Weights of Observations

Measures the relative worth of an observation as compared to other measurements.

ADVANTAGES:

1. Weights used to control size of corrections (correction is inversely proportional to weight)

2. Puts control of adjustment in surveyor’s hand.

DISADVANTAGE:

1. Must use reasonable values for weights.
Definition of a Cofactor

A cofactor is defined as:

\[ q_{ij} = \frac{\sigma_{ij}}{\sigma_o^2} \]

where:
- \( q_{ij} \) is the cofactor of the \( ij^{th} \) element,
- \( \sigma_{ij} \) is the covariance element, and
- \( \sigma_o^2 \) is the reference variance.

For a system of measurements (e.g. GPS baselines) with a covariance matrix:

\[ Q = \frac{1}{\sigma_o^2} \Sigma \]

where
- \( Q \) is the cofactor matrix of the measurement, &
- \( \Sigma \) is the covariance matrix of the measurement.
Definition of Weight

A weight is inversely proportional to the measurement’s variance:

\[ w_i = \frac{\sigma_o^2}{\sigma_i^2} \]

where \( \sigma_o \) is a scaling factor.
Definition of Weight

Letting $\sigma_o = 1$ (known as the *standard deviation of unit weight*), then

$$w_i = \frac{1}{\sigma_i^2}$$
Definition of Weight

For a System of Observations:

\[ W = Q^{-1} = \sigma_o^2 \Sigma^{-1} \]

where

\[ \Sigma = \begin{bmatrix}
\sigma_{x_1}^2 & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_n} \\
\sigma_{x_2x_1} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{x_nx_1} & \sigma_{x_nx_2} & \cdots & \sigma_{x_n}^2
\end{bmatrix} \]

\( x_1, x_2, \text{ etc. are observations or computed parameters.} \)
Definition of Weight

For statistically independent observations:

\[ W = \begin{bmatrix}
\frac{\sigma_o^2}{\sigma_{x_1}} & 0 & 0 & \cdots & 0 \\
0 & \frac{\sigma_o^2}{\sigma_{x_2}} & 0 & \cdots & 0 \\
0 & 0 & \frac{\sigma_o^2}{\sigma_{x_3}} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \frac{\sigma_o^2}{\sigma_{x_n}} \\
\end{bmatrix} = \sigma_o^2 \Sigma^{-1} \]
Weighted Mean

Example:
A distance is measured three times with an EDM. The measured values are 185.67, 185.67, and 185.68 ft. What is the weighted mean of the distance?

One method of determining the mean is:

\[
\bar{M} = \frac{185.67 + 185.67 + 185.68}{3}
\]

\[= 185.673\]
THE WEIGHTED MEAN

Note the measurement:
  a) 185.67 occurs twice,
  b) 185.68 occurs once

Thus assign weights of:
  a) 2 to 185.67 measurement, and
  b) 1 to 185.68

Then the weighted mean is:

\[
\bar{M} = \frac{2 \times 185.67 + 1 \times 185.67}{2 + 1}
\]

= 185.673
THE WEIGHTED MEAN

\[
\overline{M} = \frac{\sum_{i=1}^{n} w_i z_i}{\sum_{i=1}^{n} w_i}
\]
Standard Error for Weighted Observations

\[ \sigma^2 = \sqrt{\frac{\sum_{i=1}^{n} w_i \varepsilon_i^2}{n}} \]
Standard Deviation for Weighted Observations

\[ \sigma^2 = \sqrt{\frac{\sum_{i=1}^{n} w_i v_i^2}{(n - 1)}} \]
Standard Deviation of Weight, $w$

$$S_i = \sqrt{\frac{\sum_{i=1}^{n} w_i \nu_i^2}{(n - 1)} \times \frac{1}{\sqrt{w_i}}}$$

$$= \sqrt{\frac{\sum_{i=1}^{n} w_i \nu_i^2}{w_i (n - 1)}}$$
Standard Deviation of the Weighted Mean

\[ S_{\bar{M}} = \sqrt{\frac{\sum_{i=1}^{n} w_i v_i^2}{\sum_{i=1}^{n} w_i(n - 1)}} \]
Weights in Leveling

The variance of a difference in elevation obtained from differential leveling is:

$$\sigma_{\Delta h}^2 = D^2 \left[ 2N \left( \sigma_{\alpha D}^2 + \sigma_{\alpha}^2 \right) \right]$$  \hspace{1cm} (a)

Thus the weight for the observation is:

$$w_i = 1/\sigma_{\Delta h}^2$$

If $l_i$ is the distance between two benchmarks, then

$$N = \frac{l_i}{2D}$$  \hspace{1cm} (b)

where

$N$ is the number of setups, and
$D$ is the distance per sight.
Weights in Leveling (cont.)

Substituting \((b)\) into \((a)\), yields:

\[
\sigma_{\Delta h}^2 = l_i D \left[ \sigma_{r/D}^2 + \sigma_\alpha^2 \right] \quad (c)
\]

Note that \(D\), \(\sigma_{r/D}\), and \(\sigma_\alpha\) are constants:

Thus letting \(k = D \left[ \sigma_{r/D}^2 + \sigma_\alpha^2 \right]\), \((c)\) becomes:

\[
\sigma_{\Delta h}^2 = l_i k
\]

And the weight of the observations become:

\[
w_1 = \frac{1}{l_1 k}; \quad w_2 = \frac{1}{l_2 k}; \quad w_3 = \frac{1}{l_3 k}
\]

\[\vdots\]

\text{etc.}
Weights in Leveling

Since weights are relative, weights for differential leveling lines can be:

1. Inversely proportional to the lengths:

\[ w_i = \frac{1}{l_i} \]

2. Inversely proportional to the number of setups:

\[ w_i = \frac{1}{\text{number of setups}} \]
## EXAMPLE
(PAGE 174)

<table>
<thead>
<tr>
<th>Route</th>
<th>Length (miles)</th>
<th>ΔElev</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>+25.35</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>+25.41</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>+25.38</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>+25.30</td>
</tr>
</tbody>
</table>
**WEIGHTS FOR ROUTES ARE:**

<table>
<thead>
<tr>
<th>ROUTE</th>
<th>Length (miles)</th>
<th>$w_i = 12 \times 1/ l_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>2</td>
</tr>
</tbody>
</table>

Note that 12 is used to make weights integers, but step is not necessary.

**Weighted mean for elevation difference:**

\[
\bar{M} = \frac{12 (25.35) + 6 (25.41) + 4 (25.38) + 2 (25.30)}{12 + 6 + 4 + 2}
\]

\[
= \frac{608.78}{24}
\]

\[
= 25.366 \text{ ft.}
\]
RESIDUALS

<table>
<thead>
<tr>
<th>ROUTE</th>
<th>( w )</th>
<th>( \Delta \text{Elev} )</th>
<th>Adj. Elev.</th>
<th>( v )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td>25.35</td>
<td>25.366</td>
<td>0.016</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>25.41</td>
<td>25.366</td>
<td>-0.044</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>25.38</td>
<td>25.366</td>
<td>-0.014</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>25.30</td>
<td>25.366</td>
<td>0.066</td>
</tr>
</tbody>
</table>

Standard deviation of unit weight:

\[
S_o = \sqrt{\frac{\sum w \ v^2}{n-1}}
\]

\[= \pm 0.09 \text{ ft}\]

Standard deviation of the weighted mean:

\[
S_{\bar{M}} = \sqrt{\frac{\sum w \ v^2}{(n-1) \sum w}}
\]

\[= \pm 0.018 \text{ ft}\]
Standard Deviation for the Weighted Observations

\[ S_1 = \sqrt{\frac{0.024184}{3 \times 12}} = \pm 0.026 \text{ ft} \]

\[ S_2 = \sqrt{\frac{0.024184}{3 \times 6}} = \pm 0.037 \text{ ft} \]

\[ S_3 = \sqrt{\frac{0.024184}{3 \times 4}} = \pm 0.045 \text{ ft} \]

\[ S_4 = \sqrt{\frac{0.024184}{3 \times 2}} = \pm 0.063 \text{ ft} \]
ANGLE WEIGHTS

Assuming all conditions are the same, and that the variance of a single angle is:

\[ S^2 \]

Then the variance of the mean of an angle turned \( n \) times is:

\[ S^2_{\alpha} = \frac{S^2}{n} \]
Since weights are inversely proportional to the variances and are relative, the weights for 3 angles measured under the same conditions are:

\[ w_1 = n_1, \]
\[ w_2 = n_2, \]
\[ w_3 = n_3 \]

OR

\[ w_i = n_i \]

where \( n \) is the number of times an angle is repeated.
## ANGLE EXAMPLE

<table>
<thead>
<tr>
<th>DAY</th>
<th>ANGLE</th>
<th>S</th>
<th>( w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>96° 36' 15&quot;</td>
<td>±12.2&quot;</td>
<td>( \frac{1}{12.2^2} )</td>
</tr>
<tr>
<td>2</td>
<td>96° 36' 22&quot;</td>
<td>±6.7&quot;</td>
<td>( \frac{1}{6.7^2} )</td>
</tr>
<tr>
<td>3</td>
<td>96° 36' 25&quot;</td>
<td>±8.9&quot;</td>
<td>( \frac{1}{8.9^2} )</td>
</tr>
<tr>
<td>4</td>
<td>96° 36' 32&quot;</td>
<td>±9.5&quot;</td>
<td>( \frac{1}{9.5^2} )</td>
</tr>
</tbody>
</table>
SOLUTION: {WORK WITH SECONDS ONLY}

\[
\overline{M} = \frac{\frac{15}{12.2^2} + \frac{22}{6.7^2} + \frac{25}{8.9^2} + \frac{32}{9.5^2}}{\frac{1}{12.2^2} + \frac{1}{6.7^2} + \frac{1}{8.9^2} + \frac{1}{9.5^2}}
\]

\[= 23.9^\circ\]

So adjusted angle is:

96° 36' 24"
# STATISTICS

**RESIDUALS:**

<table>
<thead>
<tr>
<th>Day</th>
<th>Angle</th>
<th>ν</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15&quot;</td>
<td>9&quot;</td>
</tr>
<tr>
<td>2</td>
<td>22&quot;</td>
<td>2&quot;</td>
</tr>
<tr>
<td>3</td>
<td>25&quot;</td>
<td>-1&quot;</td>
</tr>
<tr>
<td>4</td>
<td>32&quot;</td>
<td>-8&quot;</td>
</tr>
</tbody>
</table>

Standard deviation of the mean:

\[
S_M = \sqrt{\frac{9^2}{12.2^2} + \frac{2^2}{6.7^2} + \frac{-1^2}{8.9^2} + \frac{-8^2}{9.5^2}}
\]

\[
= \pm 2.9''
\]