On the Robustness of Large 1-D Network of Double Integrator Agents

He Hao, Huibing Yin and Zhen Kan

***Abstract*— We study the robustness to external disturbances of large 1-D network of double-integrator agents with distributed control. We provide precise quantitative comparison of certain H_∞ norm between two common control architectures: predecessor-following and symmetric bidirectional. In particular, we show that the scaling laws of the H_∞ norm for predecessor-following architecture is $O(\alpha^N)$ ($\alpha > 1$), but only $O(N^3)$ for symmetric bidirectional architecture, where N is the number of agents in the network. The results for symmetric bidirectional architecture are obtained by using a PDE model to approximate the closed-loop dynamics of the network for large N. Numerical calculations show that the PDE approximation provides accurate predictions even when N is small. In addition, we examine the robustness of asymmetric bidirectional architecture. Numerical simulations show that with judicious asymmetry in the velocity feedback, the robustness of the network can be improved considerably over symmetric bidirectional and predecessor-following architectures.**

I. Introduction

Distributed control of vehicular formation is relevant to a wide range of applications such as automated highway system, collective behavior of bird flocks and animal swarms, and formation flying of aerial, ground, and autonomous agents for energy savings, surveillance, mine-sweeping, etc. [1]–[4]. A fundamental issue in distributed control is that as the number of agents in the formation increases, the performance of the closed-loop degrades. Several recent works have focused on the fundamental limitations of large vehicular formation with distributed control; [5], [6] have studied the stability margin of the platoon, while [7]–[11] have examined the system's sensitivity to external disturbances.

In this paper we study the robustness (sensitivity to disturbances) of a large 1-D network of double-integrator agents with distributed control, in which each agent is modeled as a double integrator. The control objective is to make the network track a desired trajectory while maintaining a rigid formation geometry. The desired trajectory of the entire network is determined by a leader in front of the formation, and the desired formation geometry is specified as constant inter-agent spacings between each pair of agents.

Two decentralized control architectures that are commonly examined in the literature are *predecessor-following* and *symmetric bidirectional*. In the predecessor-following architecture, the control action on each agent only depends on the relative information from its immediate predecessor, that is, the agent in front of it. In the symmetric bidirectional architecture, it depends equally on the relative information from its immediate predecessor and follower. The predecessor-following architecture has extremely high sensitivity to external disturbances (see [12], [13] and references therein). This is typically referred to as string instability [14] or slinky-type effect [15], [16]. Seiler *et. al.* showed that string instability with the predecessor-following architecture is independent of the design of the controller on each agent, but a fundamental artifact of the architecture [8]. String instability can be ameliorated by non-identical controllers at the agents but at the expense of the control gains increasing without bound as the number of the agents increases [16], [17].

The high sensitivity to disturbance of predecessor-following architecture led to the examination of the symmetric bidirectional architecture for its perceived advantage in rejecting disturbances, especially with absolute velocity feedback [12]. It was shown later that symmetric bidirectional architectures also suffers from high sensitivity to disturbances when only relative measurements are used [8], [9], [18]. Indeed, such high sensitivity to disturbances persists even for more general architectures, where every agent uses information from more than two neighbors [10], [11].

Although a rich literature exists on sensitivity to disturbances with predecessor-following and symmetric bidirectional architectures, to the best of our knowledge, a precise comparison of the performance between these two architectures - in terms of quantitative measures of robustness is lacking. This paper addresses exactly this problem. In particular, we establish how certain H_∞ norm, that quantifies the system's robustness, scale with the size of the network for each of these two architectures. More precisely, we examine the *amplification factor*, which is defined as the H_∞ norm of the transfer function from the disturbances on all the following agents to their position tracking errors.

For the predecessor-following architecture, we show that the amplification factor scales as $O(\alpha^N)$ for some $\alpha > 1$. Thus, as the size of the network increases, the amplification of disturbance increases geometrically. We then show that with symmetric bidirectional architecture, the amplification factor is only $O(N^3)$. In addition, the resonance frequency in this architecture is $O(1/N)$. Thus, among the two control architectures, the symmetric bidirectional architecture performs far better than the predecessor-following architecture in terms of sensitivity to disturbance, especially as the network size becomes large.

The analysis for the symmetric bidirectional architecture is carried out with a PDE approximation of the closed-

He Hao and Zhen Kan are with Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL 32611, USA. Email: hehao,kanzhen0322@ufl.edu. Huibing Yin is with the Coordinated Science Laboratory, Department of Mechanical Science and Engineering, University of Illinois, Urbana-Champaign, IL 61801, USA. Email: yin3@illinois.edu.

loop dynamics. A PDE approximation is frequently used in the analysis of many-particle systems in statistical physics and traffic-dynamics, large spring-mass systems on lattice and synchronization of coupled-oscillators [19]–[21]. In our previous work [6], [22], PDE models provide an insightful and convenient framework to study the stability margin of large vehicular formations. The PDE models used here are based on the PDE model derived in [22]. Although the PDE is derived under the assumption that N is large, numerical results show that it makes an accurate approximation even when N is small (e. g. $N = 10$).

In this paper, we assume each agent has a double-integrator dynamics and the network is homogeneous: each agent in the network has the same open-loop dynamics and uses the same control law. The assumption of double-integrator dynamics comes from the fact that single-integrator models fail to reproduce the slinky-type effects [11] and higher order dynamics will result in instability for sufficient large N [9], [23]. And also, heterogeneity in agent mass and control gains has little effect on the stability margin and sensitivity to disturbance of the network [10], [18], [22]. However, we show by numerical simulation that *asymmetry* has a substantial effect on the robustness of the 1-D network, where *asymmetry* refers to that the information from the front and back neighbors are weighted prejudicially. Judicious asymmetry in the velocity feedback can improve the robustness of the 1-D network considerably over symmetric control.

The rest of this paper is organized as follows. Section II presents the problem statement. Section III describes the PDE model of the 1-D network with symmetric bidirectional architecture. Analysis of the amplification factor for both symmetric bidirectional and predecessor-following architectures as well as the conjecture for asymmetric bidirectional architecture appear in Section IV. The paper ends with summary and design guidelines in Section V.

II. Problem Statement

We consider the formation control of $N+1$ homogeneous agents (1 leader and N followers) which are moving in 1-D Euclidean space, as shown in Figure 1 (a). The position of the i-th agent is denoted by $p_i \in \mathbb{R}$. The dynamics of each agent are modeled as a double integrator:

$$m_i \ddot{p}_i = u_i + w_i, \quad i \in \{1, 2, \cdots, N\}, \tag{1}$$

where m_i is the mass, u_i is the control input and w_i is the external disturbance on the i-th agent. This is a commonly used model for vehicle dynamics in studying vehicular formations, and results from feedback linearization of nonlinear vehicle dynamics [11], [16], [24]. The disturbance on each agent is assumed to be $w_i = a_i \sin(\omega t + \theta_i)$.

The control objective is that agents maintain a rigid formation geometry while following a constant-velocity type desired trajectory. The desired geometry of the formation is specified by constant desired inter-agent spacing $\Delta_{(i-1,i)}$ for $i \in \{1, \cdots, N\}$, where $\Delta_{(i-1,i)}$ is the desired value of

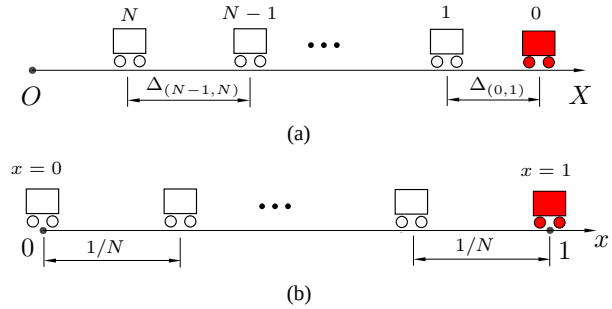

Fig. 1. Desired geometry of a 1-D network of double-integrator agents with 1 "leader" and N "followers", which are moving in 1-D Euclidean space. The filled agent in the front of the network represents the leader, it is denoted by "0". (a) is the original graph of the network in the p coordinate and (b) is the redrawn graph of the same network in the $\tilde{p}$ coordinate.

$p_{i-1}(t) - p_i(t)$. Each agent i knows the desired gaps $\Delta_{(i-1,i)}$, $\Delta_{(i,i+1)}$. The desired trajectory of the network is specified in terms of a leader whose dynamics are independent of the other agents. The leader is indexed by 0, and its trajectory is denoted by $p_0^*(t) = v^* t + \Delta_{(0,N)}$, where v^* is a positive constant, which is the cruise velocity of the network. The desired trajectory of the i-th agent, $p_i^*(t)$, is given by $p_i^*(t) = p_0^*(t) - \Delta_{(0,i)} = p_0^*(t) - \sum_{j=1}^{i} \Delta_{(j-1,j)}$. To facilitate analysis, we define the following position tracking error:

$$\tilde{p}_i := p_i - p_i^*. \tag{2}$$

In this paper, we consider the following decentralized control law, where the control on the i-th agent depends on the relative position and velocity measurements from its immediate predecessor and possibly its immediate follower:

$$\begin{aligned} u_i = &- k_i^f (p_i - p_{i-1} + \Delta_{i,i-1}) - k_i^b (p_i - p_{i+1} - \Delta_{i+1,i}) \\ &- b_i^f (\dot{p}_i - \dot{p}_{i-1}) - b_i^b (\dot{p}_i - \dot{p}_{i+1}), \\ u_N = &- k_i^f (p_N - p_{N-1} + \Delta_{N,N-1}) - b_i^f (\dot{p}_N - \dot{p}_{N-1}), \end{aligned} \tag{3}$$

where $i \in \{1, \cdots, N-1\}$ and k_i^f, k_i^b (respectively, b_i^f, b_i^b) are the front and back position (respectively, velocity) gains of the i-th vehicle. Note that the information needed to compute the control action can be easily accessed by on-board sensors, since only relative information is used.

Definition 1: The control law (3) is *symmetric* if each vehicle uses the same front and back control gains: $k_i^f = k_i^b = k_0, b_i^f = b_i^b = b_0$, and is called *homogeneous* if $k_i^f = k_j^f$, $k_i^b = k_j^b$ and $b_i^f = b_j^f$, $b_i^b = b_j^b$ for every pair (i, j) where $i, j \in \{1, 2, \cdots, N-1\}$. □

Results in [10], [18], [22] show that heterogeneity in vehicle mass and control gains has little effect on the sensitivity to disturbance and stability margin of the network. Therefore we focus on *homogeneous* platoons:

$$\begin{aligned} k_i^f &= (1+\varepsilon_k)k_0, \quad k_i^b = (1-\varepsilon_k)k_0, \\ b_i^f &= (1+\varepsilon_b)b_0, \quad b_i^b = (1-\varepsilon_b)b_0, \\ m_i &= 1, \qquad\qquad i \in \{1, 2, \cdots, N\}, \end{aligned} \tag{4}$$

where $\varepsilon_k \in [0, 1]$ and $\varepsilon_b \in [0, 1]$ are the amounts of asymmetry in the position and velocity gains respectively.

Definition 2: We call the architecture corresponding to $\varepsilon_k = \varepsilon_b = 0$ the *symmetric bidirectional*, since the control action on each vehicle depends equally on the information from its immediate predecessor and follower, the architecture corresponding to $\varepsilon_k = \varepsilon_b = 1$ the *predecessor-following*, since the control action on each vehicle only depends on the information from its immediate predecessor, and the architecture corresponding to other cases *asymmetric bidirectional*. □

In this paper, we study how the sensitivity to external disturbances scale with respect to the number of agents N in the network. We define the following metric.

Definition 3: The *amplification factor* AF is defined as the H_∞ norm of the transfer function from the disturbances acting on all the followers to their position tracking errors. □

To study the amplification factor, we assume there are sinusoidal disturbances acting on all the followers but not the leader, and study the H_∞ norm of the transfer function from the disturbances $W = [w_1, w_2, \cdots, w_N] \in \mathbb{R}^N$ on all the followers to their position tracking errors $E = [\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_N] \in \mathbb{R}^N$, where $w_i = a_i \sin(\omega t + \theta_i)$ and $\tilde{p}_i$ is defined in (2). Since there is no disturbance on the leader, its desired trajectory is given by $p_0^*(t) = v^* t + \Delta_{(0,N)}$. Using the position tracking error defined in (2), for the predecessor-following architecture, the closed-loop dynamics can be expressed as

$$\ddot{\tilde{p}}_i = -k_i^f(\tilde{p}_i - \tilde{p}_{i-1}) - b_i^f(\dot{\tilde{p}}_i - \dot{\tilde{p}}_{i-1}) + w_i, \tag{5}$$

where $i \in \{1, \cdots, N\}$. For the bidirectional architecture, the closed-loop dynamics can be written as

$$\begin{aligned} \ddot{\tilde{p}}_i =& -k_i^f(\tilde{p}_i - \tilde{p}_{i-1}) - k_i^b(\tilde{p}_i - \tilde{p}_{i+1}) \\ & - b_i^f(\dot{\tilde{p}}_i - \dot{\tilde{p}}_{i-1}) - b_i^b(\dot{\tilde{p}}_i - \dot{\tilde{p}}_{i+1}) + w_i, \\ \ddot{\tilde{p}}_N =& -k_i^f(\tilde{p}_N - \tilde{p}_{N-1}) - b_i^f(\dot{\tilde{p}}_N - \dot{\tilde{p}}_{N-1}) + w_N, \end{aligned} \tag{6}$$

where $i \in \{1, \cdots, N-1\}$.

For both architectures, the closed-loop dynamics can be represented in the following state-space form:

$$\dot{X} = AX + BW, \quad E = CX, \tag{7}$$

where X is the state vector, which is defined as $X := [\tilde{p}_1, \dot{\tilde{p}}_1, \cdots, \tilde{p}_N, \dot{\tilde{p}}_N] \in \mathbb{R}^{2N}$, W is input vector (external disturbances) and E is the output vector (position tracking errors).

Recall that the H_∞ norm of a transfer function $G(s) = C(sI - A)^{-1}B$ from W to E is defined as:

$$||G(j\omega)||_{H_\infty} = \sup_{\omega \in \mathbb{R}^+} \sigma_{\max}[G(jw)] = \sup_{W} \frac{||E||_{\mathcal{L}_2}}{||W||_{\mathcal{L}_2}}, \tag{8}$$

where $\sigma_{\max}$ denotes the maximum singular value. [1] For the predecessor-following architecture, the dynamics of each agent only depend on the information from its predecessor. Due to this special coupled structure, a closed-form transfer function can be derived, we can derive estimates for the amplification factor by using standard matrix theory. However, for bidirectional architecture, it is in general difficult to find a closed-form formula for the amplification factor from the state-space domain. We take an alternate route and propose a PDE model, which is seen as a continuum approximation of the coupled-ODE model (6), to analyze and study the H_∞ norms of the 1-D network of double-integrator agents.

[1]In this paper, the $\mathcal{L}_2$ norm is well-defined in the extended space $\mathcal{L}_e^2 = \{u | u_\tau \in \mathcal{L}^2, \forall\ \tau \in [0, \infty)\}$, where $u_\tau(t) = (i)\ u(t)$, if $0 \le t \le \tau; (ii)\ 0$, if $t > \tau$. See [25, Chapter 5]. With a little abuse of notation, we suppress the subscript and write $\mathcal{L}^2 = \mathcal{L}_e^2$.

III. PDE MODELS OF THE NETWORK WITH SYMMETRIC BIDIRECTIONAL ARCHITECTURE

The analysis in the symmetric bidirectional architecture relies on PDE models, which are seen as a continuum approximation of the closed loop dynamics (6) in the limit of large N, by following the steps involved in a finite-difference discretization in reverse. To facilitate analysis, we redraw the graph of the 1-D network of double-integrator agents, so that the position of the agents in the graph are always located in the interval $[0, 1]$, irrespective of the number of agents. The i-th agent in the "original" graph, is now drawn at position $(N - i)/N$ in the new graph. Figure 1 shows an example.

With symmetric control gains $k_i^f = k_i^b = k_0, b_i^f = b_i^b = b_0$, the closed-loop dynamics (6) can be written as

$$\begin{aligned} \ddot{\tilde{p}}_i =& \frac{k_0}{N^2} \frac{(\tilde{p}_{i-1} - 2\tilde{p}_i + \tilde{p}_{i+1})}{1/N^2} + \frac{b_0}{N^2} \frac{(\dot{\tilde{p}}_{i-1} - 2\dot{\tilde{p}}_i + \dot{\tilde{p}}_{i+1})}{1/N^2} \\ & + a_i \sin(\omega t + \theta_i). \end{aligned} \tag{9}$$

The starting point for the PDE derivation is to consider a function $\tilde{p}(x, t) : [0, 1] \times [0,\ \infty) \to \mathbb{R}$ that satisfies:

$$\tilde{p}_i(t) = \tilde{p}(x, t)|_{x=(N-i)/N}, \tag{10}$$

so that functions that are defined at discrete points i will be approximated by functions that are defined everywhere in $[0, 1]$. The original functions are thought of as samples of their continuous approximations. Use the following finite difference approximations:

$$\begin{aligned} \left[\frac{\tilde{p}_{i-1} - 2\tilde{p}_i + \tilde{p}_{i+1}}{1/N^2}\right] &= \left[\frac{\partial^2 \tilde{p}(x,t)}{\partial x^2}\right]_{x=(N-i)/N}, \\ \left[\frac{\dot{\tilde{p}}_{i-1} - 2\dot{\tilde{p}}_i + \dot{\tilde{p}}_{i+1}}{1/N^2}\right] &= \left[\frac{\partial^3 \tilde{p}(x,t)}{\partial x^2 \partial t}\right]_{x=(N-i)/N}. \end{aligned}$$

Under the assumption that N is large but finite, Eq. (9) can be seen as finite difference discretization of the following PDE:

$$\begin{aligned} \frac{\partial^2 \tilde{p}(x,t)}{\partial t^2} =& \frac{k_0}{N^2} \frac{\partial^2 \tilde{p}(x,t)}{\partial x^2} + \frac{b_0}{N^2} \frac{\partial^3 \tilde{p}(x,t)}{\partial x^2 \partial t} \\ & + a(x) \sin(\omega t + \theta(x)), \end{aligned} \tag{11}$$

where $a(x), \theta(x) : [0, 1] \to \mathbb{R}$ are defined according to the following stipulations:

$$a_i = a(x)|_{x=\frac{N-i}{N}}, \qquad \theta_i = \theta(x)|_{x=\frac{N-i}{N}}. \tag{12}$$

The boundary conditions of PDE (11) depend on the arrangement of leader in the graph. For our case, the boundary

conditions are of the Dirichlet type at $x = 1$ where the leader is, and Neumann at $x = 0$:

$$\frac{\partial \tilde{p}}{\partial x}(0,t) = 0, \qquad \tilde{p}(1,t) = 0. \tag{13}$$

The PDE model (11) is a forced wave equations with Kelvin-Voigt damping. It is an approximation of the coupled-ODE model in the sense that a finite difference discretization of the PDEs yield (6) [26], [27].

IV. Robustness (sensitivity to disturbances)

A. Symmetric bidirectional architecture

We first present the result on amplification factor for the 1-D network of double-integrator agents with symmetric bidirectional architecture.

Theorem 1: Consider the PDE model (11)-(13) of the 1-D network with symmetric bidirectional architecture, the amplification factor AF^{sb} and resonance frequency ω_r^{sb} have the asymptotic formula

$$AF^{sb} \approx \frac{8N^3}{\sqrt{k_0} b_0 \pi^3}, \quad \omega_r^{sb} \approx \frac{\sqrt{k_0}\pi}{2N}. \tag{14}$$

These formulae hold for large N. □

Proof of Theorem 1. For a multi-input-multi-output system, the H_∞ norm is defined as the supremum of the maximum singular value of the transfer function matrix $G(j\omega)$ over all frequency $\omega \in \mathbb{R}^+$. Equivalently, it can be interpreted in a sinusoidal, steady-state sense as follows (see [28]). For any frequency ω, any vector of amplitudes $a = [a_1, \cdots, a_N]$ with $\|a\|_2 \le 1$, and any vector of phases $\theta = [\theta_1, \cdots, \theta_N]$, the input vector

$$\begin{aligned} W &= [w_1, \cdots, w_N] \\ &= [a_1 \sin(\omega t + \theta_1), \cdots, a_N \sin(\omega t + \theta_N)] \end{aligned} \tag{15}$$

yields the steady-state response of E of the form

$$\begin{aligned} E &= [\tilde{p}_1, \cdots, \tilde{p}_N] \\ &= [b_1 \sin(\omega t + \psi_1), \cdots, b_N \sin(\omega t + \psi_N)]. \end{aligned} \tag{16}$$

The H_∞ norm of $G(j\omega)$ can be defined as

$$\|G(j\omega)\|_{H_\infty} = \sup \|b\|_2 = \sup_{\omega \in \mathbb{R}^+, a, \theta \in \mathbb{R}^N} \frac{\|E\|_{\mathcal{L}_2}}{\|W\|_{\mathcal{L}_2}}. \tag{17}$$

Therefore, in the PDE counterpart, the H_∞ norm is determined by

$$H_\infty = \sup_{\omega \in \mathbb{R}^+, a(x), \theta(x)} \frac{||\tilde{p}(x,t)||_{\mathcal{L}_2}}{\|a(x)\sin(\omega t + \theta(x))\|_{\mathcal{L}_2}}, \tag{18}$$

where $a(x)$ and $\theta(x)$ are piecewise smooth functions defined in $[0, 1]$.

PDE (11)-(13) is a nonhomogeneous PDE with homogeneous boundary conditions, the solution of $\tilde{p}(x,t)$ can be solved by eigenfunction expansion, see [26, Chapter 8]. Before we proceed, notice that the forcing term satisfies

$$a(x)\sin(\omega t + \theta(x)) = a_1(x)\sin(\omega t) + a_2(x)\cos(\omega t),$$

where $a_1(x) = a(x)\cos(\theta(x))$ and $a_2(x) = a(x)\sin(\theta(x))$. From the superposition property of linear system, the output is the sum of the outputs corresponding to inputs $a_1(x)\sin(\omega t)$ and $a_2(x)\cos(\omega t)$ respectively. We first consider the response of the PDE with input $a_1(x)\sin(\omega t)$. The PDE is now given by

$$\frac{\partial^2 \tilde{p}(x,t)}{\partial t^2} = \frac{k_0}{N^2}\frac{\partial^2 \tilde{p}(x,t)}{\partial x^2} + \frac{b_0}{N^2}\frac{\partial^3 \tilde{p}(x,t)}{\partial x^2 \partial t} + a_1(x)\sin(\omega t).$$

To proceed, we first consider the following homogeneous PDE with homogeneous boundaries (13)

$$\frac{\partial^2 \tilde{p}(x,t)}{\partial t^2} = \frac{k_0}{N^2}\frac{\partial^2 \tilde{p}(x,t)}{\partial x^2} + \frac{b_0}{N^2}\frac{\partial^3 \tilde{p}(x,t)}{\partial x^2 \partial t}. \tag{19}$$

The above PDE can be solved by the method of separation of variables, we assume solution of the form $\tilde{p}(x,t) = \sum_{\ell=1}^{\infty} \phi_\ell(x) h_\ell(t)$. Substituting the solution into the above PDE (19), we get the following space-dependent ODE

$$\frac{1}{N^2}\frac{d^2\phi_\ell(x)}{dx^2} + \lambda_\ell \phi_\ell(x) = 0, \tag{20}$$

where $\lambda_\ell = (2\ell - 1)^2\pi^2/(4N^2)$ and $\phi_\ell(x) = \cos((2\ell - 1)\pi x/2)$ are the eigenvalue and its corresponding eigenfunction of the Sturm-Liouville eigenvalue problem (20) with following boundary conditions, which come from (13),

$$\frac{d\phi_\ell}{dx}(0) = 0, \qquad \phi_\ell(1) = 0. \tag{21}$$

Notice that the eigenvalue λ_1 is the smallest eigenvalue, it is called the principal mode of the damped wave equation (19). Since the eigenfunctions are complete (because of Sturm-Liouville Theory), any piecewise smooth functions can be expanded in a series of these eigenfunctions, see [26]. Therefore, $a_1(x)$ can be expanded as a series in terms of $\phi_\ell(x)$, i.e. $a_1(x) = \sum_{\ell=1}^{\infty} d_\ell \phi_\ell(x)$. Substituting the series into the above PDE and using $\tilde{p}(x,t) = \sum_{\ell=1}^{\infty} \phi_\ell(x) h_\ell(t)$, we have the following time-dependent ODEs:

$$\frac{d^2 h_\ell(t)}{dt^2} + b_0\lambda_\ell \frac{dh_\ell(t)}{dt} + k_0\lambda_\ell h_\ell(t) = d_\ell \sin(\omega t), \tag{22}$$

where $\ell \in \{1, 2, \cdots\}$ and d_ℓ is given by

$$d_\ell = 2\int_0^1 a_1(x)\phi_\ell(x)\ dx. \tag{23}$$

Again, for each mode λ_ℓ, the steady-state response $h_\ell(t)$ is given by

$$\begin{aligned} h_\ell(t) &= \frac{d_\ell}{\sqrt{\omega^4 + (b_0^2\lambda_\ell^2 - 2k_0\lambda_\ell)\omega^2 + k_0^2\lambda_\ell^2}} \sin(\omega t + \psi_\ell) \\ &= A_\ell d_\ell \sin(\omega t + \psi_\ell), \end{aligned} \tag{24}$$

for some constant ψ_ℓ. Following straightforward algebra, the maximum amplitude A_ℓ and its resonance frequency for each mode is

$$A_\ell = \begin{cases} \frac{8N^3}{(2\ell-1)^3 b_0 \pi^3} \frac{1}{\sqrt{k_0 - (2\ell-1)^2 b_0^2 \pi^2/(16N^2)}}, & \text{if } \ell \le \ell_0 \\ \frac{1}{\lambda_\ell k_0}, & \text{otherwise,} \end{cases} \tag{25}$$

$$\omega_\ell = \begin{cases} \frac{(2\ell-1)\pi}{2N}\sqrt{k_0-(2\ell-1)^2 b_0^2\pi^2/(8N^2)}, & \text{if } \ell \le \ell_0 \\ 0, & \text{otherwise,} \end{cases} \tag{26}$$

where $\ell_0 = \frac{2\sqrt{2k_0}N+\pi}{2\pi}$.
When N is large, it's not difficult to see from (25) that, the maximum of A_ℓ is achieved at $\omega = \omega_1$. Therefore, for a finite $\mathcal{L}_2$ norm of $a_1(x)$, to achieve the largest $\mathcal{L}_2$ norm of $\tilde{p}(x,t)$, $a_1(x)$ should be equal to the eigenfunction of the first mode $a_1(x) = \phi_1(x)$, i.e. the projection of $a_1(x)$ onto other eigenfunctions is zero $d_\ell = 0$ $(\ell = 2, 3, \cdots)$. Similarly, the following relationship $a_2(x) = \phi_1(x)$ should hold for input $a_2(x)\cos(\omega t)$, which implies $\theta(x) = \theta_0$ is constant, since $a_1(x) = a(x)\cos(\phi(x))$ and $a_2(x) = a(x)\sin(\phi(x))$.
Consequently, the output with the maximum $\mathcal{L}_2$ norm is given by

$$\tilde{p}(x,t) = A_1\phi_1(x)\sin(\omega t + \psi_1). \tag{27}$$

Therefore, the H_∞ norm of the system is obtained

$$H_\infty = A_1\frac{\|\phi_1(x)\sin(\omega t+\psi_1)\|_{\mathcal{L}_2}}{\|\phi_1(x)\sin(\omega t+\theta_0)\|_{\mathcal{L}_2}} = A_1. \tag{28}$$

Using the assumption that N is large in (25) and (26), we compete the proof. ■

B. Disturbance amplification with predecessor-following architecture

In this section, we present the result of disturbance amplifications with predecessor-following architecture.

Theorem 2: Consider an N-agent network with predecessor-following architecture. The amplification factor AF^{pf} is asymptotically approximated by

$$AF^{pf} \approx \beta\sqrt{\frac{\alpha^{2N}-1}{\alpha^2-1}}, \tag{29}$$

where $\alpha = |T(j\omega_r^{pf})| > 1$, $\beta = |S(j\omega_r^{pf})|$, in which

$$T(s) = \frac{2b_0 s + 2k_0}{s^2+2b_0 s+2k_0}, \quad S(s) = \frac{1}{s^2+2b_0 s+2k_0},$$

and ω_r^{pf} is the resonance frequency

$$\omega_r^{pf} \approx \frac{\sqrt{\sqrt{k_0^4+4k_0^3b_0^2}-k_0^2}}{b_0}.$$

These formulae hold for large N. □

The proof follows a similar line of attack as the work in [8]. Interested readers are referred to Corollary 1 of [29] for an explicit proof.

C. Disturbance amplification with asymmetric bidirectional architecture

For the asymmetric bidirectional architecture, we consider the following control gains, which stabilize the network [22]:

1) Equal amount of asymmetry, i.e. $0 < \varepsilon_k = \varepsilon_b < 1$. In this case, it was shown in Theorem 3.5 of [30] that certain H_∞ norm (which is different from the amplification factor) grows exponentially in N. We show by numerical simulations that the amplification factor AF^{as} with equal asymmetry are approximately $O(\gamma^N)$ $(\gamma > 1)$, see Section IV-D. The asymmetric bidirectional architecture with equal asymmetry in the position and velocity feedback thus suffers from high sensitivity to disturbances, as the predecessor-following architecture. However, it doesn't imply asymmetric bidirectional architectures is not preferable, as shown below.

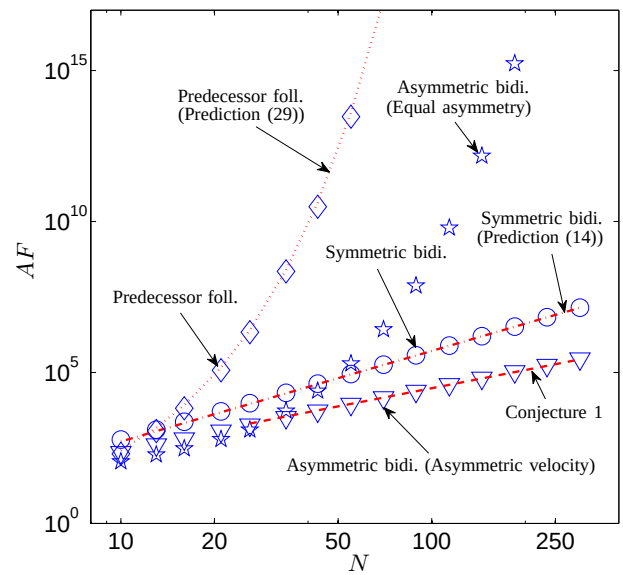

Fig. 2. Numeric comparison of the amplification factor AF between the predecessor-following and bidirectional architectures.

2) Asymmetric velocity feedback, i.e. $\varepsilon_k = 0, 0 < \varepsilon_b < 1$. It was shown in [22] that the stability margin, which is defined as the absolute value of the real part of the least stable eigenvalue of the state matrix A, can be improved considerably by using the asymmetric velocity feedback over symmetric control. We conjecture that the robustness can also be ameliorated significantly with asymmetric velocity feedback, which is witnessed by extensive numerical simulations.

Conjecture 1: Consider an N-agent network with asymmetric bidirectional architecture. When there is small asymmetry in the velocity feedback, i.e. $\varepsilon_k = 0, 0 < \varepsilon_b \ll 1$, the amplification factor AF^{av} asymptotically satisfies

$$AF^{av} \approx O(N^2). \quad \square$$

D. Numerical verification

In this section, we compare the robustness of the network with different control architectures. In addition, we verify the analytic predictions in Theorem 1 and Theorem 2 with their numerically computed values. All numerical calculations are performed in Matlab©. Figure 2 shows the comparison of amplification factor between the predecessor-following and bidirectional architectures. We can see that the amplification factor grows geometrically in the predecessor-following architecture and asymmetric bidirectional architecture with equal asymmetry. In contrast, in the symmetric bidirectional

architecture, these amplifications grow much slower than the two architectures aforementioned. In addition, the asymmetric velocity feedback architecture gives the best robustness performance. Besides, we see that the numerical result of the amplification factor in the asymmetric velocity feedback architecture coincides with our conjecture. Moreover, the analytic predictions match the numerical results very well, which verified our analysis in Theorem 1 and Theorem 2. In all cases, the control gains used are $k_0 = 1$ and $b_0 = 0.5$. The amounts of asymmetry in the cases of equal asymmetry and asymmetric velocity feedback are given by $\varepsilon_k = \varepsilon_b = 0.2$ and $\varepsilon_k = 0, \varepsilon_b = 0.2$, respectively.

V. Summary and Design Guidelines

We studied the robustness to external disturbances of large 1-D networks of double-integrator agents with two decentralized control architectures: predecessor-following and bidirectional. In particular, we examined how the amplification factor scale with N, the number of agents in the network. The analysis of the amplification factor with symmetric bidirectional architecture relied on a PDE model, which approximates the closed-loop dynamics of the network for large N. Numerical calculations showed that the PDE model made an accurate prediction to the scaling laws of amplification factor even when N is small.

Comparing Conjecture 1 with those results in Theorem 1 and Theorem 2 as well as Theorem 3.5 of [30] (equal asymmetry), we see that asymmetric velocity feedback yields the best robustness performance compared to other architectures. The next preferable choice is the symmetric bidirectional architecture. The predecessor-following and asymmetric bidirectional with equal amount of asymmetry are the worst choices for control design in terms of robustness, their amplification factors growing extremely fast with N. In conclusion, the asymmetric velocity feedback is the preferred choice for control design to get a good robustness.

Acknowledgment

The authors would like to express their gratitudes to Dr. Prabir Barooah and Dr. Prashant G. Mehta for their help and inspiring suggestions.

References

[1] J. K. Hedrick, M. Tomizuka, and P. Varaiya, "Control issues in automated highway systems," *IEEE Control Systems Magazine*, vol. 14, pp. 21 – 32, December 1994.

[2] E. Wagner, D. Jacques, W. Blake, and M. Pachter, "Flight test results of close formation flight for fuel savings," in *AIAA Atmospheric Flight Mechanics Conference and Exhibit*, 2002, AIAA-2002-4490.

[3] H. G. Tanner and D. K. Christodoulakis, "Decentralized cooperative control of heterogeneous vehicle groups," *Robotics and autonomous systems*, vol. 55, no. 11, pp. 811–823, 2007.

[4] P. M. Ludwig, "Formation control for multi-vehicle robotic minesweeping," Master's thesis, Naval postgraduate school, 2000.

[5] P. Barooah, P. G. Mehta, and J. P. Hespanha, "Mistuning-based decentralized control of vehicular platoons for improved closed loop stability," *IEEE Transactions on Automatic Control*, vol. 54, no. 9, pp. 2100–2113, September 2009.

[6] H. Hao, P. Barooah, and P. G. Mehta, "Stability margin scaling laws of distributed formation control as a function of network structure," *IEEE Transactions on Automatic Control*, vol. 56, pp. 923 – 929, April 2011.

[7] B. Bamieh, M. R. Jovanović, P. Mitra, and S. Patterson, "Effect of topological dimension on rigidity of vehicle formations: fundamental limitations of local feedback," in *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, 2008, pp. 369–374.

[8] P. Seiler, A. Pant, and J. K. Hedrick, "Disturbance propagation in vehicle strings," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1835–1841, October 2004.

[9] P. Barooah and J. P. Hespanha, "Error amplification and disturbance propagation in vehicle strings," in *Proceedings of the 44th IEEE conference on Decision and Control*, December 2005.

[10] R. Middleton and J. Braslavsky, "String instability in classes of linear time invariant formation control with limited communication range," *IEEE Transactions on Automatic Control*, vol. 55, no. 7, pp. 1519–1530, 2010.

[11] S. Darbha and P. R. Pagilla, "Limitations of employing undirected information flow graphs for the maintenance of rigid formations for heterogeneous vehicles," *International journal of engineering science*, vol. 48, no. 11, pp. 1164–1178, 2010.

[12] L. Peppard, "String stability of relative-motion PID vehicle control systems," *Automatic Control, IEEE Transactions on*, vol. 19, no. 5, pp. 579–581, 1974.

[13] K. Chu, "Decentralized control of high-speed vehicular strings," *Transportation Science*, vol. 8, no. 4, p. 361, 1974.

[14] S. Darbha and J. K. Hedrick, "String stability of interconnected systems," *IEEE Transactions on Automatic Control*, vol. 41, no. 3, pp. 349–356, March 1996.

[15] Y. Zhang, B. Kosmatopoulos, P. Ioannou, and C. Chien, "Using front and back information for tight vehicle following maneuvers," *IEEE Transactions on Vehicular Technology*, vol. 48, no. 1, pp. 319–328, 1999.

[16] S. Darbha, J. Hedrick, C. Chien, and P. Ioannou, "A comparison of spacing and headway control laws for automatically controlled vehicles," *Vehicle System Dynamics*, vol. 23, no. 8, pp. 597–625, 1994.

[17] M. E. Khatir and E. J. Davison, "Decentralized control of a large platoon of vehicles using non-identical controllers," in *Proceedings of the 2004 American Control Conference*, 2004, pp. 2769–2776.

[18] I. Lestas and G. Vinnicombe, "Scalability in heterogeneous vehicle platoons," in *American Control Conference*, 2007, pp. 4678–4683.

[19] D. Helbing, "Traffic and related self-driven many-particle systems," *Review of Modern Physics*, vol. 73, pp. 1067–1141, 2001.

[20] M. Pellicer and J. Sola-Morales, "Analysis of a viscoelastic spring-mass model," *Journal of mathematical analysis and applications*, vol. 294, no. 2, pp. 687–698, 2004.

[21] H. Yin, P. G. Mehta, S. P. Meyn, and U. V. Shanbhag, "Synchronization of coupled oscillators is a game," in *Proc. of 2010 American Control Conference*, Baltimore, MD, 2010, pp. 1783–1790.

[22] H. Hao and P. Barooah, "Control of large 1D networks of double integrator agents: role of heterogeneity and asymmetry on stability margin," in *IEEE Conference on Decision and Control*, December 2010.

[23] S. K. Yadlapalli, S. Darbha, and K. R. Rajagopal, "Information flow and its relation to stability of the motion of vehicles in a rigid formation," *IEEE Transactions on Automatic Control*, vol. 51, no. 8, August 2006.

[24] S. Stankovic, M. Stanojevic, and D. Siljak, "Decentralized overlapping control of a platoon of vehicles," *Control Systems Technology, IEEE Transactions on*, vol. 8, no. 5, pp. 816–832, 2000.

[25] H. Khalil, *Nonlinear Systems 3rd*. Prentice hall Englewood Cliffs, NJ, 2002.

[26] R. Haberman, *Elementary applied partial differential equations: with Fourier series and boundary value problems*. Prentice-Hall, 2003.

[27] L. Evans, *Partial Differential Equations: Second Edition (Graduate Studies in Mathematics)*. American Mathematical Society, 2010.

[28] G. Balas and A. Packard, "The structured singular value (μ) framework," *The Control Handbook*, pp. 671–687.

[29] H. Hao and P. Barooah, "Decentralized control of large vehicular formations: stability margin and sensitivity to external disturbances," *Arxiv preprint arXiv:1108.1409*, 2011. [Online]. Available: http://arxiv.org/abs/1108.1409

[30] F. Tangerman and J. Veerman, "Asymmetric Decentralized Flocks," *accepted to IEEE Transactions on Automatic Control*, 2011. [Online]. Available: http://www.mth.pdx.edu/~veerman/publ04.html