Sequence Sets With Optimal Integrated Periodic Correlation Level

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***Abstract*—Sequence sets with low periodic correlations are used in many areas, such as asynchronous code-division multiple access (CDMA) systems, medical imaging, radar and sonar. Lower bounds on the integrated sidelobe level (ISL) and the peak sidelobe level (PSL) of periodic sequence sets, under a power constraint, have been previously derived in the literature. In this letter, we obtain the ISL and PSL lower bounds using a different framework. The main contribution of the letter consists in using this framework to derive closed-form expressions for *all* power constrained periodic sequence sets that meet the ISL lower bound.**

***Index Terms*—Integrated sidelobe level (ISL) bound, ISL optimal sequence sets, periodic correlation.**

I. Introduction

CONSIDER a set of M sequences $\{x_k(n)\}$ $(k = 1,\ldots,M)$, each of which is of length N $(n = 1,\ldots,N)$. The periodic cross-correlation between the $k^{\rm th}$ and $s^{\rm th}$ sequence at time lag l is defined by

$$\begin{aligned} r_{ks}(l) &= \sum_{n=1}^{N} x_k(n)x_s^*(n-l \bmod N) \\ &= r_{sk}^*(-l) = r_{sk}^*(N-l) \\ &\quad k,s = 1,\ldots,M, \quad l = 0,\ldots,N-1 \end{aligned} \tag{1}$$

where * denotes the complex conjugate and $(a \bmod b) \triangleq a - \lfloor a/b \rfloor b$ ($\lfloor a/b \rfloor$ is the biggest integer that is smaller than or equal to a/b). When $k = s$, the correlation above becomes the autocorrelation of the $k^{\rm th}$ sequence. If not specified otherwise, only the following power constraint is imposed on the sequence set:

$$\sum_{n=1}^{N} |x_k(n)|^2 = N, \quad k = 1,\ldots,M. \tag{2}$$

Sequence sets with low auto- and cross-correlations are desired in many applications, including asynchronous CDMA systems [1], ultra-sonic imaging [2] and radar and sonar [3]. The two metrics that are most commonly used to measure the correlation levels are the integrated sidelobe level (ISL) and the peak sidelobe level (PSL)

$$\mathrm{ISL} = \sum_{k=1}^{M}\sum_{l=1}^{N-1} |r_{kk}(l)|^2 + \sum_{k=1}^{M}\sum_{\substack{s=1\\ s\neq k}}^{M}\sum_{l=0}^{N-1} |r_{ks}(l)|^2 \tag{3}$$

$$\begin{aligned} \mathrm{PSL} = \max\{|r_{ks}(l)|\} \quad & k,s = 1,\ldots,M \\ & l = 0,\ldots,N-1 \quad (l \neq 0 \quad \text{if} \quad k = s). \end{aligned} \tag{4}$$

A lower bound on PSL (denoted as $B_{\rm PSL}$) has been derived by Welch in [4] and several sequence sets have been proposed to asymptotically (in N) meet the Welch bound, e.g., [5]–[7]. Compared to $B_{\rm PSL}$, the corresponding ISL lower bound (denoted as $B_{\rm ISL}$) was less discussed in the literature (with [5], [8] being a notable exception) and there appears to be hardly any detailed discussion on sequence sets meeting $B_{\rm ISL}$. In this letter we focus on ISL-related aspects. We use the periodic cyclic algorithm new (PeCAN) framework of [9] to obtain $B_{\rm ISL}$ and $B_{\rm PSL}$. The main contribution of the letter consists in using this framework to derive closed-form expressions for *all* power constrained periodic sequence sets that meet the ISL lower bound.

Notation: We use bold lowercase and uppercase letters to denote vectors and matrices, respectively. $(\cdot)^H$ denotes the conjugate transpose, $(\cdot)^T$ the transpose, $\|\cdot\|_F$ the Frobenius matrix norm, $\|\cdot\|$ the Euclidean vector norm, and $\mathbf{I}_n$ the $n \times n$ identity matrix.

II. The PeCAN Framework

The PeCAN framework introduced in [9] was used to generate a single sequence with practically zero periodic auto-correlation sidelobes. In this section we outline its extension to the case of a set of sequences. The extended framework lays the ground for the derivations in the next sections.

Let $\mathbf{X}_k$ denote the following right circulant matrix associated with the $k^{\rm th}$ $(k = 1,\ldots,M)$ sequence

$$\mathbf{X}_k = \begin{bmatrix} x_k(1) & x_k(2) & \cdots & x_k(N) \\ x_k(N) & x_k(1) & \cdots & x_k(N-1) \\ \vdots & & & \vdots \\ x_k(2) & x_k(3) & \cdots & x_k(1) \end{bmatrix}_{N\times N} \tag{5}$$

and let $\{\mathbf{X}_k\}$ be stacked together as follows:

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_M \end{bmatrix}. \tag{6}$$

It is not difficult to see that each $r_{ks}(l)$ appears N times in the matrix $\mathbf{X}\mathbf{X}^H$ and therefore that the ISL metric defined in (3) can be expressed compactly as

$$\mathrm{ISL} = \frac{1}{N}\|\mathbf{X}\mathbf{X}^H - N\mathbf{I}_{MN}\|_F^2. \tag{7}$$

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The right circulant matrix $\mathbf{X}_k$ can be written as $\mathbf{X}_k = \mathbf{F}^H \mathbf{D}_k \mathbf{F}$, where $\mathbf{F}$ is the $N \times N$ FFT matrix and $\mathbf{D}_k$ is a diagonal matrix whose diagonal elements are given by the FFT of $\{x_k(n)\}_{n=1}^N$. Making use of this expression for $\mathbf{X}_k$ and of (7), the ISL metric can be written as

$$\mathrm{ISL} = \frac{1}{N} \sum_{p=1}^{N} \|\mathbf{y}_p \mathbf{y}_p^H - N\mathbf{I}_M\|^2 \tag{8}$$

$$= \frac{1}{N} \sum_{p=1}^{N} \left(\|\mathbf{y}_p\|^2 - N\right)^2 + N^2(M-1) \tag{9}$$

where

$$\mathbf{y}_p = \begin{bmatrix} y_1(p) \\ \vdots \\ y_M(p) \end{bmatrix}, \quad y_k(p) = \sum_{n=1}^{N} x_k(n) e^{-j2\pi/N(p-1)(n-1)}. \tag{10}$$

We refer the reader to [9] and [10] for more details on the above derivation. The formula in (9) for the ISL metric lies at the basis of the derivations in the following sections.

Under the unimodular constraint on $\{x_k(n)\}$ (i.e., $|x_k(n)| = 1$), the ISL metric in (9) can be minimized with respect to the phases of $\{x_k(n)\}$ by a cyclic algorithm similar to the one proposed in [10], to which, once again, we refer for details. The so-obtained algorithm is called PeCAN like in [9] and [10], since it is a direct extension of the one in the cited references. Note that the PeCAN algorithm generates *unimodular* sequence sets, whereas the derivations in the next two sections impose only the general power constraint in (2).

III. New Derivation of B_{ISL} and B_{PSL}

Our problem is basically to minimize the ISL in (9) under the power constraint in (2). It follows from the Parseval's equality that the power constraint can be written as

$$\sum_{p=1}^{N} |y_k(p)|^2 = N \sum_{n=1}^{N} |x_k(n)|^2 = N^2, \quad k = 1, \ldots, M. \tag{11}$$

Let

$$z_{kp} = |y_k(p)|^2, \quad k = 1, \ldots, M; \quad p = 1, \ldots, N \tag{12}$$

and observe that we can minimize (9) by solving the following problem:

$$\min_{\{z_{kp}\}} \quad f = \frac{1}{N} \sum_{p=1}^{N} \left(\sum_{k=1}^{M} z_{kp} - N \right)^2 \tag{13}$$

$$\text{s.t.} \quad \sum_{p=1}^{N} z_{kp} = N^2, \; k = 1, \ldots, M z_{kp} \geq 0, \; \forall k, p. \tag{14}$$

Using (14) leads to a simpler expression for f

$$f = \frac{1}{N} \sum_{p=1}^{N} \left[\left(\sum_{k=1}^{M} z_{kp} \right)^2 - 2N \sum_{k=1}^{M} z_{kp} + N^2 \right] = g + N^2 - 2MN^2 \tag{15}$$

where

$$g = \frac{1}{N} \sum_{p=1}^{N} \sum_{k=1}^{M} \sum_{s=1}^{M} z_{kp} z_{sp}. \tag{16}$$

Define an $M \times N$ matrix $\mathbf{Z}$ whose $(k,p)^{\mathrm{th}}$ element is z_{kp}. Then g can be written as

$$g = \frac{1}{N} \sum_{k=1}^{M} \sum_{s=1}^{M} \left(\sum_{p=1}^{N} Z_{kp} (Z^T)_{ps} \right) = \frac{1}{N} \sum_{k=1}^{M} \sum_{s=1}^{M} (ZZ^T)_{ks} = \frac{1}{N} \mathbf{u}^T \mathbf{Z} \mathbf{Z}^T \mathbf{u} \tag{17}$$

where

$$\mathbf{u} = [1 \quad 1 \quad \cdots \quad 1]^T \quad (M \times 1). \tag{18}$$

Furthermore, using $\mathbf{Z}$ we can rewrite the constraint $\sum_{p=1}^{N} z_{kp} = N^2$ in (14) as

$$\mathbf{Z}\mathbf{v} = N^2 \mathbf{u} \tag{19}$$

where

$$\mathbf{v} = [1 \quad 1 \quad \cdots \quad 1]^T \quad (N \times 1). \tag{20}$$

Next note that

$$\mathbf{u}^T \mathbf{Z} \mathbf{\Pi} \mathbf{Z}^T \mathbf{u} \geq 0 \tag{21}$$

where

$$\mathbf{\Pi} = \mathbf{I} - \frac{1}{N} \mathbf{v}\mathbf{v}^T \tag{22}$$

is a positive semi-definite matrix (since it is the orthogonal projector onto the null space of $\mathbf{v}$). It follows from (17), (19), (21), and (22) that

$$g = \frac{1}{N} \mathbf{u}^T \mathbf{Z} \mathbf{Z}^T \mathbf{u} \geq \frac{1}{N} \mathbf{u}^T \mathbf{Z} \frac{\mathbf{v}\mathbf{v}^T}{N} \mathbf{Z}^T \mathbf{u} = N^2 \|\mathbf{u}\|^2 \|\mathbf{u}\|^2 = M^2 N^2. \tag{23}$$

From (9), (13), (15), and (23), we obtain the equations

$$\mathrm{ISL} = f + N^2(M-1)$$
$$f = g + N^2 - 2MN^2, \quad g \geq M^2 N^2 \tag{24}$$

which lead to the following lower bound on ISL:

$$\mathrm{ISL} \geq N^2 M(M-1) \triangleq B_{\mathrm{ISL}}. \tag{25}$$

Also note from (3) that

$$\mathrm{ISL} \leq M(N-1) \cdot \mathrm{PSL}^2 + M(M-1)N \cdot \mathrm{PSL}^2 \tag{26}$$

which, together with (25), lead to the following lower bound on PSL:

$$\mathrm{PSL} \geq N \sqrt{\frac{M-1}{NM-1}} \triangleq B_{\mathrm{PSL}}. \tag{27}$$

The above B_{PSL} coincides with the PSL bound introduced in [4] and B_{ISL} agrees with the ISL bound implicitly discussed in [8].

IV. Optimal ISL Sequence Sets

A byproduct of the previous discussion is the following result:

$$\text{ISL} = g - MN^2;$$
$$g = \frac{1}{N}\sum_{p=1}^{N}\sum_{k=1}^{M}\sum_{s=1}^{M} z_{kp}z_{sp} \geq M^2N^2 \tag{28}$$

from which it is easy to see that

$$z_{kp} = N, \quad k = 1, \ldots, M;\ p = 1, \ldots, N \tag{29}$$

which satisfies the constraint in (14), is a particular minimizer of ISL. Because z_{kp} denotes $|y_k(p)|^2$, the ISL criterion depends only on $|y_k(p)|$ which is the absolute value of FFT of the k^{th} sequence at the frequency $(p-1)/N$. As an example, if we choose zero phases for all $\{y_k(p)\}$, then (29) leads to

$$y_k(p) = \sqrt{N}, \quad k = 1, \ldots, M; \quad p = 1, \ldots, N \tag{30}$$

which corresponds to the following sequence set:

$$\begin{aligned} x_k(n) &= \frac{1}{N}\sum_{p=1}^{N} y_k(p)e^{j2\pi/N(n-1)(p-1)} \\ &= \begin{cases} \sqrt{N}, & n = 1 \\ 0, & n = 2, \ldots, N \end{cases} \qquad k = 1, \ldots, M. \end{aligned} \tag{31}$$

Apparently the ISL of the sequence set $\{x_k(n)\}$ in (31) is $M(M-1)N^2$ which meets the ISL lower bound in (25). Yet such a sequence set is not too useful because the crosscorrelation at lag zero between any two sequences is as high as the in-phase autocorrelation and the peak-to-average power ratio (PAR) for any sequence in this set, which is defined as

$$\text{PAR}_k = \frac{\max_n |x_k(n)|^2}{\frac{1}{N}\sum\limits_{n=1}^{N}|x_k(n)|^2}, \quad k = 1, \ldots, M \tag{32}$$

is the largest possible (i.e., $\text{PAR}_k = N$).

Next, we derive *all* solutions to the minimization problem in (13). Let $\mathbf{z}_0$ denote an $MN \times 1$ vector whose elements are all N [corresponding to the particular solution in (29)] and let $\mathbf{z} = \text{vec}(\mathbf{Z}^T)$ which is an $MN \times 1$ vector that contains the columns of $\mathbf{Z}^T$ stacked on top of each other. Then we can write $\mathbf{Z}^T\mathbf{u}$ and $\mathbf{Zv}$ (that appear in (17) and (19), respectively) in the following way:

$$\begin{aligned} \mathbf{Z}^T\mathbf{u} &= [\mathbf{I}_N \quad \cdots \quad \mathbf{I}_N]_{N\times MN} \cdot \mathbf{z} \\ &= \mathbf{Az} \quad (\mathbf{A} \triangleq \mathbf{u}^T \otimes \mathbf{I}_N) \\ \mathbf{Zv} &= \begin{bmatrix} \mathbf{v}^T & & 0 \\ & \ddots & \\ 0 & & \mathbf{v}^T \end{bmatrix}_{M\times MN} \cdot \mathbf{z} \\ &= \mathbf{Bz} \quad (\mathbf{B} \triangleq \mathbf{I}_M \otimes \mathbf{v}^T). \end{aligned} \tag{33}$$

If an $MN \times 1$ vector $\boldsymbol{\delta}$ satisfies

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}_{(M+N)\times MN} \cdot \boldsymbol{\delta} = 0 \tag{34}$$

then $\mathbf{z} = \boldsymbol{\delta}$ leads to $g = 0$ in (17) and $\mathbf{Zv} = 0$ in (19). From this observation and the fact that $\mathbf{z}_0$ is a particular solution to (13), it follows that all solutions to (13) are given by

$$\mathbf{z} = \mathbf{z}_0 + \rho\boldsymbol{\delta} \tag{35}$$

where ρ is sufficiently small so that $\mathbf{z} > 0$ (element-wise). Note also that $\mathbf{z}_0$ belongs to the range space of $\mathbf{A}^T$, which implies that $\mathbf{z}_0^T\boldsymbol{\delta} = 0$ and thus that $\mathbf{z}_0$ is the minimum-norm solution of (13) (indeed, from (35), we have that $||\mathbf{z}||^2 = ||\mathbf{z}_0||^2 + \rho^2||\boldsymbol{\delta}||^2 \geq ||\mathbf{z}_0||^2$).

The problem that remains is to determine $\boldsymbol{\delta}$, or equivalently, the null space of $[\mathbf{A}^T\ \mathbf{B}^T]^T$. We assume that $M \geq 2$ and $N \geq 3$ so that $[\mathbf{A}^T\ \mathbf{B}^T]^T$ is a "fat" matrix that has a nontrivial null space. Owing to the special structure of $\mathbf{A}$ and $\mathbf{B}$, a closed-form basis for the null space of $[\mathbf{A}^T\ \mathbf{B}^T]^T$ can be obtained as follows. Define

$$\begin{aligned} \mathbf{G} &= \begin{bmatrix} 1 & & \\ -1 & \ddots & \\ & \ddots & 1 \\ & & -1 \end{bmatrix}_{N\times(N-1)} \\ \mathbf{H} &= \begin{bmatrix} \mathbf{G} & & \\ & \ddots & \\ & & \mathbf{G} \\ -\mathbf{G} & \cdots & -\mathbf{G} \end{bmatrix} \end{aligned} \tag{36}$$

(the dimension of $\mathbf{H}$ is $NM \times (N-1)(M-1)$) and then it is easy to verify that

$$\mathbf{AH} = \mathbf{0} \quad \text{and} \quad \mathbf{BH} = \mathbf{0}. \tag{37}$$

Because $\text{rank}(\mathbf{H}) = (N-1)(M-1)$, $\mathbf{H}$ spans a subspace of dimension $(N-1)(M-1)$ that belongs to the null space of $[\mathbf{A}^T\ \mathbf{B}^T]^T$. To show that $\mathbf{H}$ spans the entire null space of $[\mathbf{A}^T\ \mathbf{B}^T]^T$, note that

$$[\mathbf{A}^T \quad \mathbf{B}^T] = \begin{bmatrix} \mathbf{I}_N & \mathbf{v} & & \\ \mathbf{I}_N & & \mathbf{v} & \\ \vdots & & & \ddots \\ \mathbf{I}_N & & & & \mathbf{v} \end{bmatrix}_{MN\times(M+N)}. \tag{38}$$

The first $M+N-1$ columns of this matrix are obviously linearly independent but not all of its $M+N$ columns are so; indeed, we have that

$$[\mathbf{A}^T \quad \mathbf{B}^T] \begin{bmatrix} -\mathbf{v} \\ \mathbf{1}_{N\times1} \\ \vdots \\ \mathbf{1}_{N\times1} \end{bmatrix} = \mathbf{0}. \tag{39}$$

Hence, $\text{rank}([\mathbf{A}^T\ \mathbf{B}^T]^T) = M+N-1$ and consequently the dimension of the null space of this matrix is $MN - (M+N-1) = (N-1)(M-1) = \text{rank}(\mathbf{H})$. The proof that $\mathbf{H}$ spans the entire null space of $[\mathbf{A}^T\ \mathbf{B}^T]^T$ is thus concluded.

In summary, the vector $\boldsymbol{\delta}$ can be written as $\boldsymbol{\delta} = \mathbf{Hw}$, where the $(N-1)(M-1) \times 1$ vector $\mathbf{w}$ is arbitrary. Furthermore, it follows then from (35) that all solutions to (13) are given by

$$\mathbf{z} = \mathbf{z}_0 + \rho\mathbf{Hw}. \tag{40}$$

For any given $\mathbf{w}$ and ρ, we compute $\mathbf{z}$ (i.e., $\{z_{kp}\}$) using (40) and $y_k(p) = \sqrt{z_{kp}}e^{j\phi_{kp}}$ where the phases $\{\phi_{kp}\}$ can be chosen randomly. Then the IFFT of $\{y_k(p)\}_{p=1}^{N}$ gives the k^{th} sequence $(k = 1, \ldots, M)$. We call such a sequence set (meeting the ISL lower bound) an ISL-optimal sequence set.

Note that in the single-sequence case (i.e., $M = 1$), $[\mathbf{A}^T\ \mathbf{B}^T]^T$ has a trivial null space, and thus there is only one solution to (13): $\mathbf{z} = \mathbf{z}_0$. This observation agrees with the known fact that for any single ISL-optimal sequence (such as the Frank or Chu sequences, see e.g., [3]), the magnitude of its FFT is the same (equal to $\sqrt{N}$ if the sequence power is constrained to be N) at all frequency points. In the case of $M > 1$, a similar characterization (which appears to be novel)

TABLE I
PECAN SEQUENCE SET, $N = 10,\ M = 2$

$\mathbf{x}_1$	$\mathbf{x}_2$
$-0.6535 + 0.7569j$	$0.7905 + 0.6125j$
$-0.3837 - 0.9235j$	$0.6590 + 0.7521j$
$-0.7717 - 0.6360j$	$-0.1753 - 0.9845j$
$-0.7989 - 0.6015j$	$-0.3145 - 0.9493j$
$0.9106 - 0.4133j$	$0.8426 - 0.5385j$
$-0.0453 + 0.9990j$	$0.9004 + 0.4351j$
$0.2875 - 0.9578j$	$-0.6840 - 0.7295j$
$0.1281 - 0.9918j$	$-0.4809 + 0.8768j$
$-0.8022 + 0.5971j$	$0.9599 - 0.2804j$
$-0.9137 - 0.4063j$	$-0.4742 + 0.8804j$

TABLE II
SEQUENCE SET FROM (40), $N = 10,\ M = 2$

$\mathbf{x}_1$	$\mathbf{x}_2$
$0.3767 - 1.4607j$	$0.7378 + 0.1074j$
$-0.4307 - 0.0888j$	$1.4517 + 0.4993j$
$0.4292 - 0.9076j$	$0.4929 - 0.5050j$
$0.2901 + 1.1674j$	$-0.8559 + 0.2080j$
$0.9873 + 0.9642j$	$-0.1185 - 0.8191j$
$0.9064 + 0.4486j$	$1.6596 - 0.1516j$
$-0.1659 + 0.1825j$	$-0.0035 - 0.6168j$
$0.8787 - 0.7447j$	$-0.5303 + 0.5207j$
$-0.2686 + 0.4341j$	$0.9002 - 0.4568j$
$-0.5939 + 0.3849j$	$-0.0312 + 0.6313j$

can be shown to hold: a sequence set is ISL-optimal if and only if $\|\mathbf{y}_1\| = \cdots = \|\mathbf{y}_N\|$ (see (10) for the definition of $\mathbf{y}_p$).

Finally we remark on the fact that if a set of M sequences meets B_{ISL}, then these M sequences are complementary sequences (see, e.g., [8]), i.e., $\sum_{k=1}^{M} r_{kk}(l) = 0$ for $l = 1, \ldots, N-1$. Hence the construction method outlined in this section for ISL-optimal sequence sets can also be used to obtain M complementary sequences.

V. NUMERICAL EXAMPLES AND CONCLUDING REMARKS

We use the PeCAN algorithm in Section II and the closed-form construction in Section IV to generate ISL-optimal sequence sets. Table I shows a PeCAN sequence set with $N = 10,\ M = 2$ while Table II shows a sequence set of the same size constructed from (40). Both sequence sets achieve the ISL lower bound: $B_{\mathrm{ISL}} = 200$. The PeCAN sequence set is unimodular and its PSL is 3.53; the other sequence set has $\mathrm{PAR}_1 = 2.28$, $\mathrm{PAR}_2 = 2.78$ and its PSL is 3.51.

Note that the PeCAN algorithm uses a random initialization (see [9] and [10]) and that different initializations lead to different unimodular sequence sets that are all expected to meet the ISL lower bound. Furthermore, because the PeCAN algorithm relies on FFT operations (see [9] and [10]), it can generate large sequence sets up to $NM \sim 10^5$. Fig. 1 shows the correlation level of a PeCAN sequence set with $N = 10000$ and $M = 10$. The correlation level at lag l is defined as $10\log\left(\sum_{k,s}|r_{ks}(l)|^2/MN^2\right)$. This PeCAN sequence set also meets the ISL lower bound which is $B_{\mathrm{ISL}} = 9 \times 10^9$.

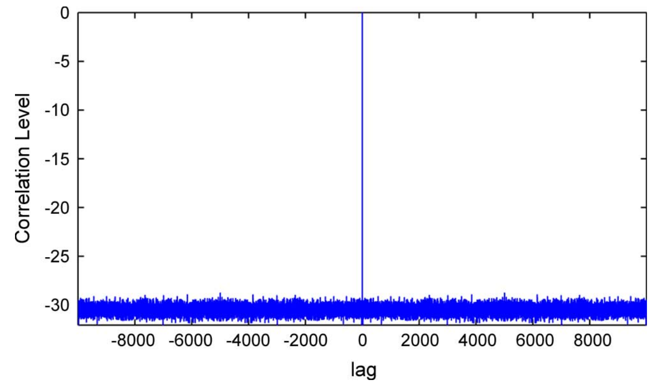

Fig. 1. Correlation level of an ISL-optimal sequence set generated by the PeCAN algorithm ($N = 10000,\ M = 10$).

Compared to the PeCAN algorithm which is iterative, the construction method in Section IV has a closed form and thus there is essentially no length limit for the generated sequence sets. Moreover, because $\rho, \mathbf{w}$ and $\{\phi_{kp}\}$ (see (40)) can be randomly chosen, this method can also generate many different sequence sets that all meet B_{ISL}. As a matter of fact, the PeCAN sequence sets are just special cases of (40).

Note that the closed-form construction method based on (40) does not deliver unimodular sequence sets as PeCAN does. This means that we may need to try different choices of ρ, $\mathbf{w}$ or $\{\phi_{kp}\}$ to obtain sequence sets with low PAR's. A systematic way for choosing ρ, $\mathbf{w}$ and $\{\phi_{kp}\}$ in the construction method of this letter to achieve, besides optimal ISL, other useful features (such as low PAR and low PSL) remains an interesting topic for future research.

REFERENCES

[1] N. Suehiro, "A signal design without co-channel interference for approximately synchronized CDMA systems," *IEEE J. Sel. Areas Commun.*, vol. 12, no. 6, pp. 837–841, Jun. 1994.
[2] V. Diaz, J. Urena, M. Mazo, J. Garcia, E. Bueno, and A. Hernandez, "Using golay complementary sequences for multi-mode ultrasonic operation," in *Proc. IEEE 7th Int. Conf. on Emerging Technologies and Factory Automation*, UPC Barcelona, Catalonia, Spain, Oct. 1999.
[3] N. Levanon and E. Mozeson, *Radar Signals*. New York: Wiley, 2004.
[4] L. R. Welch, "Lower bounds on the maximum correlation of signals," *IEEE Trans. Inform. Theory*, vol. IT-20, no. 3, pp. 397–399, May 1974.
[5] D. V. Sarwate, "Bounds on crosscorrelation and autocorrelation of sequences," *IEEE Trans. Inform. Theory*, vol. IT-25, no. 6, pp. 720–724, Nov. 1979.
[6] P. V. Kumar and O. Moreno, "Prime-phase sequences with periodic correlation properties better than binary sequences," *IEEE Trans. Inform. Theory*, vol. 37, no. 5, pp. 603–616, May 1991.
[7] Y. K. Han and K. Yang, "New M-ary sequence families with low correlation and larger size," *IEEE Trans. Inform. Theory*, vol. 55, no. 4, pp. 1815–1823, Apr. 2009.
[8] D. V. Sarwate, "Meeting the Welch bound with equality," in *Sequences and Their Applications (Proc. SETA '98)*, London, U.K., 1999, pp. 79–102.
[9] P. Stoica, H. He, and J. Li, "On designing sequences with impulse-like periodic correlation," *IEEE Signal Process. Lett.*, vol. 16, no. 8, pp. 703–706, Aug. 2009.
[10] H. He, P. Stoica, and J. Li, "Designing unimodular sequence sets with good correlations—Including an application to MIMO radar," in *IEEE Trans. Signal Process.* [Online]. Available: http://plaza.ufl.edu/haohe/papers/MIMO-CAN.pdf