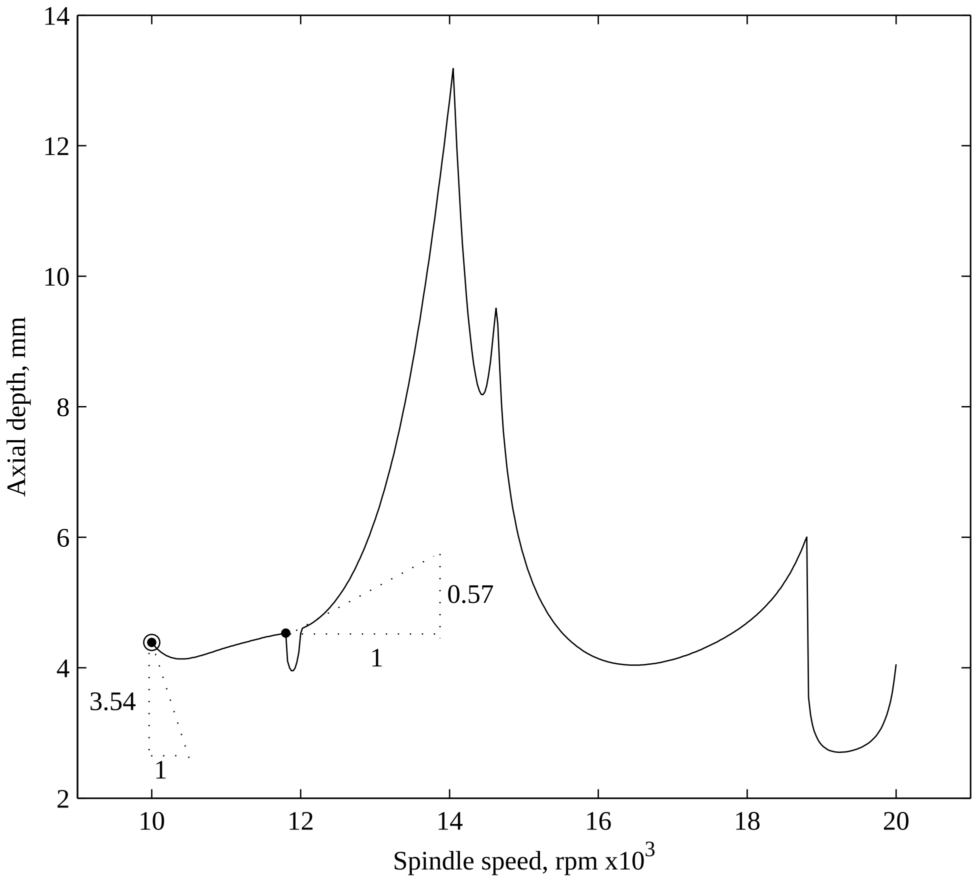

Fig. 2. Stability boundary calculated using the cutting conditions listed in Table 1. Since the stable boundary is determined by the maximum magnitude of an eigenvalue of the dynamic map, a slope discontinuity occurs at the cusp where two eigenvalues swap places as the one with the largest magnitude.

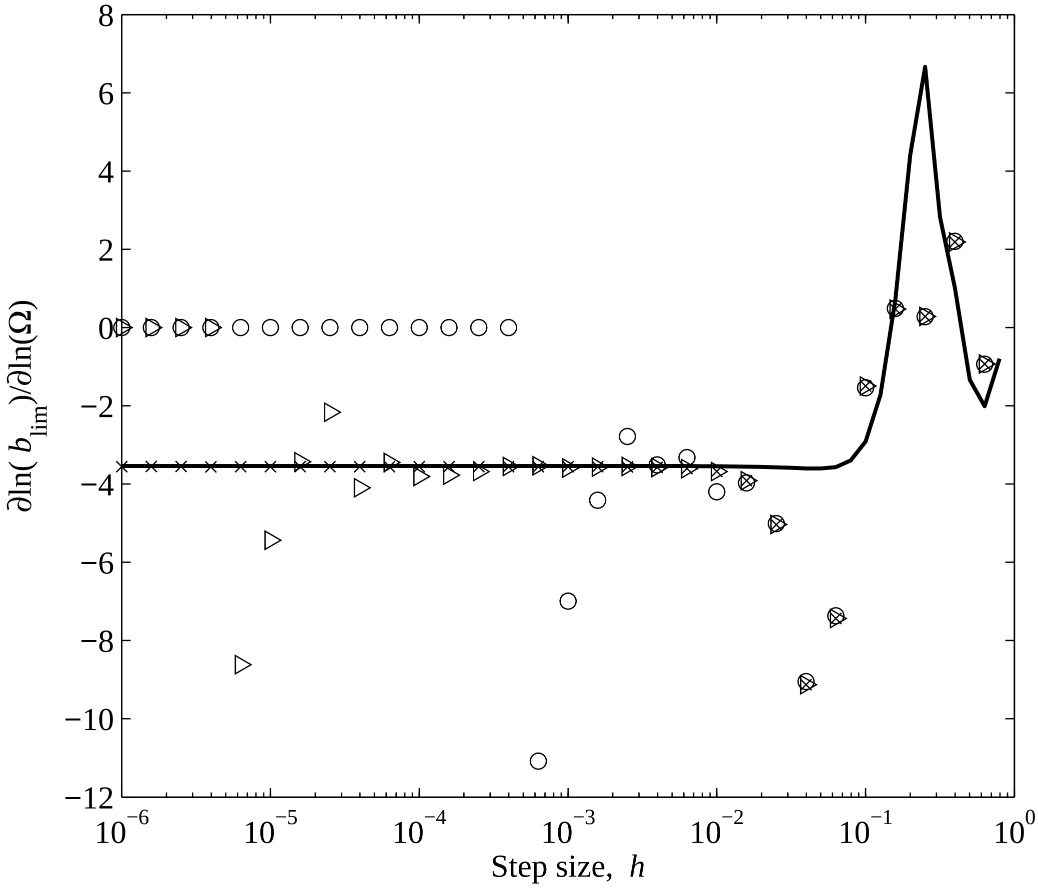

Fig. 3. The logarithmic derivative of axial depth with respect to spindle speed versus step size percentage at a spindle speed of 10000 rpm. — Semi-analytical, $\varepsilon = 1 \times 10^{-2}$. $\circ$ Central finite difference, $\varepsilon = 1 \times 10^{-2}$; $\triangleright$ central finite difference, $\varepsilon = 1 \times 10^{-4}$; $\times$ central finite difference, $\varepsilon = 1 \times 10^{-7}$. The semi-analytical method provides a large accurate step size range even with a larger value of ε.

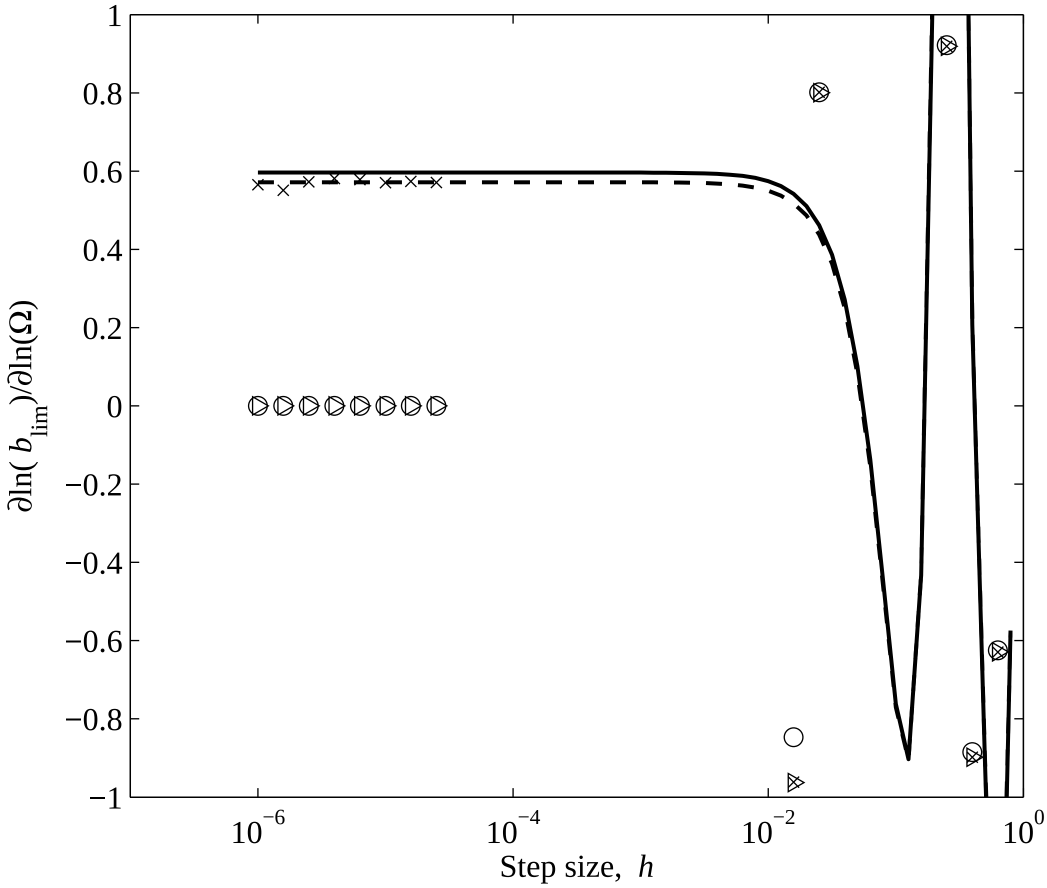

Fig. 4. The logarithmic derivative of axial depth with respect to spindle speed versus step size percentage near a derivative discontinuity (11800 rpm). — Semi-analytical, $\varepsilon = 1 \times 10^{-2}$; −− Semi–analytical, $\varepsilon = 1 \times 10^{-4}$. $\circ$ Central finite difference, $\varepsilon = 1 \times 10^{-2}$; $\triangleright$ central finite difference, $\varepsilon = 1 \times 10^{-4}$; $\times$ central finite difference, $\varepsilon = 1 \times 10^{-7}$. The semi-analytical results are minimally affected by the slope discontinuity. For the central finite difference, however, only a tolerance error of 1×10^{-7} gives a stable derivative calculation. Further, a smaller step size range than Fig. 3 (1×10^{-6} to 2×10^{-5}) is obtained.

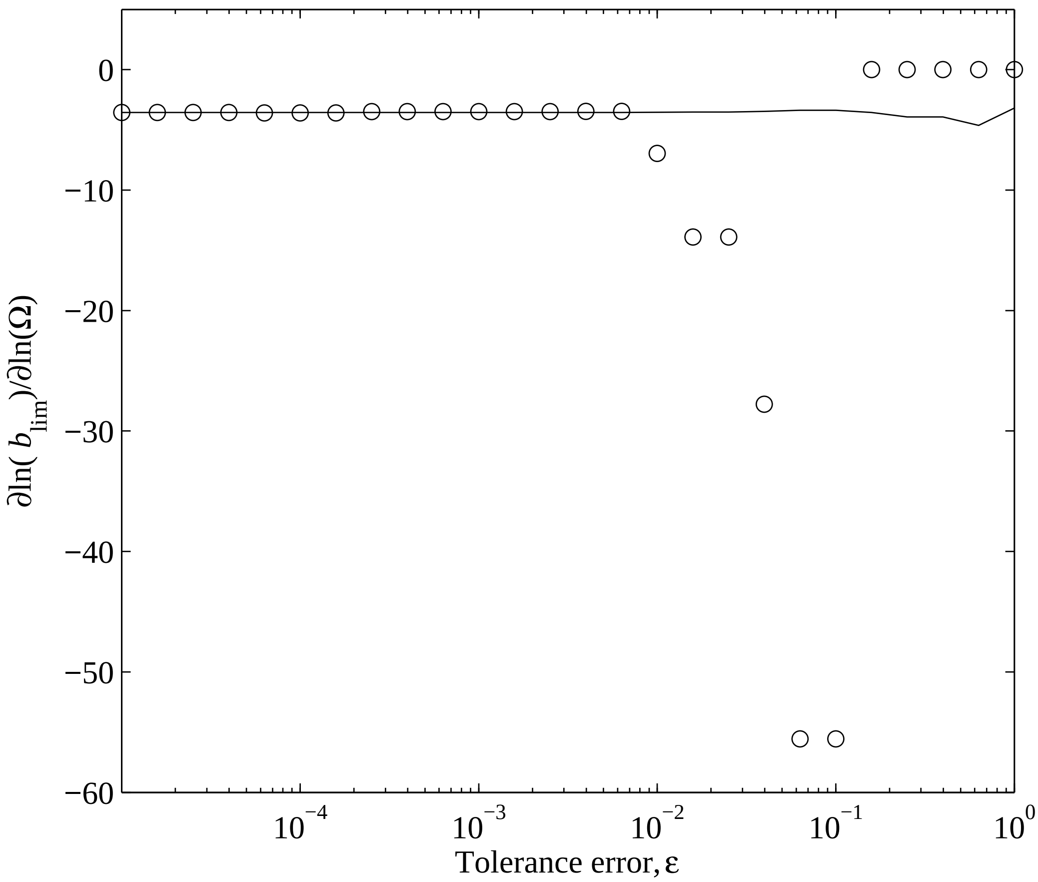

Fig. 5. Comparison between the calculated sensitivity using overall finite difference and semi-analytical methods for the same step size $h = 1 \times 10^{-3}$ at a spindle speed of 10000 rpm. — Semi-analytical; ○ central finite difference. The semi-analytical method gives accurate calculation of b_{lim} sensitivity even for large ε.

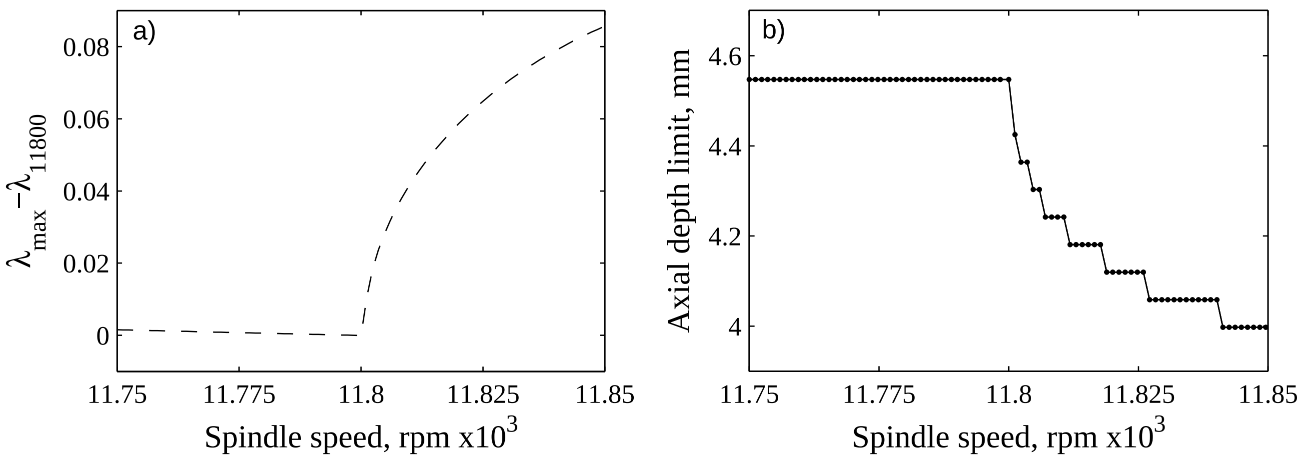

Fig. 6. Source of numerical error in sensitivity computation. a) Smooth variation of $\lambda_{\max}$. b) Staircase variation in $b_{\lim}$: — number of iterations, $n = 15$; $\cdot$ $\varepsilon = 1 \times 10^{-2}$.

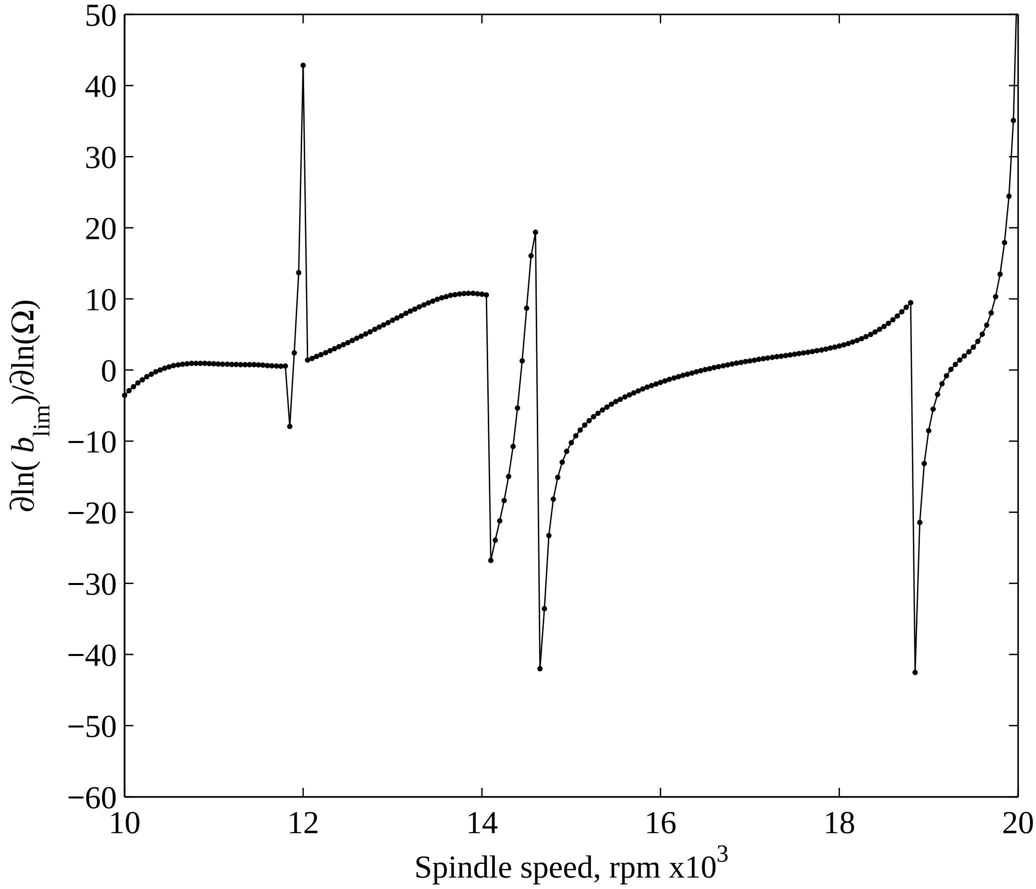

Fig. 7. Comparison between the calculated sensitivity using central finite difference and semi-analytical methods. — Semi-analytical; · central finite difference. Good agreement is observed provided the conditions are carefully selected (see Table 2).