

# Lecture 6

## Predicate Logic 2

# Today

- Evaluation with respect to a model
- Some Specific Uses of Quantifiers

# Predicate Logic

... what we have done so far

## 1 SYNTAX of FOL

- **Vocabulary:** individual variables and constants, predicate variables and constants, logical connectives, quantifiers and auxiliary symbols
- **Syntactic Rules:** generate (i.e., recursively define) the set of well-formed formulas of FOL,

including the formation of well-formed formulas by means of the existential and universal quantifiers:

IV. If  $\phi$  is a well-formed formula and  $x$  is an individual variable, then  $\exists x\phi$ ,  $\forall x\phi$  and are well-formed formulas.

# Predicate Logic

... what we have done so far

## 1 SYNTAX of FOL

- **Vocabulary:** individual variables and constants, predicate variables and constants, logical connectives, quantifiers and auxiliary symbols
- **Syntactic Rules:** generate (recursively define) the set of well-formed formulas of FOL

included the formation of well-formed formulas by means of the existential and universal quantifiers:

IV. If  $\phi$  is a well-formed formula and  $x$  is an individual variable, then  $\exists x\phi$ ,  $\forall x\phi$  and  $\neg\phi$  are well-formed formulas.

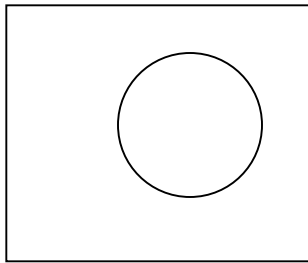
## 2 SEMANTICS of FOL

... what we'll do today

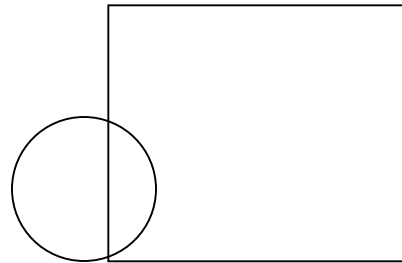
# Predicate Logic

- Basic intuition about meaning in natural language

*The circle is inside the square.*



true



false

Portner, Paul. 2005. What is Meaning? Blackwell Publishing.

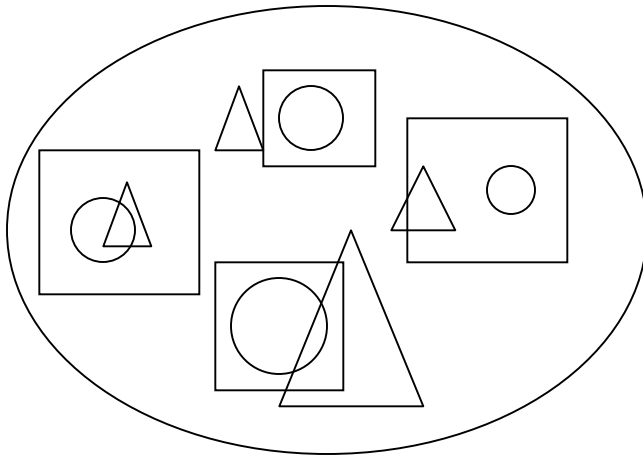
October 20, 2008

Hana Filip

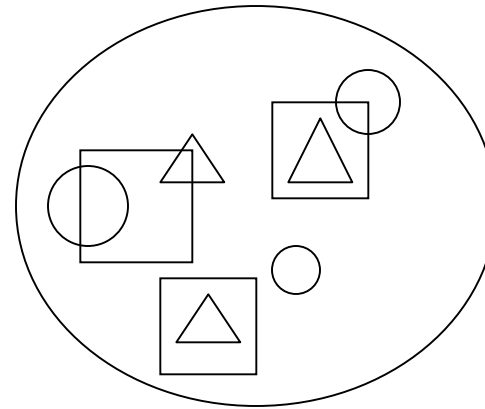
5

# Predicate Logic

*The circle is inside the square.*



**true (set of possible worlds)**



**false (set of possible worlds)**

- There is a number of scenarios or **possible worlds** making the sentence true (and false).
- In fact, there are infinitely many other possible worlds (not drawn above) making the above sentence true (and false).
- We can divide them into the ‘true set’ and the ‘false set.’

# Predicate Logic

- Conclusion: The knowledge of meaning of sentences in natural languages involves (at least) the knowledge of the conditions under which a sentence is true, and the conditions under which it is false, so that we can judge whether a given possible world makes that sentence true or false.
- A theory of meaning that takes this idea as its basic premise is called a TRUTH-CONDITIONAL THEORY OF MEANING (see also Lecture 3).

# Predicate Logic

- You can know the meaning of a sentence -- the conditions that make it true (and also false) -- without knowing whether it is *actually* true or false, i.e., without knowing its truth-value (true or false).
- You know what it would take for a sentence like

*There are other intelligent beings in the universe*

to be true, even if you may never find out whether it is *actually* true or not.

# Predicate Logic

- The meaning of a sentence is called a **PROPOSITION**.
- A sentence **EXPRESSES** or **DENOTES** a proposition.
- The proposition expressed by a sentence amounts to its **TRUTH-CONDITIONS**.

# Toy English

## Gottlob Frege (1848-1925)

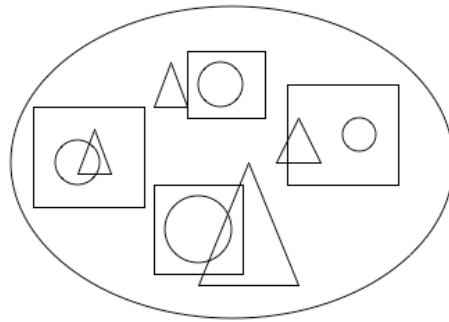


(1) The **REFERENCE** (denotation) of a sentence is its **truth value**: A sentence will designate  $t$  if true, and  $f$  if false. (The True and The False, which Frege took to be objects.)

(2) To know the meaning of a declarative sentence is to know its truth conditions, i.e., what makes the sentence true. It is the **SENSE (a proposition)** of a sentence. A sentence expresses its sense (which is a proposition).

# Predicate Logic

The proposition denoted by *The circle is inside the square* is the one indicated by



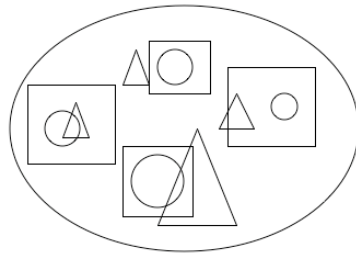
- The set of possible worlds represented by this diagram is only a small subset of a large set of worlds (in fact infinitely many possible worlds) that make this sentence true.
- A proposition is the set of possible worlds (states of affairs) in which it is true.

# Predicate Logic

- **A proposition is the set of possible worlds (states of affairs) in which it is true.**

This captures the idea that the knowledge of what it takes to make a sentence true is exactly what you need to know in order to decide if a given possible world is in the ‘true set.’

- While the sentence *The circle is inside the square* is true in the set of worlds like

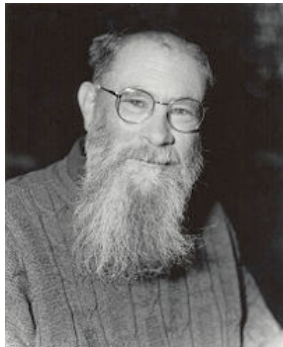


It could have also been true in other worlds, because ‘[t]here are many ways things could have been’ (Lewis 1973) besides the way depicted in the above diagram and still make the above sentence true. Portner, Paul. 2005. What is Meaning? Blackwell Publishing.

# Predicate Logic

**David Lewis.** 1973. *Counterfactuals* (an analysis of counterfactual conditionals in terms of the theory of possible worlds, cp. *If that match had been scratched, it would have lighted.*)

“There are many ways things could have been besides the way they actually are. On the face of it, this sentence is an existential quantification. It says that there exist many entities of a certain description, to wit, **‘ways things could have been.’** I believe permissible paraphrase of what I believe; taking the paraphrase at its face value, I therefore believe in the existence of entities which might be called ‘ways things could have been.’ I prefer to call them **‘possible worlds’**” (p.84).



September 28, 1941 Oberlin, Ohio  
October 14, 2001 Princeton, New Jersey

# Predicate Logic

- **A parallel between the meaning of sentences and common nouns (like *dog*)**
- *Dog* denotes the set of all the dogs that there are in our actual world, that there were in the past and will be in the future, dogs in fiction, etc. I.e., *dog* denotes dogs in the actual world and also in all the possible worlds, in addition to the actual one.

To know the meaning of *dog* amounts to being able to pick out in any possible world the set of dogs in it: “Give me a possible world, and I’ll give you the set of dogs in that world.” (It is a function from possible worlds to sets of dogs.)

- Meaning of a sentence: To know the meaning of a sentence amounts to knowing what it takes to make a sentence true, which is exactly what you need to know in order to decide if a given possible world makes it true.

A sentence denotes the set of possible worlds in which it is true: “Give me a possible world, and I’ll tell you whether it makes a given sentence true.” (It is a function from possible worlds to truth values.)

# Predicate Logic

A semantics of predicate logic is made up of three components:

- (i) the model  $M$  which specifies the structure of the relevant world or universe of discourse  $U$  and that also specifies the values of nonlogical constants,
- (ii) the assignment to variables, and
- (iii) the recursively specified semantic rules.

# Predicate Logic

A model  $M$  consists of

- a universe of discourse  $U$   
set of individuals we are talking about
- an interpretation function  $I$  that assigns values to individual constants and to predicate constants.

An interpretation function  $I$  assigns:

- (i) to each individual constant a member of  $U$
- (ii) to each one-place predicate a subset of  $U$
- (iii) to each two-place predicate a subset of  $U \times U$   
(i.e., a subset of the Cartesian product of  $U$  and  $U$ , a set of ordered pairs); and in general
- (iv) to each  $n$ -place predicate a subset of  $U \times U \times \dots \times U = U^n$   
(a set of ordered  $n$ -tuples of elements from  $U$ )

# Predicate Logic

- When the universe of discourse  $U$  and semantic values for the nonlogical constants and predicates are fully specified, we have a **MODEL** for predicate logic.
- Once we specify a model, we can determine the truth value of any proposition (expressed by a sentence) relative to this model  $M$ .
- ***A sentence is true with respect to (a particular) model  $M$ .***
- Model-theoretic semantics assumes the concept of ***the interpretation of an expression in a model.***

# Predicate Logic

## INTERPRETATION OF QUANTIFIER-FREE FORMULAS

Interpreting formulas in predicate logic involves

- selecting a specific situation to be described and
- assigning specific values to our nonlogical constants, i.e., individual and predicate constants.

Let  $M = \langle U, I \rangle$

where

$U = \{\text{Mary, Sue, John}\}$

$U$  is a set of individuals Mary, Sue, John.

These are the individuals we want to talk about.

# Predicate Logic

The function  $I$  assigns values (also extensions) to individual constants and predicate constants:

$I$  assigns an extension in  $U$  to the individual constants:

$$I(m) = \text{Mary}$$

$$I(s) = \text{Sue}$$

$$I(j) = \text{John}$$

$I$  assigns an extension in  $U$  to the predicate constants:

$$I(G) = \{\text{Mary}, \text{Sue}\}$$

$$I(B) = \{\text{John}\}$$

$$I(S) = \{\text{Mary}\}$$

$$I(L) = \{\langle \text{Mary}, \text{John} \rangle, \langle \text{Sue}, \text{Mary} \rangle, \langle \text{John}, \text{John} \rangle, \langle \text{Mary}, \text{Sue} \rangle\}.$$

# Predicate Logic

The interpretation function  $I$  assigns the individual Mary to the individual constant  $m$ .

I.e., the interpretation of  $m$  with respect to the model  $M$  is the person called Mary. This is then written as

$$\llbracket m \rrbracket^M = \text{Mary}$$

For any expression  $\alpha$  (Greek letter ‘alpha’), we use  $\llbracket \alpha \rrbracket^M$  to denote the semantic value of  $\alpha$  with respect to the model  $M$ .

**If  $\alpha$  is a nonlogical constant (individual or predicate), then  $\llbracket \alpha \rrbracket^M = I(\alpha)$ .**

(see also de Swart, p.93, (i), note that the above applies to both individual constants and variables)

# Predicate Logic

If  $\alpha$  is a two-place predicate, I assigns it a subset of ordered pairs of elements of U:

$$I(\alpha) \subseteq U \times U$$

[the interpretation of  $\alpha$  is a subset of the Cartesian product of U and U]

The Cartesian product of the set X and the set Y,  $X \times Y$ , is the set of all possible ordered pairs whose first component **is a member of X** and whose second component **is a member of Y**:

$$X \times Y = \{ \langle x, y \rangle \mid x \in X \text{ and } y \in Y \}.$$

In our model, the denotation of the two-place predicate  $L$  is:

$$I(L) = \{ \langle \text{Mary}, \text{John} \rangle, \langle \text{Sue}, \text{Mary} \rangle, \langle \text{John}, \text{John} \rangle, \langle \text{Mary}, \text{Sue} \rangle \}.$$

$$\llbracket L \rrbracket^M = \{ \langle x, y \rangle \in U \times U \mid L(x, y) \}$$

[the set of ordered pairs  $x$  and  $y$  that are elements of the Cartesian product of U and U such that  $x$  stands in the relation  $L$  to  $y$ ]

# Predicate Logic

The interpretation of formulas proceeds from the bottom up.

Example 1: Evaluate the formula  $G(m)$  in  $M$ . Let  $I(G) = \{\text{Mary, Sue}\}$ .

- The general rule for evaluation of atomic propositions is as follows (see also de Swart, p.79 and 93):

If  $X(t_1, \dots, t_n)$  is an atomic proposition, and  $t_1, \dots, t_n$  are terms (individual constants or individual variables), then the valuation function

$V_M(X(t_1, \dots, t_n)) = 1$  if and only if  $\langle \llbracket t_1 \rrbracket^M, \dots, \llbracket t_n \rrbracket^M \rangle \in I(X)$ , else 0.

- $\llbracket G(m) \rrbracket^M = 1$  iff  $\llbracket m \rrbracket^M \in \llbracket G \rrbracket^M$   
 $\llbracket m \rrbracket^M \in \llbracket G \rrbracket^M$  iff  $I(m) \in I(G)$ .  
 $I(m) \in I(G)$  iff  $\text{Mary} \in \{\text{Mary, Sue}\}$ .

This is the case, hence  $\llbracket G(m) \rrbracket^M = 1$ .

# Predicate Logic

Example 2: Evaluate the formula  $G(m) \wedge L(m,j)$  in  $M$ .

Let  $I(G) = \{\text{Mary, Sue}\}$  and

$I(L) = \{\langle \text{Mary, John} \rangle, \langle \text{Sue, Mary} \rangle, \langle \text{John, John} \rangle, \langle \text{Mary, Sue} \rangle\}$ .

$$\begin{aligned} \llbracket G(m) \wedge L(m,j) \rrbracket^M = 1 & \text{ iff } \llbracket G(m) \rrbracket^M = 1 \quad \text{and} \quad \llbracket L(m,j) \rrbracket^M = 1 \\ & \text{ iff } \llbracket m \rrbracket^M \in \llbracket G \rrbracket^M \quad \text{and} \quad \langle \llbracket m \rrbracket^M, \llbracket j \rrbracket^M \rangle \in \llbracket L \rrbracket^M \\ & \text{ iff } I(m) \in I(G) \quad \text{and} \quad \langle I(m), I(j) \rangle \in I(L) \end{aligned}$$

This is the case, hence:  $\llbracket G(m) \wedge L(m,j) \rrbracket^M = 1$

# Predicate Logic

- We have to relativize our interpretations to the model  $M$  and also the variable assignment  $g$ , because complex expressions can contain both constants and variables.

If  $\alpha$  is an individual variable, then  $\llbracket \alpha \rrbracket^{M,g} = g(\alpha)$ .

(see also de Swart, p.93, (ii) )

Let  $g$  be the variable assignment:

$\{ \langle x, \text{John} \rangle, \langle y, \text{Sue} \rangle, \langle z, \text{Mary} \rangle, \langle x3, \text{Sue} \rangle, \dots \}$

In fact, all expressions  $\alpha$  are now interpreted with respect to a model  $M$  and an assignment function  $g$  that assigns values to variables.

# Predicate Logic

Example: Evaluate the formula  $G(x)$  with respect to our model  $M$  and a variable assignment  $g$ .

Let  $I(G) = \{\text{Mary}, \text{Sue}\}$  and

let  $g = \{\langle x, \text{John} \rangle, \langle y, \text{Sue} \rangle, \langle z, \text{Mary} \rangle\}$ .

$\llbracket G(x) \rrbracket^{M,g} = 1$     iff  $\llbracket x \rrbracket^{M,g} \in \llbracket G \rrbracket^{M,g}$ ,  
iff  $g(x) \in I(G)$ .

This is not the case, since  $\text{John} \notin \{\text{Mary}, \text{Sue}\}$ .

Hence  $\llbracket G(x) \rrbracket^{M,g} = 0$ .

[if the individual John is assigned to  $x$ ]

# Predicate Logic

- MEANING RULES FOR QUANTIFIERS

Generally, in interpreting any formulas with variables, it is crucial that we consider alternative assignments of values to the variables, even when the facts in our model are fixed.

For example,

- $\forall xB(x)$  is true just in case  $B(x)$  is always true when any member of the universe of discourse  $U$  whatsoever is assigned to  $x$ , not just John. Our function  $g$  only assigns *John* to  $x$ :  $g = \{\langle x, \text{John} \rangle, \langle y, \text{Sue} \rangle, \langle z, \text{Mary} \rangle\}$ .
- $\exists xG(x)$  is true with respect to a model  $M$  and the variable assignment function  $g$  iff there is at least one assignment to the variable  $x$  of an individual in  $U$  such that the open sentence (propositional function)  $G(x)$  with that individual assigned to the variable  $x$  is true.

# Predicate Logic

- We have shown that the given the assignment to variables provided by the function  $g$  does not fulfill these conditions, since we found out that it makes the matrix  $G(x)$  false.

If  $g = \{ \langle x, \text{John} \rangle, \langle y, \text{Sue} \rangle, \langle z, \text{Mary} \rangle \}$  and

if  $I(G) = \{ \text{Mary}, \text{Sue} \}$ , then

$\llbracket G(x) \rrbracket^{M,g} = 0$ , because  $\text{John} \notin I(G)$ .

- Now we are instructed to look beyond the given variable-assignment  $g$  and search among all possible assignment-functions for an **alternative  $g'$  to  $g$**  which assigns to  $x$  another individual which has the property  $G$ . We need to see whether there is another, and at least one, such individual. We call
- **a variable-assignment function  $g'$  an  $x$ -alternative to  $g$**   
if it is identical to  $g$  with respect to all variables other than  $x$ , differing from  $g$  if at all only in the value it assigns to  $x$ .

# Predicate Logic

- **A variant of a variable assignment**
- Let  $g$  be a variable assignment,  $x$  be a variable, and  $e$  be an element of the domain  $U$ . Then  $g[x/e]$  is that variable assignment that is like  $g$  with the possible exception that  $g[x/e](x) = e$ .

That is,  $g$  and  $g[x/e]$  differ at most in the value for the variable  $x$ .

- Some value assignments may assign the same individual to more than one variable. There is nothing wrong with a value assignment that assigns the same individual from  $U$  to every variable.

We have  $g = \{ \langle x, John \rangle, \langle y, Sue \rangle, \langle z, Mary \rangle \}$ .

- The value assignment  $g$  is not to be considered part of the originally defined model  $M$ . It has nothing to do with how we interpret the nonlogical *constant* basic expressions of the language.

# Predicate Logic

The interpretation rule for quantifiers:

If  $\phi$  is a formula and  $x$  is a variable, then

$\llbracket \exists x\phi \rrbracket^{M,g} = 1$  iff for some element  $e$ ,  $e \in U$ ,  $\llbracket \phi \rrbracket^{M,g[x/e]} = 1$

$\llbracket \forall x\phi \rrbracket^{M,g} = 1$  iff for every element  $e$ ,  $e \in U$ ,  $\llbracket \phi \rrbracket^{M,g[x/e]} = 1$

# Predicate Logic

## Example with the $\exists x$ quantifier

$\llbracket \exists x G(x) \rrbracket^{M,g}$

This formula says that at least one individual  $e$  in the domain  $U$ ,  $e \in U$ , satisfies the predicate  $G$ :  $\llbracket G(x) \rrbracket^{M,g} = 1$

Let  $g$  be as above:  $g = \{ \langle x, \text{John} \rangle, \langle y, \text{Sue} \rangle, \langle z, \text{Mary} \rangle \}$ .

Let  $g'$  be a variable assignment,  $x$  be a variable, and  $e$  be an element of the domain  $U$ . Then  $g[x/e]$  is that variable assignment  $g'$  that is like  $g$  with the possible exception that  $g[x/e](x) = e$ .

Let us try assigning Mary to  $x$ :

$g[x/\text{Mary}] = \{ \langle x, \text{Mary} \rangle, \langle y, \text{Sue} \rangle, \langle z, \text{Mary} \rangle, \langle x3, \text{Sue} \rangle, \dots \}$

# Predicate Logic

We have  $\llbracket G(x) \rrbracket^{M, g[x/Mary]} = 1$  iff  $\llbracket (x) \rrbracket^{M, g[x/Mary]} \in \llbracket G \rrbracket^{M, g[x/Mary]}$

iff  $g[x/Mary](x) \in I(G)$

iff  $Mary \in \{Mary, Sue\}$  (True)

Recall that we have:  $I(G) = \{Mary, Sue\}$

Hence, we have found one individual of  $U$  that satisfies the property  $G$ , as required by the quantifier interpretation rule above.

Hence  $\llbracket \exists x G(x) \rrbracket^{M, g[x/Mary]} = 1$ .

# Predicate Logic

## Example with the $\forall x$ quantifier

$$\forall x[G(x) \rightarrow S(x)]$$

This formula will be true if and only if for each individual  $e$  in the domain  $U$  we successively interpret the open sentence (propositional function)  $[G(x) \rightarrow S(x)]$  assigning some  $e$  to  $x$ , and it comes out true on every successive assignment of some  $e$  to  $x$ .

Since we have  $U = \{\text{Mary, Sue, John}\}$ , i.e., these are the individuals in our domain of discourse, we have three individuals  $e$  to test:

- (1)  $e = \text{Mary}$
- (2)  $e = \text{John}$
- (3)  $e = \text{Sue}$

# Predicate Logic

## Case (1) $e = \text{Mary}$

$\llbracket G(x) \rightarrow S(x) \rrbracket^{M, g[x/\text{Mary}]} = 0$  iff (see INDIRECT PROOF and the truth table for ‘ $\rightarrow$ ’)

$\llbracket G(x) \rrbracket^{M, g[x/\text{Mary}]} = 1$  and  $\llbracket S(x) \rrbracket^{M, g[x/\text{Mary}]} = 0$ .

$\llbracket G(x) \rrbracket^{M, g[x/\text{Mary}]} = 1$  iff

$\llbracket (x) \rrbracket^{M, g[x/\text{Mary}]} \in \llbracket G \rrbracket^{M, g[x/\text{Mary}]}$

$\llbracket (x) \rrbracket^{M, g[x/\text{Mary}]} \in \llbracket G \rrbracket^{M, g[x/\text{Mary}]}$  iff  $g[x/\text{Mary}](x) \in I(G)$

$g[x/\text{Mary}](x) \in I(G)$  iff  $\text{Mary} \in \{\text{Mary}, \text{Sue}\}$ .

This is the case, hence  $\llbracket G(x) \rrbracket^{M, g[x/\text{Mary}]} = 1$

$\llbracket S(x) \rrbracket^{M, g[x/\text{Mary}]} = 0$  iff

$\llbracket (x) \rrbracket^{M, g[x/\text{Mary}]} \notin \llbracket S \rrbracket^{M, g[x/\text{Mary}]}$

$\llbracket (x) \rrbracket^{M, g[x/\text{Mary}]} \notin \llbracket S \rrbracket^{M, g[x/\text{Mary}]}$  iff  $g[x/\text{Mary}](x) \notin I(S)$ .

$g[x/\text{Mary}](x) \notin I(S)$  iff  $\text{Mary} \notin \{\text{Mary}\}$ .

This is not case, hence  $\llbracket S(x) \rrbracket^{M, g[x/\text{Mary}]} \neq 0$ , hence 1.

**Hence, we know that  $\llbracket G(x) \rightarrow S(x) \rrbracket^{M, g[x/\text{Mary}]} \neq 0$ , hence = 1.**



# Predicate Logic

Case (3)  $e = \text{Sue}$

$\llbracket G(x) \rightarrow S(x) \rrbracket^{M,g[x/\text{Sue}]} = 0$ , iff

...

$\text{Sue} \in \{\text{Mary}, \text{Sue}\}$  and  $\text{Sue} \notin \{\text{Mary}\}$

True

False

**Hence, we know that  $\llbracket G(x) \rightarrow S(x) \rrbracket^{M,g[x/\text{Sue}]} = 0$ .**

We were not able to show that  $\llbracket G(x) \rightarrow S(x) \rrbracket^{M,g[x/e]} = 1$  for all  $e \in U$ .

Hence, we know that  $\llbracket \forall x[G(x) \rightarrow S(x)] \rrbracket^{M,g[x/e]} = 0$ .

# Predicate Logic

- We know that  $\llbracket G(x) \rightarrow S(x) \rrbracket^{M,g[x/\text{Mary}]} \neq 0$ , hence = 1.
- We know that  $\llbracket G(x) \rightarrow S(x) \rrbracket^{M,g[x/\text{John}]} \neq 0$ , hence = 1.
- We know that  $\llbracket G(x) \rightarrow S(x) \rrbracket^{M,g[x/\text{Sue}]} = 0$ .

Hence, we were not able to show that  $\llbracket G(x) \rightarrow S(x) \rrbracket^{M,g[x/e]} = 1$   
**FOR ALL**  $e \in U$ .

Hence, we know that  $\llbracket \forall x[G(x) \rightarrow S(x)] \rrbracket^{M,g[x/e]} = 0$ .

# Predicate Logic

## EXAMPLES WITH MORE THAN ONE QUANTIFIER

- Expressions containing quantifiers within the scope of other quantifiers add an extra degree of complexity in the evaluation. The same rules apply, but the expression is evaluated from the outside in. (See also Partee et al. 1990, p. 146.)
- It is important to distinguish scope relations carefully.
- To find out whether

$$\forall x \exists y L(x,y)$$

is true we must “try out” all values for  $x$ , and for each of these values of  $x$  we must try to find some value for  $y$  that makes  $L(x,y)$  true.

# Predicate Logic

$\forall x \exists y L(x,y)$

is true iff for every individual in the domain there is an individual that bears the relation  $L$  to it:

$\llbracket \forall x \exists y L(x,y) \rrbracket^{M,g} = 1$  iff for every  $e$ ,  $e \in U$ ,  $\llbracket \exists y L(x,y) \rrbracket^{M,g[x/e]} = 1$

This formula will be true wrt  $M$  and  $g$  just in case  $\exists y L(x,y)$  is true wrt  $M$  and to all  $g'$ , where  $g'$  may differ from  $g$  in the value assigned to  $x$ .

We have three assignments to consider:

- |                       |            |                     |
|-----------------------|------------|---------------------|
| (1) $e = \text{Mary}$ | written as | $g[x/\text{Mary}]$  |
| (2) $e = \text{John}$ | written as | $g[x/\text{John}]$  |
| (3) $e = \text{Sue}$  | written as | $g[x/\text{Sue}]$ . |

Therefore, we have to compute three cases:

$\llbracket \exists y L(x,y) \rrbracket^{M,g[x/\text{Mary}]}$

$\llbracket \exists y L(x,y) \rrbracket^{M,g[x/\text{John}]}$

$\llbracket \exists y L(x,y) \rrbracket^{M,g[x/\text{Sue}]}$

# Predicate Logic

Notice that initially  $L(x,y)$  is false with respect to our model  $M$  and  $g$ , because we have

$$g = \{ \langle x, \text{John} \rangle, \langle y, \text{Sue} \rangle, \langle z, \text{Mary} \rangle \}.$$

$$I(L) = \{ \langle \text{Mary}, \text{John} \rangle, \langle \text{Sue}, \text{Mary} \rangle, \langle \text{John}, \text{John} \rangle, \langle \text{Mary}, \text{Sue} \rangle \}.$$

So we get

$$\langle \text{John}, \text{Sue} \rangle \notin I(L).$$

$\exists y L(x,y)$  will be true with respect to our model  $M$  and  $g$  iff we can find some alternative assignment  $g'$  differing only in the value assigned to  $y$  that makes  $L(x,y)$  true wrt  $M$  and  $g'$ .

# Predicate Logic

(1)  $e = \text{Mary}$  written as  $g[x/\text{Mary}]$

$\llbracket \exists y L(x,y) \rrbracket^{M,g[x/\text{Mary}]} = 1$  iff there is  $e, e \in U$ , such that  $\llbracket L(x,y) \rrbracket^{M,g[x/\text{Mary}]} [y/e] = 1$ .

We have such  $e$ , namely  $e = \text{John}$ , such that  $\llbracket L(x,y) \rrbracket^{M,g[x/\text{Mary}]} [y/e] = 1$ , as

$\langle \llbracket x \rrbracket^{M,g[x/\text{Mary}]} [y/e], \llbracket y \rrbracket^{M,g[x/\text{Mary}]} [y/e] \rangle \in \llbracket L \rrbracket^{M,g[x/\text{Mary}]} [y/e]$

iff

$\langle g[x/\text{Mary}] [y/\text{John}] (x), g[x/\text{Mary}] [y/\text{John}] (y) \rangle \in I(L)$

$\langle g[x/\text{Mary}] [y/\text{John}] (x), g[x/\text{Mary}] [y/\text{John}] (y) \rangle \in I(L)$  iff  $\langle \text{Mary}, \text{John} \rangle \in I(L)$

$\langle \text{Mary}, \text{John} \rangle \in I(L)$  iff

$\langle \text{Mary}, \text{John} \rangle \in \{ \langle \text{Mary}, \text{John} \rangle, \langle \text{Sue}, \text{Mary} \rangle, \langle \text{John}, \text{John} \rangle, \langle \text{Mary}, \text{Sue} \rangle \}$ .

True.

# Predicate Logic

(2)  $e = \text{John}$  written as  $g[x/\text{John}]$

We have  $\llbracket \exists y L(x,y) \rrbracket^{M, g[x/\text{John}]} = 1$  iff

there is  $e, e \in U$ , such that  $\llbracket L(x,y) \rrbracket^{M, g[x/\text{John}]} [y/e] = 1$

We have  $e = \text{John}$  such that  $\llbracket L(x,y) \rrbracket^{M, g[x/\text{John}]} [y/\text{John}] = 1$ .

$I(L) = \{ \langle \text{Mary}, \text{John} \rangle, \langle \text{Sue}, \text{Mary} \rangle, \langle \text{John}, \text{John} \rangle, \langle \text{Mary}, \text{Sue} \rangle \}$ .

# Predicate Logic

(3)  $e = \text{Sue}$  written as  $g[x/\text{Sue}]$ .

We have  $\llbracket \exists y L(x,y) \rrbracket^{M, g[x/\text{Sue}]} = 1$  iff  
there is  $e, e \in U$ , such that  $\llbracket L(x,y) \rrbracket^{M, g[x/\text{Sue}]} [y/e] = 1$

We have  $e = \text{Mary}$  such that  $\llbracket L(x,y) \rrbracket^{M, g[x/\text{Sue}]} [y/\text{Mary}] = 1$ .

$I(L) = \{ \langle \text{Mary}, \text{John} \rangle, \langle \text{Sue}, \text{Mary} \rangle, \langle \text{John}, \text{John} \rangle, \langle \text{Mary}, \text{Sue} \rangle \}$ .

# Predicate Logic

Hence, for each and every individual  $e$  in  $U$  we could find at least one individual such that they stand in the  $L$  relation to each other.

This means that  $\llbracket \forall x \exists y L(x,y) \rrbracket^{M,g} = 1$ .

# Predicate Logic

## EXAMPLE WITH A DIFFERENT ORDER OF QUANTIFIERS

$\llbracket \exists x \forall y L(x,y) \rrbracket^{M,g} = 1$  iff

there is  $e, e \in U$ , such that  $\llbracket \forall y L(x,y) \rrbracket^{M,g[x/e]} = 1$

iff for every  $e', e' \in U$ ,

$\llbracket L(x,y) \rrbracket^{M,g[x/e][y/e']} = 1$

There is at least one person such that everybody likes that person.

# Predicate Logic

We have three cases to consider

(1)  $e = \text{Mary}$

(2)  $e = \text{John}$

(3)  $e = \text{Sue}$

And we have:  $I(L) = \{ \langle \text{Mary}, \text{John} \rangle, \langle \text{Sue}, \text{Mary} \rangle, \langle \text{John}, \text{John} \rangle, \langle \text{Mary}, \text{Sue} \rangle \}$ .

Take  $e = \text{Mary}$ , is it the case that for every  $e'$ ,  $e' \in U$ ,  $\llbracket L(x,y) \rrbracket^{M,g[x/\text{Mary}]} [y/e'] = 1$  ?

No. One counterexample is  $e' = \text{Mary}$ :  $\langle \text{Mary}, \text{Mary} \rangle \notin I(L)$

Take  $e = \text{Sue}$ , is it the case that for every  $e'$ ,  $e' \in U$ ,  $\llbracket L(x,y) \rrbracket^{M,g[x/\text{Sue}]} [y/e'] = 1$  ?

No. One counterexample is  $e' = \text{John}$ :  $\langle \text{Sue}, \text{John} \rangle \notin I(L)$

Take  $e = \text{John}$ , is it the case that for every  $e'$ ,  $e' \in U$ ,  $\llbracket L(x,y) \rrbracket^{M,g[x/\text{John}]} [y/e'] = 1$  ?

No. One counterexample is  $e' = \text{Mary}$ :  $\langle \text{John}, \text{Mary} \rangle \notin I(L)$

Hence we were not able to find a  $e$ , as required.

Hence  $\llbracket \exists x \forall y L(x,y) \rrbracket^{M,g} = 0$ .

# Predicate Logic

## INFERENCE PATTERNS

### Existential Generalization

$$\frac{P(c)}{\exists xP(x)}$$

$c$  is an individual constant which has an individual in the universe of discourse as its semantic value.

Example: The existential generalization motivates that (b) entails (a) (see your HW2):

- a. Some Italian is a violinist.  $\exists x[\text{Italian}(x) \wedge \text{Violinist}(x)]$
- b. Bernardo is an Italian violinist.  $\text{Italian}(b) \wedge \text{Violinist}(b)$

Bernardo instantiates at least one Italian violinist.

Therefore, if (b) is true, then (a) must be, as well.

[*Bernardo is an Italian violinist and it's not true that at least one person is an Italian violinist* is a contradiction. I.e., we cannot add a negation of what (a) expresses to (b) without a contradiction. Hence, (a) is an entailment of (b).]

# Predicate Logic

## INFERENCE PATTERNS

### Universal Instantiation

$$\frac{\forall xP(x)}{P(c)}$$

$c$  is an arbitrarily chosen individual constant standing for some individual that is assigned to each free occurrence of  $x$  in  $P(x)$ .

Every human is mortal.

$$\forall x[\text{Human}(x) \rightarrow \text{Mortal}(x)]$$

Socrates is a human.

$$\text{Human}(s)$$

Socrates is mortal.

$$\text{Mortal}(s)$$

# Predicate Logic

## SOME SPECIFIC USES OF QUANTIFIERS

**Reflexive pronouns:**

Everyone admires himself.

$\forall x[\text{person}(x) \rightarrow A(x,x)]$

# Predicate Logic

**Non-restrictive / appositive relative clause:**

Someone, who is late, is to be punished.

$\exists x[L(x) \wedge P(x)]$

# Predicate Logic

**Restrictive relative clause:**

Someone who is late is to be punished.

$$\forall x[Lx \rightarrow Px]$$

# Predicate Logic

He who wants something badly enough will get it.

\* $\forall x[[P(x) \wedge \exists y[T(y) \wedge W(x,y)]] \rightarrow G(x,y)]$

problem:

$G(x,y)$  does not fall within the scope of  $\exists y$ , so the  $y$  in  $G(x,y)$  is unbound, free.

\* $\forall x[P(x) \wedge \exists y[T(y) \wedge [W(x,y) \rightarrow G(x,y)]]]$

problem: ‘for every person there is something with a given property’

$\forall x[P(x) \rightarrow \forall y[T(y) \rightarrow [W(x,y) \rightarrow G(x,y)]]] \equiv$

$\forall x \forall y[[P(x) \wedge T(y) \wedge W(x,y)] \rightarrow G(x,y)] \equiv$

$\forall y[T(y) \rightarrow \forall x[P(x) \rightarrow [W(x,y) \rightarrow G(x,y)]]]$

# Predicate Logic

There is a unique apple. (= There is exactly one apple)

$$\exists x[\text{apple}(x) \wedge \forall y[\text{apple}(y) \rightarrow x=y]]$$

$$\equiv \exists x \forall y[\text{apple}(x) \leftrightarrow x=y]$$

# Predicate Logic

There is at least one object

$$\exists xPx$$

There are at least two objects

$$\exists x\exists y[x\neq y]$$

There are at least two books

$$\exists x\exists y[x\neq y \wedge \text{book}(x) \wedge \text{book}(y)]$$

There are at least three objects

$$\exists x\exists y\exists z[x\neq y \wedge y\neq z \wedge x\neq z]$$

# Predicate Logic

## HOMework DUE OCTOBER 27

De Swart, pp. 94-96:

1. Exercise 5. (If you need practice to do this exercise, work through at least some problems in Barwise & Etchemendy, Chapter 9 - 11.)
2. Exercise 6 (skip 6.1 (i) and 6.2 (iv)).
3. Exercise 7 (skip (i) and (ii) ).