A STUDY ON NUMERICAL INSTABILITY OF INVISCID TWO-FLUID MODEL NEAR ILL-POSEDNESS CONDITION

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ABSTRACT

The two-fluid model is widely used in studying gas-liquid flow inside pipelines because it can qualitatively predict the flow field at low computational cost. However, the two-fluid model becomes ill-posed when the slip velocity exceeds a critical value, and computations can be quite unstable before flow reaches the unstable condition. In this study computational stability of various convection schemes for the two-fluid model is analyzed. A pressure correction algorithm for inviscid flow is carefully implemented to minimize its effect on numerical stability. Von Neumann stability analysis for the wave growth rates by using the 1st order upwind, 2nd order upwind, QUICK, and the central difference schemes shows that the central difference scheme is more accurate and more stable than the other schemes. The 2nd order upwind scheme is much more susceptible to instability at long waves than the 1st order upwind and inaccurate for short waves. The instability associated with ill-posedness of the two-fluid model is significantly different from the instability of the discretized two-fluid model. Excellent agreement is obtained between the computed and predicted wave growth rates. The connection between the ill-posedness of the two-fluid model and the numerical stability of the algorithm used to implement the inviscid two-fluid model is elucidated.

INTRODUCTION

Gas-liquid flow inside a pipeline is prevalent in the handling and transportation of fluids. A reliable flow model is essential to the prediction of the flow field inside the pipeline. To fully simulate the system, Navier–Stokes equations in three-dimensions are required. However, it is very expensive to simulate flows in a long pipe with today’s computer limitations. To reduce the computational cost and obtain basic and essential flow properties, such as gas volume fraction, liquid and gas velocity, pressure, a one-dimensional model is necessary. The two-fluid model is considered to give a realistic prediction for the gas-liquid flow inside a pipeline.

The two-fluid model, also known as the separated flow model, consists of two sets of conservation equations for mass, momentum and energy for the gas phase and the liquid phase. It was proposed by Wallis [1], and further refined by Ishii [2]. Although it has success in simulating two-phase flow in a pipeline, the two-fluid model suffers an ill-posedness problem. When relative velocity between liquid and gas exceeds a critical value, the governing equations do not possess real characteristics [3] [4] [5]. This ill-posedness condition suggests that the results of the two-fluid model at that condition do not reflect the real flow situation inside the pipe. The two-fluid model only gives meaningful results when the relative velocity between gas and liquid phase is less than a critical value, which depends on gravity and liquid level, among other flow properties. However, this critical value coincides with the stability condition of inviscid Kelvin-Helmholtz instability (IKH) analysis [6]. Because the instability of IKH analysis results in the flow regime transition from stratified flow to slug flow or annular flow [7], ill-posedness of two-fluid model is interpreted as to trigger the flow regime transition [7] [8].

The computational methods for solving the two-fluid model have been investigated by many researchers. In this study, it is further assumed that both liquid and gas phases are incompressible because most of stratified flows are at relatively low speed compared with the speed of sound. To solve the incompressible two-fluid equations, one approach is to simplify the governing system to only two equations for liquid phase volume fraction and liquid velocity and neglect the transient terms in the gas mass and momentum equations [7] [9]. A more effective method is to use a pressure correction scheme [10]. Issa and Kempf [6], and Issa and Woodburn [11] applied the pressure correction scheme for the two-fluid model and simulated stratified flow and slug flow inside a pipe.
When two-fluid model becomes ill-posed, the solution becomes unstable. A good discretized model should be capable of capturing the incipience of the instability point. However, numerical instability may not be the same as the instability caused by the ill-posedness. Lyczkowski et al. [12] used von Neumann stability analysis to study a compressible two-fluid model with their numerical scheme and found that numerical instability and ill-posedness may not be identical. However, their two-fluid model lacked the gravitational term and the study focused on one discretization scheme and is thus incomplete. Stewart [13], Ohkawa and Tomiyama [14] attempted to analyze the numerical stability of an incompressible two-fluid model with a simplified model equation as an alternative. Their study showed that higher order upwind schemes yield a more unstable numerical solution than the 1st order upwind scheme.

In this study, a pressure correction scheme is presented that is designed to increase the stability of the numerical scheme when the flow is near the ill-posedness condition. The von Neumann stability analysis is employed to study the stability of the discretized two-fluid model with different interpolation schemes for the convection term. For the wave amplification factor using the 1st order upwind, 2nd order upwind, QUICK, and central difference schemes, the central difference scheme is more accurate and more stable. Excellent agreement for the growth of wave amplitude is obtained between the analysis and the actual computation under various configurations.

NOMENCLATURE

- \( c \) = wave speed
- \( E \) = common amplitude factor
- \( G \) = amplification factor
- \( g \) = gravity
- \( k \) = wavenumber
- \( H \) = hydraulic depth
- \( p \) = pressure
- \( N \) = grids number
- \( t \) = time
- \( u \) = velocity
- \( x \) = space coordinate
- \( \alpha \) = volume fraction
- \( \beta \) = angle of inclination from the horizontal
- \( \varepsilon \) = amplitude factor
- \( \phi \) = phase angle
- \( \lambda \) = characteristic root
- \( \rho \) = density

Subscripts

- \( e \) = east face of control volume
- \( g \) = gas
- \( l \) = liquid
- \( i \) = interface, grid index
- \( P \) = center of main control volume
- \( w \) = west face of control volume

NUMERICAL METHOD

Governing Equations

The basis of the two-fluid model is a set of one-dimensional conservation equations for the balance of mass, momentum and energy for each phase. The one-dimensional conservation equations are obtained by integrating the flow properties over the cross-sectional area of the flow.

![Fig.1 Schematic depiction of two-phase flow in a horizontal pipe.](image)

Because the ill-posedness originates from the hydrodynamic instability of the two-fluid model, only continuity and momentum equations are considered in the inviscid two-fluid model. Surface tension is also neglected since it only acts on small scales, while the waves determining the flow structure in pipe flows are usually of long wavelength. The gas phase is assumed incompressible, as the Mach number of the gas phase is usually very low for stratified flow. Hence, the governing equations are as follows:

\[
\frac{\partial}{\partial t} (\alpha_g u_g) + \frac{\partial}{\partial x} (\alpha_g u_g^2) = 0, \quad (1) \\
\frac{\partial}{\partial t} (\alpha_l u_l) + \frac{\partial}{\partial x} (\alpha_l u_l^2) = 0, \quad (2) \\
\frac{\partial}{\partial t} (\alpha_g u_g) + \frac{\partial}{\partial x} (\alpha_l u_l) = -\frac{\alpha_g}{\rho_g} \frac{\partial p}{\partial x} - g \cos \beta H \frac{\partial \alpha_l}{\partial x} - \alpha_g \sin \beta, \quad (3) \\
\frac{\partial}{\partial t} (\alpha_l u_l) + \frac{\partial}{\partial x} (\alpha_l u_l^2) = -\frac{\alpha_l}{\rho_l} \frac{\partial p}{\partial x} - g \cos \beta H \frac{\partial \alpha_g}{\partial x} - \alpha_l \sin \beta. \quad (4)
\]

where \( t \) and \( x \) are the respective time and axial coordinates, \( \alpha \) is the volume fraction, \( u \) is the velocity, \( p \) is the pressure, \( \rho \) is the density, \( H \) is the hydraulic depth, \( g \) is the gravitational acceleration, \( \beta \) is the angle of inclination of the pipe; the subscripts \( l \) and \( g \) denote the liquid and gas, respectively, and the subscript \( i \) denotes the interface.

Computational Procedure

The governing equations (1-4) are solved iteratively. The basic procedure is to solve the continuity equation for liquid for the liquid volume fraction, and the liquid and gas phase momentum equations are used to obtain the liquid and gas phase velocities. To obtain a governing equation for the pressure, Eq. (1) and Eq. (2) are first combined to form a total mass conservation,

\[
\frac{\partial}{\partial x} (\alpha_g u_g) + \frac{\partial}{\partial x} (\alpha_l u_l) = 0. \quad (5)
\]

Substituting the liquid and gas momentum equations into (5) yields a constraint on pressure. SIMPLE type of pressure correction scheme [10] is then used.

A finite volume method is employed to discretize the governing equations. A staggered grid (Fig. 2) is adopted to obtain a compact stencil for pressure [15]. On the staggered grids, the flow properties such as volume fraction, density and pressure are located at the center of the main control volume,
and the liquid and gas velocities are located at the cell face of the main control volume.

The Euler backward scheme is employed for the transient term. The discretized liquid continuity equation becomes

$$\frac{\Delta x}{\Delta t} \left[ \left( \alpha_l \right)_e - \left( \alpha_l \right)_w \right] + \left( \alpha_l \right)_e \left( u_l \right)_e - \left( \alpha_l \right)_w \left( u_l \right)_w = 0, \quad (6)$$

where the superscript 0 represents the values at the last time step. The subscript $P$ denotes the center of the main control volume, while subscripts $e$ and $w$ denote the east and west faces of main control volume, respectively. The liquid velocity at the cell face is known, and the volume fraction at the cell face can be evaluated using various interpolation schemes. Among them, central difference (CDS), 1st order upwind (FOU), 2nd order upwind (SOU) and QUICK schemes are commonly used. Eq. (2) for the gas phase is similarly discretized.

The liquid momentum equation is integrated on the velocity control volume. Using similar notations, one obtains

$$\frac{\Delta x}{\Delta t} \left[ \left( \alpha_l \right)_e - \left( \alpha_l \right)_w \right] + \left( \alpha_l \right)_e \left( u_l \right)_e - \left( \alpha_l \right)_w \left( u_l \right)_w = 0, \quad (7)$$

It is important to note that the interpolation schemes used in Eq. (7) must be exactly the same as those in Eq. (6) in order to reduce the dissipation and dispersion errors.

The gas phase momentum equation is similarly treated,

$$\frac{\Delta x}{\Delta t} \left[ \left( \alpha_g \right)_e - \left( \alpha_g \right)_w \right] + \left( \alpha_g \right)_e \left( u_g \right)_e - \left( \alpha_g \right)_w \left( u_g \right)_w = 0, \quad (8)$$

For the pressure correction scheme, Eq. (5) is integrated across the main control volume. The discretized equation is

$$\left( \alpha_l u_e \right)_e = \left( \alpha_l u_w \right)_w + \left( \alpha_l \right)_w \left( u_l \right)_w = 0. \quad (9)$$

Because Eq. (5) is obtained by combining Eq. (1) and Eq. (2), the discretization of Eq. (9) should be exactly the same as that of Eqs. (2, 6). The final pressure equation is obtained by substituting two momentum equations, Eqs. (7, 8) to Eq. (9).

### Characteristics and Ill-Posedness

Eqs. (1-4) form a system of 1st order PDEs and characteristic roots, $\lambda$, of the system can be found. If $\lambda$’s are real, the system is hyperbolic. Complex roots imply an elliptic system which causes the two-fluid model system to become ill-posed because only initial conditions can be specified in the temporal direction. Any infinitesimal disturbance will cause the waves to grow exponentially without bound.

The characteristic roots of Eqs. (1-4) are

$$\lambda = \frac{\left( \rho_l \frac{u_l}{\alpha_l} + \rho_g \frac{u_g}{\alpha_g} \right) \pm \sqrt{\left( \rho_l \frac{u_l}{\alpha_l} - \rho_g \frac{u_g}{\alpha_g} \right)^2 - 2 \rho_l \rho_g g \sin \beta \alpha_l \alpha_g (u_l - u_g)^2}}{\rho_l + \rho_g}. \quad (10)$$

When $g = 0$, Eq. (10) can have real roots only if $\lambda = u_l = u_g$. Otherwise, the two-fluid model is ill-posed (Gidaspow, 1974). If $g \neq 0$, the real roots (or well-posedness) requirement gives

$$\left( u_l - u_g \right)^2 < \left( \frac{\alpha_l}{\rho_l} + \frac{\alpha_g}{\rho_g} \right) \frac{\rho_l - \rho_g}{\alpha_l} g \sin \beta. \quad (11)$$

Eq. (11) gives the critical value for the two phase slip velocity beyond which the system becomes ill-posed. The two-fluid model well-posedness condition is exactly the same as from the IKH analysis on the two-fluid model [7].

### Von Neumann Analysis for Various Discretization Schemes

Von Neumann stability analysis is commonly used for analyzing the stability of a finite difference scheme.

![Grid index in staggered grid for von Neumann stability analysis](image)

For the pressure correction scheme, Eq. (5) is integrated across the main control volume. The discretized equation is

$$\left( \alpha_l u_e \right)_e = \left( \alpha_l u_w \right)_w + \left( \alpha_l \right)_w \left( u_l \right)_w = 0. \quad (9)$$

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In this derivation, the 1st order upwind (FOU) scheme is used as an example, and both liquid and gas velocities are assumed positive. Discretization of Eq. (6) using FOU leads to

$$\left( \alpha_l \right)_p - \left( \alpha_l \right)_l + \left( \alpha_l \right)_l \left( u_l \right)_l - \left( \alpha_l \right)_w \left( u_l \right)_w = 0. \quad (12)$$

Splitting the variables into base values and disturbances, the linearized equation for the disturbance $\bar{\alpha}_l$ is

$$\bar{\alpha}_l^p = \bar{\alpha}_l^w + \left( \alpha_l \right)_l \left( u_l \right)_l - \left( \alpha_l \right)_w \left( u_l \right)_w = 0. \quad (13)$$

where “$^\text{av}$” denotes disturbance variables. Expressing disturbances as

$$\bar{\alpha}_l^p = E \alpha_l \alpha_l^p, \quad (14)$$

$$\bar{\alpha}_l^w = E \alpha_l \alpha_l^w, \quad (15)$$

where $E$ is a common amplitude factor, $k$ is the wavenumber, and $I$ represents imaginary unit. Eq. (13) is simplified to

$$\left( 1 - G^{-1} \right) u_l^I + I E \alpha_l \alpha_l^I = 0. \quad (17)$$

where $G$ is the amplification factor:

$$G = \frac{E^n}{E^{n-1}}, \quad (18)$$
and $\phi$ is the phase angle:

$$\phi = k \cdot \Delta x.$$  \hfill (19)

defined over $[0, \pi]$ which represents all the resolvable wave components in the computational domain for the given grid. Short waves correspond to the region near $\phi = \pi$.

The wave growth equation for the gas phase mass conservation equation is similarly obtained:

$$\varepsilon \left( \frac{\Delta}{\Delta t} \left( 1 - G^{-1} \right) + u_k (1 - e^{-i\phi}) \right) - \varepsilon \omega \alpha \left( e^{i\phi} - e^{-i\phi} \right) = 0. \hfill (20)$$

For the liquid momentum equation, Eq. (10) is discretized with the FOU scheme to

$$\frac{\Delta x}{\Delta t} \left[ \rho_l \left( \frac{\partial v_{iln}}{\partial t} - \frac{\partial \rho_l}{\partial t} \right) + \rho_l u_k \left( \frac{\partial v_{iln}}{\partial x} + \frac{\partial u_{iln}}{\partial t} \right) \right]$$

$$= \left( \hat{p}_{iln}^* - \hat{p}_{iln} \right) + \rho_l \cos \beta \frac{H_k}{\alpha_l} \left( \hat{\alpha}_{iln}^* - \hat{\alpha}_{iln} \right). \hfill (21)$$

For the gas phase, the velocity disturbance is governed by

$$\frac{\Delta x}{\Delta t} \left( \rho_l \left( \frac{\partial v_{iln}}{\partial t} - \frac{\partial \rho_l}{\partial t} \right) + \rho_l u_k \left( \frac{\partial v_{iln}}{\partial x} + \frac{\partial u_{iln}}{\partial t} \right) \right)$$

$$= \left( \hat{p}_{iln}^* - \hat{p}_{iln} \right) + \rho_l \cos \beta \frac{H_k}{\alpha_g} \left( \hat{\alpha}_{iln}^* - \hat{\alpha}_{iln} \right). \hfill (22)$$

The pressure term can be canceled by combining Eqs. (22-23),

$$\varepsilon \left( \frac{\Delta}{\Delta t} \rho_l \left( 1 - G^{-1} \right) + \rho_l u_k \left( 1 - e^{-i\phi} \right) \right) - \varepsilon \omega \alpha \left( e^{i\phi} - e^{-i\phi} \right)$$

$$= \varepsilon \left( \frac{\Delta}{\Delta t} \rho_l \left( 1 - G^{-1} \right) + \rho_l u_k \left( 1 - e^{-i\phi} \right) \right) + \varepsilon \left( \frac{\Delta}{\Delta t} \rho_l \left( 1 - G^{-1} \right) + \rho_l u_k \left( 1 - e^{-i\phi} \right) \right)$$

$$+ \varepsilon \left( \frac{\Delta}{\Delta t} \rho_l \left( 1 - G^{-1} \right) + \rho_l u_k \left( 1 - e^{-i\phi} \right) \right) = 0. \hfill (23)$$

Eqs (17, 20, 24) can be written in the form of an amplification matrix. Non-trivial solutions for $\{ e, e^*, e^* \}$ exist only when the determinant of the matrix is zero. Hence,

$$a(G^{-1})^2 + b(G^{-1}) + c = 0.$$  \hfill (25)

where

$$a = \rho,$$  \hfill (26a)

$$b = -2 \left( \frac{\rho_g}{\alpha_g} (1 + CFL_1 \Delta \phi) + \frac{H_k}{\alpha_l} (1 + CFL_1 \Delta \phi) \right),$$  \hfill (26b)

$$c = \frac{\rho_g}{\alpha_g} \left( 1 + CFL_1 \Delta \phi \right)^2 + \frac{H_k}{\alpha_l} \left( 1 + CFL_1 \Delta \phi \right)^2,$$  \hfill (26c)

and CFL is the Courant number

$$CFL_1 = \frac{\Delta t}{u_k}, \text{ and } CFL_2 = \frac{\Delta t}{u_g}. \hfill (27)$$

The values of $\Delta \phi$ are given in Table 1. From Eq. (25), the amplification factor can be easily found,

$$G = \frac{2a}{-b \pm \sqrt{b^2 - 4ac}} \hfill (28)$$

Stability requires $|G| \leq 1$ for all $\phi$.

**Table. 1. $\Delta \phi$ for different discretization schemes.**

<table>
<thead>
<tr>
<th>Scheme</th>
<th>$\Delta \phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1\textsuperscript{st} order upwind</td>
<td>$1 - e^{-i\phi}$</td>
</tr>
<tr>
<td>Central difference</td>
<td>$e^{i\phi} - e^{-i\phi}$</td>
</tr>
<tr>
<td>2\textsuperscript{nd} order upwind</td>
<td>$3 - 4e^{-i\phi} + e^{-2i\phi}$</td>
</tr>
<tr>
<td>QUICK</td>
<td>$3e^{i\phi} + 3 - 7e^{-i\phi} + e^{-2i\phi}$</td>
</tr>
</tbody>
</table>

**Inviscid Kelvin-Helmholtz (IKH) Analysis**

IKH analysis provides a stability condition for the two-fluid model as well as useful information on the growth rate of disturbance in the two-fluid model. Eqs. (1-4) are linearized and substituted for the perturbed liquid volume fraction, liquid and gas phase velocities, and pressure in the form of $\varepsilon \exp[I(\omega x - k z)]$ in which $\varepsilon$ is the amplitude, $\omega$ is the angular frequency, and $k$ is the wavenumber. The following system is obtained for the disturbance amplitude:

$$\begin{vmatrix}
\omega - u_k & -\alpha_k & 0 & 0 \\
\omega - u_k & 0 & \alpha_k & 0 \\
-k \frac{H_l}{\alpha_l} & g \cos \beta & 0 & 0 \\
-k \frac{H_l}{\alpha_l} & 0 & \omega - u_k & 0 \\
\end{vmatrix} \begin{vmatrix}
\varepsilon \\
\varepsilon \\
\varepsilon \\
\varepsilon \\
\end{vmatrix} = 0. \hfill (29)$$

The dispersion relation is obtained as:

$$c = \frac{\omega}{k} = \frac{\rho_k u_k + \rho_l u_l}{\alpha_l}, \hfill (30)$$

where $c$ is the wave speed. The negative imaginary part of $\omega$ determines the growth rate of disturbance. Eq. (30) is identical to Eq. (10), only with $\lambda$ being replaced by $c$. Details of IKH analysis can be found in [7].

**Initial and Boundary Conditions for Numerical Solution**

In von Neumann stability analysis, a periodic boundary condition is implicitly assumed. In computations, such periodic boundary conditions are also employed in order to provide a valid comparison.

The von Neumann stability analysis is for the growth of an infinitesimal perturbation. In computations, a small initial perturbation must be properly introduced without generating additional, higher harmonic noise. In this study, the solution of the inviscid Kelvin-Helmholtz analysis is used as the initial condition. Thus, if $k$ and $\varepsilon$ are specified at $t=0$, corresponding
values for $\alpha$, $\varepsilon_i$, $\varepsilon_k$ and $\varepsilon_p$ must be consistent with Eqs. (29-30).

**RESULTS AND DISCUSSIONS**

**Computational Stability Assessment based on von Neumann Stability Analysis**

Comparison of stability of the 1st order upwind, 2nd order upwind, central difference, and QUICK schemes are conducted first for flow conditions before near, and after the instability.

It is well known that for ordinary convection-diffusion equations the 1st order upwind scheme is less accurate with high numerical diffusion, and high order schemes, such as SOU, CDS, and QUICK, have lower numerical diffusion.

In this study, for illustration purposes, water and air are considered and the pipe diameter is taken to be 0.078m. The computational domain is 1m long, the grid number is $N=200$. The pipe incline angle is $\beta = 0$. The unperturbed values are: $\alpha_l = 0.5$, $u_l = 1\ m/s$, $u_g = 17\ m/s$ and CFL of liquid is 0.1.

Stability condition based on Eq. (10) or (30) for the above parameters is $\Delta U = u_g - u_l < 16.0768 \ m/s$. Thus, the two-fluid model for this condition is well-posed. It serves as an ideal testing case to assess the performances of various schemes since the system is quite close to being ill-posed. There are two values of $G$ given by Eq. (28) and the larger one determines the instability, so that only the larger growth rate is used here.

Fig. 4 compares the amplification factors $G$ of four discretization schemes. The solid line is the amplification factor by IKH analysis ($G=1$). The dotted line is for the CDS scheme. It is slightly lower than one but quite close to one with a slight diffusion error at high wavenumber range. This implies the CDS is an ideal scheme to compute the two-fluid model. The dashed line is for the FOU scheme which possesses excessive numerical diffusion at high $k$. Furthermore, $G > 1$ at low $k$. Thus the computation using FOU is unstable under this flow condition. The dashed and dotted line is for the SOU scheme. Although SOU is regarded as a better scheme than FOU with less numerical diffusion, its performance in the two-fluid model is very poor. For large $k$, the numerical diffusion of SOU is much larger than that of FOU. For small $k$, the amplification factor of SOU is much larger than that of FOU. Dashed double dotted line is the amplification factor of the QUICK scheme. Its numerical diffusion at high $k$ is lower than that of FOU and SOU, but it is still much larger than that of CDS. At small $k$, $G$ is slightly larger than 1 indicating that QUICK is unstable as well. The reason that the amplification factor of CDS is close to the analytical amplification factor is probably due to a lack of 2nd order diffusion error and low dispersion error. Compared with SOU, numerical diffusion of FOU scheme is much higher and dispersion is slightly lower. Overall performance of FOU is better than that of SOU which suggests that the dispersion error in the two-fluid model is much more important than it is in the simple convection-diffusion equation. The interpolation of QUICK is essentially linear interpolation with the upwind correction. Therefore its numerical diffusion and stability is worse than that of CDS, but better than that of FOU and SOU.

Next, the effect of the slip velocity $\Delta U = u_g - u_l$ on the numerical stability is discussed. Fig. 5 shows the amplification factors of the CDS scheme for a range of values of $\Delta U$.

When $\Delta U$ is smaller than that given by the IKH stability, the amplification factors of all the harmonics in the computational domain are less than one. However, if $\Delta U$ is higher than the IKH stability criteria, the two-fluid model is analytically ill-posed, and $G$ for small $k$ exceeds one, as shown by the curve for $\Delta U = 16.1 \ m/s$ in Fig. 5. From numerical results, a neutral stability condition of CDS is found to be near $\Delta U = 16.0773 \ m/s$ for the condition used in Fig. 5, which is quite close to IKH stability condition of 16.0768 m/s. As $\Delta U$ further increases, $G$ increases too. The range of unstable harmonic wavenumber becomes wide. The amplification factor of CDS scheme matches that of IKH only at very low wavenumber. In the high $k$ range, numerical damping causes $G$ to be much lower than one.

Fig. 4. Comparisons of amplification factors of various numerical schemes. $N = 200$, $\alpha_l = 0.5$, $u_l = 1\ m/s$, $u_g = 17\ m/s$ and $CFL_l = 0.1$.

When $\Delta U$ is smaller than that given by the IKH stability, the amplification factors of all the harmonics in the computational domain are less than one. However, if $\Delta U$ is higher than the IKH stability criteria, the two-fluid model is analytically ill-posed, and $G$ for small $k$ exceeds one, as shown by the curve for $\Delta U = 16.1 \ m/s$ in Fig. 5. From numerical results, a neutral stability condition of CDS is found to be near $\Delta U = 16.0773 \ m/s$ for the condition used in Fig. 5, which is quite close to IKH stability condition of 16.0768 m/s. As $\Delta U$ further increases, $G$ increases too. The range of unstable harmonic wavenumber becomes wide. The amplification factor of CDS scheme matches that of IKH only at very low wavenumber. In the high $k$ range, numerical damping causes $G$ to be much lower than one.
scheme is similar to the FOU. The stability condition for SOU is \( \Delta U \leq 13.73 \text{m/s} \) and for QUICK it is \( \Delta U = 16.03 \text{m/s} \).

Fig. 6 Growth rate of FOU scheme for different values of \( \Delta U \).

\[ \frac{\Delta U}{T} \leq \Delta t \]

Fig. 7 shows the effect of the liquid velocity on the amplification factors of CDS with \( \Delta U = 16 \text{m/s} \) and \( \Delta t/\Delta x = 0.1 \). For \( u_t = 0.01 \text{m/s} \) and \( u_l = 0.1 \text{m/s} \), \( G \) decreases monotonically with the phase angle. Damping appears at high \( k \). When \( u_l \) increases, \( G \) at high \( k \) range rises significantly, leaving a high damping saddle at the intermediate \( k \) range. On the other hand, if \( \Delta U \) is constant, \( CFL_t/CFL_l \) is much larger than one when \( u_t \) is small so that it is hard to keep both \( CFL_t \) and \( CFL_l \) in the moderate range, which is essential to the computational stability.

Fig. 8 shows the effect of \( u_t \) on \( G \) for the FOU scheme with \( \Delta U = 16 \text{m/s} \), \( \Delta t/\Delta x = 0.1 \). The behavior of FOU is much different with that of CDS. When \( u_t \) is small, most harmonics are unstable. For a larger \( u_t \), excessive numerical diffusion makes the computations stable.

Fig. 9 Amplification factors of CDS scheme for different value of \( \Delta t/\Delta x \). \( N = 200 \), \( u_t = 1 \text{m/s} \), \( \Delta U = 16 \text{m/s} \), and \( u_l = 0.5 \).

Figs. 9-10 show the effect of \( \Delta t/\Delta x \) on \( G \) for the CDS and FOU schemes. Both show increasing numerical diffusion with increasing \( \Delta t/\Delta x \) resulting in a decrease in \( G \). This can be...
explained by examining Eq. (26c), where the last term is related to the gravitational effect. It is well known that gravity stabilizes the flow. Thus increasing $\Delta t/\Delta x$ computationally helps the stability.

### Scheme Consistency Test

In this scheme consistency test, growth rates of harmonics at different grid densities are compared. The computational domain is 1m long. At $t=0$, an infinitesimal sinusoidal disturbance with $k = 2\pi$ is introduced. Initial conditions of volume fraction, liquid and gas velocities and pressure are compatible with the results of IKH analysis, Eq. (29).

Fig. 11 shows the comparison of wave growth at different grids. The cell face interpolation scheme is CDS, $u_l = 1m/s$, $u_g = 17.5m/s$, $\beta = 0. CFL_i = 0.1$, and $a_l = 0.5$. The grid numbers are N=100, N=200, N=400. Because CFL, $u_l$, and $u_g$ are constant in this comparison, $\Delta t/\Delta x$ is a constant. This ensures that $\Delta t$ goes to zero as $\Delta x$ approaches zero. An analytical solution for wave growth by IKH analysis is plotted in Fig. 11 for comparison with the numerical results. With $N$ increasing from 100 to 400, the error between the exact and analytical solution for wave growth by IKH analysis is plotted in Fig. 11 for comparison with the numerical results. With $N$ increasing from 100 to 400, the error between the exact and numerical solutions decreases as required by consistency.

Although the error of solution at grids $N=100$ is slightly larger than that at $N=200$ and $N=400$, the error of solution at $N=200$ is quite close to that at $N=400$. This suggests that $N=200$ is large enough for $k = 2\pi$; hence $N=200$ is used unless otherwise mentioned.

![Fig. 11. Comparison of $\dot{u}_l$ growth for different N by CDS scheme. $u_l = 1m/s$, $u_g = 17.5m/s$. $CFL_i = 0.1$, and $a_l = 0.5$.](image)

**Computational Assessment based on the Growth of Disturbance**

To validate the pressure correction scheme, comparisons between the computed wave growth with the analytical growth from the von Neumann stability analysis are presented. First we consider $k = 2\pi$. $N=200$, $u_l = 1m/s$, $u_g = 15m/s$, $a_l = 0.5$, $CFL_i = 0.05$, and the computational time is $t=4s$.

Based on IKH analysis, the disturbance does not grow. The computational scheme used is CDS. Fig. 12 shows that at $t=4s$, the amplitude of the computed wave is slightly lower than that of the analytical solution. The phases of the analytical and numerical solutions are almost identical. This demonstrates excellent performance of CDS for the two-fluid model. Fig. 13 shows the measured decay of the amplitude of the liquid velocity disturbance. The amplification factor of CDS with $k = 2\pi$ is $0.999997962$ using the von Neumann stability analysis. Since it takes 16000 steps to reach $t=4s$, the ratio of the amplitude at $t=4s$ to that at $t=0$ is $0.999997962^{16000} = 0.967918$. The actual rate using CDS is 0.96807, with an error of 0.016%

Careful examination of Fig. 13 reveals small amplitude wrinkles in the wave amplitude. The reason is that the initial condition is the analytical solution of IKH analysis, which is slightly different from the solution by the CDS dispersion equation. This mismatch of the initial conditions leads to the generation of a weak wave and very low numerical diffusion of CDS ensures that this weak wave exists for a long time.

Figure 14 shows wave growth for an ill-posed condition, with $u_l = 1m/s$, $u_g = 17.5m/s$, $a_l = 0.5$. $CFL_i = 0.1$. The relative velocity criterion is larger than 16.0768m/s so that any perturbation will grow with time. The initial disturbance is introduced with $k = 2\pi$. In Fig. 14, the computational result is obtained after 10399 time steps. The original long wave with $k = 2\pi$ is overwhelmed by a much stronger short wave. In Fig. 15, the growth history of the amplitude is presented. The initial growth stage, from 0 to 4s, corresponds to the growth of the initial long wave with $k = 2\pi$. This is further confirmed by comparing with the analytical $G$ for $k = 2\pi$. The predicted total growth is 22.84 for the amplitude ratio from $t=0$ to $t=4s$, while the computed amplitude ratio is 22.89. After the initial growth stage a short wave with higher $G$ takes over and becomes dominant in the computed solution. This is the stage of fast growth in Fig.15. Although the amplitude of the short wave is not small, the wavenumber of the short wave matches predicted values based on von Neumann analysis. For $\Delta U = 16.5m/s$ in the present computation, the wave with the highest $G$ occurs at $\phi_{max} = 0.282743$. If the 1m domain is occupied by this wave, the total number of waves is $n = N\phi_{max} / (2\pi) = 9$, which is exactly the number of waves in Fig. 14.

![Fig. 12. Propagation of $\dot{u}_l$. CDS scheme, $N = 200$, $u_l = 1m/s$, $\Delta U = 14m/s$, $CFL_i = 0.05$, $a_l = 0.5$.](image)
Fig. 13. Amplitude of liquid velocity disturbance. CDS scheme, $N = 200$, $u_i = 1 m/s$, $\Delta U = 14 m/s$, $CFL_i = 0.05$, $a_i = 0.5$.

Next, a comparison between results of the FOU scheme and predictions from the von Neumann analysis is presented. The parameters of computation are: $N = 200$, $u_i = 0.5 m/s$, $\Delta U = 16 m/s$, $CFL_i = 0.02$, and $a_i = 0.5$. The flow is stable based on IKH stability analysis, but unstable based on the von Neumann stability analysis. The growth rate of FOU is shown in Fig. 16. The harmonic with $G_{\text{max}} = 1.00201$ is $\phi_{\text{max}} = 0.586903$. It is anticipated that this harmonic will grow from the round-off error and eventually dominate the computation. In the computation, a small amplitude sinusoidal wave with $k = 2\pi$ is introduced at $t=0$. Fig. 17 shows the liquid velocity variation after 11800 time steps. Obviously, a short wave has overwhelmed the initial long wave. Because the short waves originate from machine level error which has a broad spectral distribution, the amplitude and frequency of the waves are not uniform. However, the dominant wave component in Fig. 18 is $n = 19$ by counting number of waves in the 1 m computational domain. Since $\phi_{\text{max}} = 0.586903$ for the current condition, it implies $n_{\text{max}} = 19$. This agrees very well with the computational observation. Furthermore, for $G_{\text{max}} = 1.00201$, the amplitude can grow by a factor of $2.92 \times 10^{10}$ in 11800 steps. Since the initial amplitude of machine level noise is of $O(10^{-10})$, it is reasonable to expect the amplitude of the dominant short wave to be on the order of $O(10^{-6})$ after 11800 time steps, which is consistent with the results shown in Fig. 17.
CONCLUDING REMARKS

Numerical instability for the incompressible two-fluid model near the ill-posed condition is investigated for various discretization schemes, while the pressure correction method is used to obtain the pressure. The von Neumann stability analysis is carried out to obtain the amplification factor of a small disturbance in the discretized system. The central difference scheme has the best stability characteristics in handling the two-fluid model, followed by the QUICK scheme. It is quite interesting to note that the excessive numerical diffusion in the 1st order upwind scheme seems to promote the numerical instability in comparison with the central difference scheme. Despite its nominal 2nd order accuracy and popularity, the 2nd order upwind scheme is much more unstable than the 1st order upwind scheme for solving two-fluid model equations. Different discretization schemes for the convection term with varying degrees of the numerical diffusion and dispersion cannot cause a delay in the stability; they often promote instability in the two-fluid model.

The analytically predicted wave amplitude growth is also compared with that obtained from carefully implemented computations using various discretization schemes for the convection term. Excellent agreement between the numerical results and the predicted results is obtained for the growth of the wave amplitude and the dominant wavenumber when the computation becomes unstable.

The relation between the computational instability and the ill-posedness is discussed. In the presence of the small-amplitude long-wave disturbance, whose amplitude is much larger than the machine round-off error, the growth of the disturbance exactly matches the prediction of the von Neumann stability analysis when the computational stability condition is violated. In the meantime, a shorter wave emerges from the machine round-off error, and eventually dominates the entire disturbance, which causes the computation to blow up. This computational instability is widely interpreted as the result of ill-posedness of the two-fluid model. The results of the present study suggest that the computational instability is largely the property of the discretized two-fluid model and is strongly affected by the inherent ill-posedness of the two-fluid model differential equations. Introduction of numerical diffusion and/or dispersion can significantly change the instability of the discretized system; however, such steps often yield unfavorable computational results. For solving two-fluid models, central difference is recommended since it is much more accurate and dependable than other schemes investigated.

When presence the shear stress terms in two-fluid model, the ill-posedness condition is not affected, but the flow instability condition is changed. Major conclusions in this paper overall is still valid [17].

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