MAC2234: Quiz 5 *June 18, 2014*

Name: Sol V. Shunz, Esq.

Score	P1	P2	Q5
	E	E	E

P1. Derive the general solution to the logistic equation,

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{N}\right).$$

Write the solution in terms of y_0, the initial value of y at $t = 0$.

We need a lot of room to fully explain every step of P1, so for this week, the solutions will be 2 pages.

$$\frac{dy}{dt} = ky\left(\frac{N-y}{N}\right)$$

$$\frac{N}{y(N-y)}\frac{dy}{dt} = k$$

$$\int\left(\frac{1}{y} + \frac{1}{N-y}\right)dy = \int k\,dt$$

$$\ln y - \ln(N-y) = kt + c$$

$$e^{\ln y - \ln(N-y)} = e^{kt+c}$$

$$e^{\ln\frac{y}{N-y}} = e^{kt}e^{c}$$

Now, for convenience, we will write A for e^c.

$$\frac{y}{N-y} = Ae^{kt}$$

$$y = Ae^{kt}(N-y) = Ae^{kt}N - Ae^{kt}y$$

$$y + Ae^{kt}y = ANe^{kt}$$

$$y(1 + Ae^{kt}) = ANe^{kt}$$

$$y = \frac{ANe^{kt}}{1 + Ae^{kt}}$$

Now, for convenience, we write B for $\frac{1}{A}$.

P2. Solve the initial value problem.

$$\begin{cases} 2y + t\frac{dy}{dt} - e^{2t} = 0 \\ y\left(\frac{1}{2}\right) = \sqrt{3} \end{cases}$$

$$y = \frac{N}{1 + Be^{-kt}}$$

Describe the long term behavior of the solution.

To get the solution in terms of y_0, we must solve $y(0) = y_0$ for B.

$$y(0) = y_0 = \frac{N}{1 + Be^{-k(0)}} = \frac{N}{1+B}$$

$$y_0(1+B) = N$$

$$B = \frac{N - y_0}{y_0}$$

So, the final answer is

$$\boxed{y(t) = \frac{N}{1 + \left(\frac{N-y_0}{y_0}\right)e^{-kt}}}$$

MAC2234: Quiz 5 June 18, 2014

Name: Rev. Dr. D. Ansuerbe

Score	E P1	E P2	E Q5

P1. Derive the general solution to the logistic equation,

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{N}\right).$$

Write the solution in terms of y_0, the initial value of y at $t = 0$.

P2. Solve the initial value problem.

$$\begin{cases} 2y + t\frac{dy}{dt} - e^{2t} &= 0 \\ y\left(\frac{1}{2}\right) &= \sqrt{3} \end{cases}$$

Describe the long term behavior of the solution.

start here

$$\frac{dy}{dt} + \frac{2}{t}y = \frac{e^{2t}}{t}$$

$$\mu(t) = \int \frac{2}{t}\,dt = 2\ln t = \ln t^2$$

$$I(t) = e^{\mu(t)} = e^{\ln t^2} = t^2$$

$$t^2 y = \int t^2 \frac{e^{2t}}{t}\,dt = \int t e^{2t}\,dt$$

$u = t \quad dv = e^{2t}$

$du = dt \quad v = \frac{1}{2}e^{2t}$

$$t^2 y = \frac{e^{2t}t}{2} - \frac{1}{2}\int e^{2t}\,dt$$

$$= \frac{e^{2t}}{2}\left(t - \frac{1}{2}\right) + c$$

$$y = \frac{e^{2t}}{2t}\left(1 - \frac{1}{2t}\right) + \frac{c}{t^2}$$

(for complete solution, we solve for c)

$$y\left(\tfrac{1}{2}\right) = \frac{e^{2(1/2)}}{2(1/2)}\left(1 - \frac{1}{2(1/2)}\right) + \frac{c}{(1/2)^2}$$

$$= e(1 - 1) + \frac{c}{(1/4)}$$

$$= 4c$$

$$4c = \sqrt{3} \rightarrow c = \frac{\sqrt{3}}{4}$$

So our final answer is:

$$\boxed{y(t) = \frac{e^{2t}}{2t}\left(1 - \frac{1}{2t}\right) + \frac{\sqrt{3}}{4t^2}}$$

for this part, we evaluate $\lim_{t\to\infty} y(t)$.

$$\lim_{t\to\infty} y(t) = \left(\lim_{t\to\infty}\frac{e^{2t}}{2t}\right)(1) + 0 \quad \text{(by linearity)}$$

$$= \lim_{t\to\infty}\frac{e^{2t}}{2t}$$

$$= \infty \quad \text{(by L'Hôpital)}$$

So the solution increases without bound.