On Aperiodic-Correlation Bounds

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***Abstract*—We present a new derivation of a lower bound for an *aperiodic* correlation metric: the integrated sidelobe level (ISL) of a set of sequences under the energy constraint. Sequences (or sequence sets) with low aperiodic correlations are widely demanded in many applications, including radar/sonar range compression, medical imaging, channel estimation and multi-user spread-spectrum communications. While the lower bound has been implicitly discussed in the literature before, here we adopt a different framework to derive the bound. In particular, we make use in the derivation of our recently proposed cyclic algorithm framework, which can also be used to efficiently synthesize unimodular sequences with low correlations. We also show that by relaxing the unimodular constraint, the ISL lower bound can be approached closely.**

***Index Terms*—Aperiodic correlation, integrated sidelobe level (ISL), lower bound.**

I. Introduction

In radar and sonar applications, different targets backscatter the transmitted signal at different time instants; at the receiver end, when the matched filter is used to detect one target, the interference caused by other targets' returns is determined by the *aperiodic* autocorrelations of the transmitted signal [1], [2]. Furthermore, with regard to the emerging multi-input multi-output (MIMO) radar applications, in which multiple signals are transmitted simultaneously, both the autocorrelation of each signal and the cross-correlation of each signal pair are required to be low to achieve a high signal-to-interference ratio [3], [4]. A similar requirement occurs in spread-spectrum multi-user communications [5], [6].

The aforementioned applications, as well as many others not mentioned here, require the design of a set of sequences with low auto and cross-correlations. Let $\{x_k(n)\}$ ($k = 1, \ldots, M$ and $n = 1, \ldots, N$) denote a set of M sequences, each of which is of length N and restricted to have the same energy:

$$\sum_{n=1}^{N} |x_k(n)|^2 = N, \quad k = 1, \ldots, M. \tag{1}$$

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The aperiodic cross-correlation between the kth and sth sequence at time lag l is given by

$$r_{ks}(l) = \sum_{n=l+1}^{N} x_k(n) x_s^*(n-l) = r_{sk}^*(-l), \quad k, s = 1, \ldots, M \text{ and } l = 0, \ldots, N-1 \tag{2}$$

where $(\cdot)^*$ denotes the complex conjugate. When $k = s$, the correlation above becomes the autocorrelation of the kth sequence.

Excluding the in-phase (i.e., zero time lag) autocorrelations, all other correlations are categorized to be correlation sidelobes and correspondingly the peak sidelobe level (PSL) metric is defined as

$$\begin{aligned} &\mathrm{PSL} = \max \{|r_{ks}(l)|\}, \\ &\quad k, s = 1, \ldots, M \\ &\text{and } l = 0, \ldots, N-1 \ (l \neq 0 \text{ if } k = s). \end{aligned} \tag{3}$$

The following PSL lower bound is due to Welch [7]:

$$\mathrm{PSL} \geq N \sqrt{\frac{M-1}{2NM - M - 1}} \triangleq B_{\mathrm{PSL}}. \tag{4}$$

Another commonly used metric for correlation sidelobes is the integrated sidelobe level (ISL):

$$\mathrm{ISL} = \sum_{k=1}^{M} \sum_{\substack{p=-N+1 \\ p \neq 0}}^{N-1} |r_{kk}(p)|^2 + \sum_{k=1}^{M} \sum_{\substack{s=1 \\ s \neq k}}^{M} \sum_{p=-N+1}^{N-1} |r_{ks}(p)|^2 \tag{5}$$

for which a lower bound was implicitly derived in [6]. We present in Section II a novel derivation of the ISL lower bound using a new framework.

If the time difference $n-l$ in (2) is replaced by "$n-l$ modulo N", the definition becomes the *periodic* correlation. There is a considerable literature, such as [8]–[11], on the *periodic* correlation bound and on how to generate sequence sets that asymptotically meet the bound. However, the parallel problem in the *aperiodic* correlation case, on which this letter focuses, is remarkably more difficult and the related literature is rather limited. In Section III, we use the CAN algorithm [2], [12] to approach the ISL bound fairly closely, particularly when relaxing the unimodular constraint on the sequence(s).

Throughout this letter, vectors and matrices are labeled using bold lowercase and uppercase typefaces, respectively. $(\cdot)^H$ denotes the conjugate transpose, $(\cdot)^T$ the transpose, $\|\cdot\|$ the Euclidean vector norm and $\arg\{\cdot\}$ the phase of a complex scalar.

II. The ISL Lower Bound

It is shown in [12] that the ISL metric in (5) can be transformed to the frequency domain as:

$$\mathrm{ISL} = \frac{1}{2N}\sum_{p=1}^{2N}\left(\|\mathbf{y}_p\|^2 - N\right)^2 + (M-1)N^2 \tag{6}$$

where

$$\mathbf{y}_p = \begin{bmatrix} y_1(p) \\ \vdots \\ y_M(p) \end{bmatrix},$$
$$y_k(p) = \sum_{n=1}^{N} x_k(n) e^{-j(2\pi/2N)(p-1)(n-1)},$$
$$k = 1, \ldots, M. \tag{7}$$

Note that $\{y_k(p)\}_{p=1}^{2N}$ is the DFT of the sequence $\{x_k(n)\}_{n=1}^{N}$ padded with N zeros in the tail. In this section we start from this frequency-domain expression of ISL to derive a lower bound on it.

Let $z_{kp} = |y_k(p)|^2$. Then the energy constraint in (1) is related to $\{z_{kp}\}$ via the Parseval equality:

$$\sum_{p=1}^{2N} z_{kp} = 2N \sum_{n=1}^{N} |x_k(n)|^2 = 2N^2, \quad k = 1, \ldots, M. \tag{8}$$

Expanding (6) and plugging in (8), we obtain

$$\mathrm{ISL} = \frac{1}{2N}\sum_{p=1}^{2N}\left(\sum_{k=1}^{M} z_{kp}\right)^2 - MN^2. \tag{9}$$

Making use of the Cauchy-Schwartz inequality leads to the following result:

$$\mathrm{ISL} = \frac{1}{(2N)^2}\left[\sum_{p=1}^{2N} 1^2\right] \cdot \left[\sum_{p=1}^{2N}\left(\sum_{k=1}^{M} z_{kp}\right)^2\right] - MN^2$$
$$\geq \frac{1}{4N^2}\left[\sum_{p=1}^{2N} 1 \cdot \left(\sum_{k=1}^{M} z_{kp}\right)\right]^2 - MN^2 \tag{10}$$
$$= M^2N^2 - MN^2 \tag{11}$$

where (8) was used to get (11) from (10).

The above result on the ISL lower bound is summarized as:

$$\mathrm{ISL} \geq N^2M(M-1) \triangleq B_{\mathrm{ISL}}. \tag{12}$$

Interestingly, the PSL lower bound in (4) can be easily obtained from B_{ISL} as a corollary. It follows from the definition of ISL in (5) that

$$\mathrm{ISL} \leq 2M(N-1)\mathrm{PSL}^2 + M(M-1)(2N-1)\mathrm{PSL}^2. \tag{13}$$

Substituting (12) in (13), we obtain B_{PSL} in (4).

Remark: The equality in (10) holds if and only if $\sum_{k=1}^{M} z_{kp} = c$ for all $p = 1, \ldots, 2N$ where c is a constant. Because of the energy constraint in (8), it is easy to see that $c = NM$. In other words, a set of energy-constrained sequences $\{x_k(n)\}$ meet the ISL lower bound if and only if their $2N$-point DFT values satisfy $\|\mathbf{y}_p\|^2 = NM$ for all $p = 1, \ldots, 2N$ (see (7) for the definition of $\|\mathbf{y}_p\|$). An example of such a sequence set is given in (14) below.

TABLE I
ISL OF CAN SEQUENCE SETS

	ISL	B_{ISL}
$M = 2, N = 200$	80013.7	80000
$M = 2, N = 512$	524385.8	524288
$M = 4, N = 512$	3145752.2	3145728
$M = 4, N = 1000$	12000044.8	12000000

III. APPROACHING B_{ISL}

A natural question arises as to whether we can generate sequence sets that achieve the correlation lower bound B_{ISL} or B_{PSL}. Here we focus on trying to meet B_{ISL}.

A trivial solution to meeting B_{ISL} is the following sequence set (recall that the energy constraint in (1) is always imposed):

$$x_k(n) = \begin{cases} \sqrt{N}, & n = 1, \\ 0, & n = 2, \ldots, N, \end{cases} \quad k = 1, \ldots, M \tag{14}$$

whose correlation sidelobes are all zero except for the zero-lag cross-correlation which is N. A set of M sequences leads to $M(M-1)$ pairs of cross-correlations and thus the ISL for the above sequence set is exactly equal to the lower bound $N^2M(M-1)$. However, the sequence set in (14) is not practically useful in that its PSL is as high as the in-phase autocorrelation. Moreover, transmiting only at one time instant while keeping silent at all other times, as evidenced by the zeros for $n = 2, \ldots, N$ in (14), results in a high (in fact, the maximum possible) peak-to-average power ratio (PAR), which is once again undesirable in practice.

The CAN algorithm introduced in [12] aims to find unimodular sequence sets with low ISL. The unimodular constraint refers to every sequence element being unit modulus, i.e., $|x_k(n)| = 1$. In this case the energy constraint in (1) is automatically satisfied. Note that unimodular sequences are often preferred in practice due to hardware restrictions, such as an economical non-linear amplifier essentially working well only when the PAR is 1 or close to 1.

Although the unimodular constraint is certainly more stringent than the energy constraint, the unimodular sequence sets generated by CAN have an ISL that is fairly close to B_{ISL}, provided that there are at least two sequences in the set (the $M = 1$ situation turns out to be special and is taken care of later on). To illustrate, we show the ISL of sequence sets generated by CAN and the corresponding B_{ISL} in Table I, for various combinations of M and N. Note that the CAN algorithm is run from a random initialization, and that different random initializations lead to different sequence sets but with similarly low correlations (see [12] for details). Regarding Table I, only one such realization is presented for each pair of (M, N).

The good performance of CAN synthesized unimodular sequence sets, compared to B_{ISL}, can no longer be guaranteed when $M = 1$ in which case $B_{\mathrm{ISL}} = 0$. Hereafter in this section, only the autocorrelation of a single sequence is considered. For a sequence $\{x(n)\}_{n=1}^{N}$ with $|x(n)| = 1$ for all n, it holds that $|r(N-1)| = |x(N)x^*(1)| = 1$ and thus $\mathrm{ISL} \geq 1$. Hence, obviously B_{ISL} cannot be reached by unimodular sequences. Actually the ISL of a single sequence generated by CAN is much larger than 1 (e.g., on the order of 10^3 when $N = 200$), although a CAN sequence can possess much lower correlation sidelobes

than many well-known unimodular sequences in the literature, such as the Golomb or Frank sequence [2].

We consider below relaxing the unimodular constraint in the CAN algorithm so as to obtain lower correlations. More precisely, define the PAR of the sequence $\mathbf{x} = [\,x(1) \quad \cdots \quad x(N)\,]^T$ as:

$$\mathrm{PAR}(\mathbf{x}) = \frac{\max_n |x(n)|^2}{\frac{1}{N}\sum_{n=1}^{N} |x(n)|^2} = \max_n |x(n)|^2 \tag{15}$$

where the second equality is due to the energy constraint. The CAN algorithm generates sequences with $\mathrm{PAR} = 1$. Here we extend it to the more general case of $\mathrm{PAR} \le \rho$ where ρ can be any number between 1 and N.

Following [2] (to which we refer the reader for details), the ISL metric in (6) can be made small by solving the following minimization problem (still in the case of $M = 1$ for simplicity):

$$\begin{aligned} \min_{\{x(n)\}_{n=1}^{N};\{\psi(p)\}_{p=1}^{2N}} \quad & f = \|\mathbf{A}^H\mathbf{z} - \mathbf{v}\|^2 = \|\mathbf{z} - \mathbf{A}\mathbf{v}\|^2 \\ \text{s.t.} \quad & \|\mathbf{x}\|^2 = N, \quad \mathrm{PAR}(\mathbf{x}) \le \rho \end{aligned} \tag{16}$$

where

$$\begin{aligned} \mathbf{z} &= [\,x(1) \quad \ldots \quad x(N) \quad 0 \quad \ldots \quad 0\,]^T_{2N\times 1}\,, \\ \mathbf{v} &= \frac{1}{\sqrt{2}}[\,e^{j\psi(1)} \quad \ldots \quad e^{j\psi(2N)}\,]^T_{2N\times 1} \end{aligned} \tag{17}$$

$\{\psi(p)\}$ are auxiliary variables and $\mathbf{A}^H$ is a unitary $2N \times 2N$ DFT matrix (i.e., $\mathbf{A}^H\mathbf{x}$ gives the $2N$-point DFT of any vector $\mathbf{x}$ of length $2N$). Notice that (16) would reduce to the problem discussed in [2] if its second constraint were replaced by $\mathrm{PAR}(\mathbf{x}) = 1$.

The problem in (16) can be solved in a cyclic way. We first fix $\mathbf{z}$ and compute the $\mathbf{v}$ that minimizes f:

$$\psi(p) = \arg\{\text{the } p^{\text{th}} \text{ element of } \mathbf{A}^H\mathbf{z}\}, \quad p = 1, \ldots, 2N. \tag{18}$$

Next we fix $\mathbf{v}$ and note that the minimization problem can be cast as

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x} - \mathbf{s}\|^2 \\ \text{s.t.} \quad & \|\mathbf{x}\|^2 = N, \quad \mathrm{PAR}(\mathbf{x}) \le \rho \end{aligned} \tag{19}$$

where $\mathbf{s}$ is an $N \times 1$ vector made from the first N elements of $\mathbf{A}\mathbf{v}$.

The "nearest-vector" problem in (19) has already been tackled in [13]; herein we outline its solution intuitively. To begin with, note that the solution to (19) without the PAR constraint is given by $\hat{\mathbf{x}} = \sqrt{N}\mathbf{s}/\|\mathbf{s}\|$. Then note that the PAR constraint is equivalent to $\max_n |x(n)| \le \sqrt{\rho}$. Hence if the magnitudes of all elements in $\hat{\mathbf{x}}$ are below $\sqrt{\rho}$, then $\hat{\mathbf{x}}$ is a solution; if not, we resort to a recursive procedure as follows. The element in $\mathbf{x}$ corresponding to the largest element (in terms of magnitude) in $\mathbf{s}$, say s_α, is given by $\sqrt{\rho}\exp\{j\arg(s_\alpha)\}$. The other $N-1$ elements in $\mathbf{x}$ are obtained by solving the same problem as in (19), except that now $\mathbf{x}$ and $\mathbf{s}$ are $(N-1)\times 1$ and that the energy constraint is $\|\mathbf{x}\|^2 = N - \rho$. Since the scalar case of (19) is trivial, the final solution is guaranteed. We refer the readers to [13] for more details.

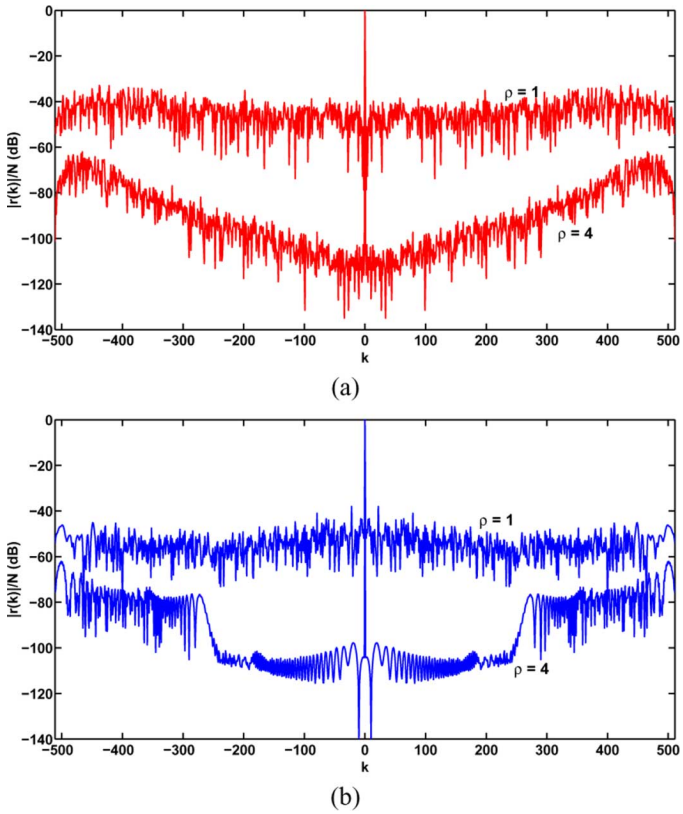

Fig. 1. (a) Autocorrelations (normalized by N and shown in dB) of two CAN sequences of length $N = 512$, one with $\mathrm{PAR} = 1$ and the other with $\mathrm{PAR} = 4$, both initialized by a randomly generated sequence. (b) The same as (a) except that the P4 sequence is used to initialize the CAN algorithm.

To summarize, we iterate between (18) and (19) until convergence (e.g., until the norm of the difference between the $\mathbf{x}$'s obtained in two consecutive iterations is less than a predefined threshold, e.g., 10^{-3}). The criterion in (16) is decreased in every iteration step so local convergence is guaranteed (i.e., the so-obtained $\mathbf{x}$ is at least a local minimum solution to (16)). The iterative process can be started from a random phase initialization of $\mathbf{x}$, e.g., $\{x(n) = e^{j\phi(n)}\}_{n=1}^{N}$, where each $\phi(n)$ is drawn independently from a uniform distribution over $[0, 2\pi]$; such an initialization is used whenever we consider random initialization below. Alternatively $\mathbf{x}$ can be initialized by any good existing sequence ("good" meaning that the sequence itself already has relatively low correlations), e.g., the P4 sequence [1]:

$$x(n) = e^{j(2\pi/N)(n-1)(n-1-N/2)}, \quad n = 1, \ldots, N. \tag{20}$$

The resulting algorithm is still named CAN in view of the fact that the CAN algorithm proposed in [2] is just a special case of (16) (corresponding to $\mathrm{PAR} = 1$) and surely an important one; no ambiguity will be introduced by using this name since from now on we will specify the PAR value whenever we apply CAN.

Consider next using CAN to generate a sequence of length $N = 512$ with energy N. Fig. 1(a) shows the autocorrelations (normalized by N and in dB) of two CAN sequences, one with

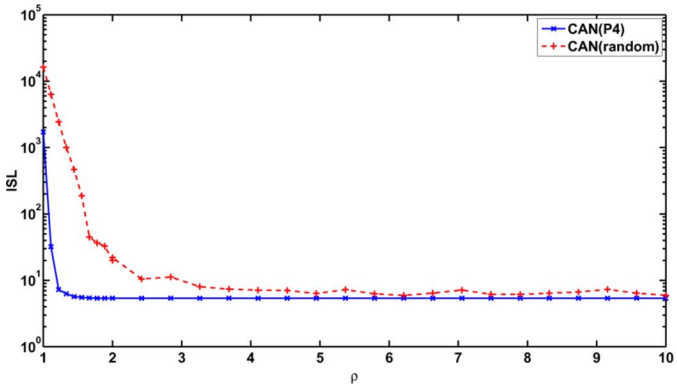

Fig. 2. ISL of CAN sequence (with length $N = 512$ and initialized either randomly or by P4) versus ρ.

$\mathrm{PAR} = 1$ and the other with $\mathrm{PAR} = 4$, both initialized by a randomly generated sequence. Fig. 1(b) is for the same setting as Fig. 1(a) except that the P4 sequence was used to initialize the CAN algorithm. Clearly ρ plays an important role: a larger ρ leads to significantly lower correlation sidelobe levels. (Note that we do not plot, for comparison, the correlations of the P4 or other well-known sequences such as Golomb or Frank, because they have higher correlation sidelobes than the CAN sequence with $\mathrm{PAR} = 1$; see [2] for examples.)

Fig. 2 illustrates the ISL of a length $N = 512$ CAN sequence with ρ ranging from 1 to 10. As before, we use either a randomly generated sequence or the P4 sequence to initialize CAN. The P4 initialization gives lower ISL than the random initialization. Interestingly, when ρ is relatively small, the decrease of ISL caused by the increase of ρ is significant. Note that in the case of P4 initialization, the ISL can be decreased by more than 2 orders of magnitude if ρ is increased just from 1 to 1.2. However, after reaching a certain point, the increase of ρ does not push ISL any lower. The ISL of the CAN sequence initialized by P4 when $\rho = 4$ is 5.38, a value relatively close to the ISL lower bound of $B_{\mathrm{ISL}} = 0$. A full interpretation is still lacking as to why the ISL of the CAN sequence does not go to zero when ρ is sufficiently large, though the possible trapping of the algorithm in local minima is a likely explanation.

IV. Concluding Remarks

In this letter, using a different framework from the one in the previous literature, we have derived a lower bound on the integrated sidelobe level (ISL) of *aperiodic* correlations of a sequence set under a total energy constraint. We have shown that, if a sequence set has more than one sequence, the ISL lower bound B_{ISL} can be nearly met by the *unimodular* sequence sets generated by the CAN algorithm we proposed in [12]. A more challenging problem corresponds to the case of a single sequence where B_{ISL} equals zero. To provide a solution to the latter problem, we have extended the CAN algorithm to push the ISL lower by relaxing the peak-to-average power ratio (PAR) from 1 to a prescribed number. A larger PAR usually leads to lower correlation sidelobes.

We finally comment on the fact that the "relaxing PAR" technique can also be utilized for zero-correlation zone (ZCZ) sequence synthesis. The WeCAN (weighted CAN) algorithm described in [12] can generate sequence sets whose correlation sidelobes are almost zero within a certain time lag interval — thus called ZCZ. Increased PAR is expected to lead to a sequence with a longer ZCZ.

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